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ON THE TWO DEFINITIONS OF Ho(pro C)*

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We show that the two constructions of a homotopy procategory, **Ho(pro C)** given by the author [8] and by Edwards and Hastings [2] yield isomorphic categories.

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Homotopy procategory   level weak equivalence

1. Introduction

In 1974, the author introduced a homotopy procategory **Ho(pro SS)** of simplicial sets, [4], as a tool in the study of the stability problem in shape theory. He developed this idea in an abstract setting in 1976, ([9], [10] and [11]). Edwards and Hastings, in 1976, produced a Quillen model category structure on categories of the form **pro** C where C itself had a (nicely behaved) model category structure (see [6]). Their category **Ho(pro** C) and the one introduced by me were clearly closely related, but no proof that they were the same has, to my knowledge, appeared in the literature. This short note rectifies this lack.

The results proved here, namely that both constructions yield the same homotopy category, has spin off in two directions. It allows workers in proper homotopy theory, and the other areas opened up by Edwards and Hastings, access to the obstruction theoretic methods I developed in [11]. It also allows the coherent homotopy theory, which formed a base for my work, to be made available for the further development of Edwards' and Hastings' ideas.

Combining the result of this note with the much deeper results of Cordier [4], who shows that the Lisica–Mardešić coherent prohomotopy category **CPHTop** [7] is isomorphic to the category **Hopro(Top)** that I defined in 1974, one obtains a full proof of the equivalence of the Edwards–Hastings homotopy procategory and the Lisica–Mardešić coherent prohomotopy category [7]. This in turn shows the equivalence of the strong shape theory of Edwards–Hastings and that of Lisica–Mardešić. The methods introduced by Cordier [2], [3] and [4], and with the author

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[5] show that it is possible to avoid much of the detailed computational work required by the Lisica–Mardešić model, replacing it by categorical machinery.

I would like to thank Jean-Marc Cordier for clearing up my earlier confusion on how much is still needed to be done to complete the proof of the equivalence, i.e. nothing more than this note!

I would also like to thank Luis Hernandez for asking me if the two constructions *did* yield the same category and for patiently listening to my attempts to prove that they did. I would also like to thank the British Council and the Universidad de Zaragoza for financial help towards the visit during which this work was done.

2. The two constructions

Although both constructions invert a class of morphisms called 'weak equivalences', it is not immediately clear how these two classes are related. To start with, therefore, we must briefly consider in detail the definition of these two classes.

(a) References [8], [9], [10] and [11]. Let $\boldsymbol{C}$ be a category and $\boldsymbol{W}$ a class of morphisms which will be called *weak equivalences*. We use the following reindexing lemma (cf. Artin–Mazur [1] App.)

Given any $f: X \to Y$ in **pro** $\boldsymbol{C}$, say with $X: \boldsymbol{I} \to \boldsymbol{C}$, $Y: \boldsymbol{J} \to \boldsymbol{C}$ there is a filtering category, $\boldsymbol{I}_f$, cofinal functors $\phi_X: \boldsymbol{I}_f \to \boldsymbol{I}$, $\phi_Y: \boldsymbol{I}_f \to \boldsymbol{J}$ and a natural transformation $\bar{f}: X\phi_X \to Y\phi_Y$ such that the diagram,

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ X\phi_X & \xrightarrow{\;\bar{f}\;} & Y\phi_Y \end{array}$$

is commutative in **pro** $\boldsymbol{C}$.

We say that $\bar{f}$ is a level map and that it is obtained by reindexing f. Of course, f will have many different reindexings. Now we define the class, S, as being the class of all $f: X \to Y$ in **pro** $\boldsymbol{C}$ such that for some reindexing $f': X' \to Y'$ of f, f' is a *level weak equivalence*, i.e. for each index a of X' and Y' (remember f' is a natural transformation), $f'(a) \in W$.

We now form the quotient category (**pro** $\boldsymbol{C}$)(S^{-1}) with canonical functor

$$\gamma: \quad \textbf{pro } \boldsymbol{C} \to (\textbf{pro } \boldsymbol{C})(S^{-1})$$

and set $\bar{S} = \{f: X \to Y \text{ in } \textbf{pro } \boldsymbol{C}: \gamma(f) \text{ is an isomorphism}\}$. We call $\bar{S}$ the *saturation* of S.

(b) Reference [6]. Edwards and Hastings work with a richer structure. In their category $\boldsymbol{C}$, there are distinguished classes of cofibrations and fibrations as well as weak equivalences. A map which is at one and the same time a weak equivalence and a cofibration is called a *trivial cofibration*, similarly for a *trivial fibration*.

A map f in **pro** C will be called a *strong trivial cofibration* if it can be reindexed to obtain a level trivial cofibration.

A map f in **pro** C will be called a *strong trivial fibration* if it can be reindexed to obtain a trivial fibration in some C^J. The description of (trivial) fibrations in C^J takes too long to summarize here (see [6, pp. 60–67]) but amongst other properties we note that if f is a trivial fibration in C^J then each $f(j)$ is a trivial fibration in C.

A map $f: X \to Y$ in a category is called a *retract* of another map $f': X' \to Y'$ if there is a commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X' & \xrightarrow{f'} & Y' \\ {\scriptstyle r_X}\downarrow & & \downarrow{\scriptstyle r_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where $r_X i_X = \mathrm{Id}_X$, $r_Y i_Y = \mathrm{Id}_Y$.

A map in **pro** C will be called a *trivial cofibration* if it is a retract of a strong trivial cofibration, similarly for a trivial fibration. A map f in **pro** C is a *weak equivalence* if $f = pi$ where p is a trivial fibration and i is a trivial cofibration.

In addition to the above Edwards and Hastings assume that C satisfies "condition N" (see [6, p. 45]). Amongst other things this assumes the existence of a functorial cylinder. (A similar requirement was made by me in [8].)

Let Σ denote the class of weak equivalences in the above sense. Edwards and Hastings (cf. [6, 2.3.8]) point out that Σ is in fact saturated as this is a consequence of the model category structure [12, p. I.5.5].

3. The comparison

Certain relationships between $\bar{S}$ and Σ are fairly easy to derive.

Firstly we note that if $f' \in \bar{S}$ and f is a retract of f' then f is in $\bar{S}$. To see this we argue as follows using the notation introduced earlier.

The morphism $\gamma(f')$ in (**pro** C)(S^{-1}) is assumed to be invertible say with inverse $\phi: Y' \to X'$. Define a morphism $\psi: Y \to X$ by

$$\psi = \gamma(r_X)\phi\gamma(i_Y)$$

We calculate $\psi\gamma(f)$,

$$\psi\gamma(f) = \gamma(r_X)\phi\gamma(i_Y)\gamma(f) = \gamma(r_X)\phi\gamma(f')\gamma(i_X)$$
$$= \gamma(r_X)\gamma(i_X) = \mathrm{Id}_X.$$

Similarly $\gamma(f)\psi = \mathrm{Id}_Y$, thus proving our claim.

Any strong trivial cofibration or strong trivial fibration is in S and hence in $\bar{S}$. By the above, any trivial fibration or trivial cofibration in the sense of Edwards and Hastings is thus also in $\bar{S}$. It follows of course that any weak equivalence in their sense is in $\bar{S}$, so $\Sigma \subseteq \bar{S}$.

To prove that the categories (**pro** C)(S^{-1}) and (**pro** C)(Σ^{-1}) are isomorphic it suffices to prove $\Sigma = \bar{S}$, we therefore now turn to proving that $S \subset \Sigma$. As this will imply that all f in S are invertible in (**pro** C)(Σ^{-1}), it will prove that $\bar{S} \subseteq \Sigma$, thus completing the proof.

Suppose f is a level weak equivalence. Using the functorial cylinder, we form the mapping cylinder of f and a factorization of f as

$$X \xrightarrow{i_f} M_f \xrightarrow{p_f} Y$$

Here i_f is a strong trivial cofibration and p_f is constructed as follows.

The mapping cylinder is formed as the pushout in the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle e_1(X)}\downarrow & & \downarrow{\scriptstyle j_f} \\ X \times I & \xrightarrow{\pi_f} & M_f \end{array}$$

The collapsing map $\sigma: X \times I \to X$ satisfies $\sigma e_1(X) = \mathrm{Id}_X$. This induces the map $p_f: M_f \to Y$ satisfying $p_f j_f = \mathrm{Id}_Y$. The cofibration i_f is $\pi_f e_0(X)$.

If we assume as in [6] that the trivial cofibrations are stable under cobase change, then as $e_1(X)$ is a trivial cofibration so if j_f. Hence p_f is a one-sided inverse to a trivial cofibration.

We thus have $f = p_f i_f$ where i_f is in Σ and p_f is a one-sided inverse to something in Σ. As Σ is saturated p_f must itself be in Σ, hence $f \in \Sigma$. Since any isomorphism is in Σ, it is clear that any weak equivalence (in my sense) is in Σ, i.e. $S \subseteq \Sigma$ so $\bar{S} \subseteq \Sigma$. This completes the proof.

Thus the two constructions of a homotopy pro-category **Ho(pro** C) as (**pro** C)(S^{-1}) and (**pro** C)(Σ^{-1}) do in fact yield the same category if C has a Quillen model category structure satisfying condition N of Edwards and Hastings.

References

[1] M. Artin and B. Mazur, Étale homotopy, Lecture Notes Math. 100 (Springer, Berlin, 1969).
[2] J.-M. Cordier, Sur la notion de diagramme homotopiquement cohérent, Cahiers Top. Géom. Diff. 23 (1982) 93–112.
[3] J.-M. Cordier, Sur les limites homotopiques de diagrammes homotopiquement cohérent, Compositio Math. 62 (1987) 367–388.
[4] J.-M. Cordier, Une représentation de **Ho(Top**^{A}), preprint, Amiens, 1985.
[5] J.-M. Cordier and T. Porter, Vogt's theorem on categories of homotopy coherent diagrams, Math. Proc. Camb. Phil. Soc. 100 (1986) 65–90.

[6] D.A. Edwards and H.M. Hastings, Čech and Steenrod Homotopy Theories with Applications to Geometric Topology, Lecture Notes Math. 542 (Springer, Berlin, 1976).
[7] J.T. Lisica and S. Mardešić, Coherent prohomotopy and strong shape, Glasnik Mat. 19 (39) (1984) 335-399.
[8] T. Porter, Stability results for topological spaces, Math. S. 140 (1974) 1-21.
[9] T. Porter, Abstract homotopy theory in procategories, Cahiers Topol. Géom. Diff. 17 (1976) 113-124.
[10] T. Porter, Coherent prohomotopical algebra, Cahiers Topol. Géom. Diff. 18 (1977) 139-179.
[11] T. Porter, Coherent prohomotopy theory, Cahiers Topol. Géom. Diff. 19 (1978) 3-46.
[12] D. Quillen, 'Homotopical Algebra', Lecture Notes Math. 43, (Springer, Berlin, 1967).