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On a modification of a discrete epidemic model

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ABSTRACT

In this paper under some conditions on the constants $A, B \in (0, \infty)$ we study the existence of positive solutions, the existence of a unique nonnegative equilibrium and the convergence of the positive solutions to the nonnegative equilibrium of the system of difference equations

$$x_{n+1} = (1 - y_n - y_{n-1})(1 - e^{-Ay_n}), \qquad y_{n+1} = (1 - x_n - x_{n-1})(1 - e^{-Bx_n})$$

where $A, B \in (0, \infty)$ and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers which satisfy the relations $x_0 + x_{-1} < 1$, $y_0 + y_{-1} < 1$, $1 - y_0 > (1 - x_0 - x_{-1})(1 - e^{-Bx_0})$, $1 - x_0 > (1 - y_0 - y_{-1})(1 - e^{-Ay_0})$.

1. Introduction

In [1] Kocic and Ladas proposed the study of the difference equation

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)(1 - e^{-Ax_n}), \quad k \in \{2, 3, \ldots\} \tag{1.1}$$

which is a special case of an epidemic model (see [2,3]).

Moreover, in [4] Zhang and Shi studied the oscillation, the behavior of the solutions of Eq. (1.1), where $A \in (0, \infty)$, $k \in \{2, 3, \ldots, \}$ and the initial values $x_{-k+1}, \ldots, x_0$ are arbitrary positive numbers such that $\sum_{j=0}^{k-1} x_{-j} < 1$.

Finally, in [5] Stevic studied Eq. (1.1), where $A \in (0, \infty)$, $k \in \{2, 3, \ldots, \}$ and the initial values $x_{-k+1}, \ldots, x_0$ are arbitrary negative numbers.

Now, in this paper under some conditions on the constants $A, B \in (0, \infty)$ we study the existence of positive solutions, the existence of a unique nonnegative equilibrium and the convergence of the positive solutions to the nonnegative equilibrium of the system of difference equations

$$x_{n+1} = (1 - y_n - y_{n-1})(1 - e^{-Ay_n}), \qquad y_{n+1} = (1 - x_n - x_{n-1})(1 - e^{-Bx_n}) \tag{1.2}$$

where the initial values x_{-1}, x_0, y_{-1}, y_0 satisfy the relations

$$\begin{aligned} &x_{-1}, x_0, y_{-1}, y_0 > 0, \qquad x_0 + x_{-1} < 1, \qquad y_0 + y_{-1} < 1, \\ &1 - y_0 > (1 - x_0 - x_{-1})(1 - e^{-Bx_0}), \qquad 1 - x_0 > (1 - y_0 - y_{-1})(1 - e^{-Ay_0}). \end{aligned} \tag{1.3}$$

It is obvious that if $A = B$ and $x_{-1} = y_{-1}, x_0 = y_0$ then system (1.2) reduces to Eq. (1.1) for $k = 2$.

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2. Main Results

In the first proposition we study the existence of the positive solutions of (1.2).

Proposition 2.1. *Consider system (1.2) where the constants A, B satisfy*

$$0 < A \le 6, \qquad 0 < B \le 6. \tag{2.1}$$

Let (x_n, y_n) be a solution of (1.2) with initial values x_{-1}, x_0, y_{-1}, y_0 satisfying (1.3). Then

$$x_n > 0, \qquad y_n > 0, \quad n = 1, 2, \ldots. \tag{2.2}$$

Proof. From (1.2) we get

$$x_1 = (1 - y_0 - y_{-1})(1 - \mathrm{e}^{-Ay_0}), \qquad y_1 = (1 - x_0 - x_{-1})(1 - \mathrm{e}^{-Bx_0}). \tag{2.3}$$

Then from (1.3), (2.1) and (2.3) we take

$$x_1 > 0, \qquad y_1 > 0. \tag{2.4}$$

In addition, from (1.2) we have,

$$x_2 = (1 - y_1 - y_0)(1 - \mathrm{e}^{-Ay_1}), \qquad y_2 = (1 - x_1 - x_0)(1 - \mathrm{e}^{-Bx_1}). \tag{2.5}$$

Using (1.3) and (2.3) it follows

$$1 - y_0 > y_1, \qquad 1 - x_0 > x_1. \tag{2.6}$$

Therefore, relations (2.1), (2.4), (2.5) and (2.6) imply that

$$x_2 > 0, \qquad y_2 > 0.$$

We prove now that

$$x_n + x_{n-1} < 1, \qquad y_n + y_{n-1} < 1, \quad n = 2, 3, \ldots. \tag{2.7}$$

From (1.2), (1.3) and (2.1) we have

$$y_2 + y_1 < (1 - x_1 - x_0)(1 - \mathrm{e}^{-Bx_1}) + (1 - x_0)(1 - \mathrm{e}^{-Bx_0}). \tag{2.8}$$

We consider the function f, for $x, y > 0,\ x + y < 1,\ 0 < B \le 6$, as follows

$$f(x, y, B) = (1 - x - y)(1 - \mathrm{e}^{-Bx}) + (1 - y)(1 - \mathrm{e}^{-By}). \tag{2.9}$$

Since f is an increasing function with respect to B we have that

$$f(x, y, B) \le (1 - x - y)(1 - \mathrm{e}^{-6x}) + (1 - y)(1 - \mathrm{e}^{-6y}), \quad x, y > 0,\ x + y < 1,\ 0 < B \le 6. \tag{2.10}$$

We set the function h as follows

$$h(x, y) = (1 - x - y)(1 - \mathrm{e}^{-6x}) + (1 - y)(1 - \mathrm{e}^{-6y}), \quad x, y \ge 0,\ x + y \le 1. \tag{2.11}$$

Then, we take the system of equations

$$\begin{aligned} \frac{\partial h}{\partial x} &= -1 + \mathrm{e}^{-6x} + 6\mathrm{e}^{-6x}(1 - x - y) = 0 \\ \frac{\partial h}{\partial y} &= -2 + \mathrm{e}^{-6x} + \mathrm{e}^{-6y} + 6\mathrm{e}^{-6y}(1 - y) = 0, \quad x, y > 0,\ x + y < 1. \end{aligned} \tag{2.12}$$

System (2.12) is equivalent to system

$$\begin{aligned} & y = \frac{7 - \mathrm{e}^{6x} - 6x}{6}, \\ & -2 + \mathrm{e}^{-6x} + (\mathrm{e}^{6x} + 6x)\mathrm{e}^{-7 + \mathrm{e}^{6x} + 6x} = 0, \quad x, y > 0,\ x + y < 1. \end{aligned} \tag{2.13}$$

We consider the function

$$g(x) = -2 + \mathrm{e}^{-6x} + (\mathrm{e}^{6x} + 6x)\mathrm{e}^{-7 + \mathrm{e}^{6x} + 6x}, \quad 0 \le x \le 1. \tag{2.14}$$

From (2.14) we have

$$g(0) = -1 + e^{-6} < 0, \qquad g(1) = -2 + e^{-6} + (e^6 + 6)e^{-1+e^6} > 0, \tag{2.15}$$

and

$$g''(x) = 36e^{-6x} + e^{-7+e^{6x}+12x}(36 + 2(6 + 6e^{6x})^2 + 36(e^{6x} + 6x) + (6 + 6e^{6x})^2(e^{6x} + 6x)) > 0, \quad 0 < x < 1. \tag{2.16}$$

Using (2.15) and (2.16) equation $g(x) = 0$ has a unique solution $\bar{x}$, such that $0 < \bar{x} < 1$. Using Newton's Method we have that

$$\bar{x} = 0.244745, \tag{2.17}$$

with the precision set to six decimal places. From (2.13) for $x = \bar{x}$ we take,

$$\bar{y} = 0.198157. \tag{2.18}$$

Furthermore, from (2.12) we take

$$\begin{aligned} \frac{\partial^2 h}{\partial x^2} &= -12e^{-6x} - 36e^{-6x}(1 - x - y) < 0, \quad \text{for } x + y < 1, \\ \frac{\partial^2 h}{\partial y^2} &= -12e^{-6y} - 36e^{-6y}(1 - y), \\ \frac{\partial^2 h}{\partial x \partial y} &= -6e^{-6x}. \end{aligned} \tag{2.19}$$

From (2.19) and for $x \geq y > 0$ and $x + y < 1$ we have

$$D(x, y) = \frac{\partial^2 h}{\partial x^2}\frac{\partial^2 h}{\partial y^2} - \left(\frac{\partial^2 h}{\partial x \partial y}\right)^2 = 36e^{-12x}\left(4e^{6(x-y)}(1 + 3(1 - x - y))(1 + 3(1 - y)) - 1\right) > 0. \tag{2.20}$$

So, from (2.17), (2.18) and (2.20) we get

$$D(\bar{x}, \bar{y}) > 0. \tag{2.21}$$

Since from (2.17) and (2.18), $(\bar{x}, \bar{y}) = (0.244745, 0.198157)$ is the unique solution of system (2.13), using (2.19) and (2.21)

$$h(x, y) \leq h(0.244745, 0.198157) = 0.986458, \quad \text{for } x, y > 0,\ x + y < 1. \tag{2.22}$$

Now, suppose that

$$y = 0, \quad 0 \leq x \leq 1 \quad (\text{resp. } x = 0,\ 0 \leq y \leq 1).$$

Then from (2.11) we get

$$\begin{aligned} &h(x, y) = (1 - x)(1 - e^{-6x}) < 1, \quad \text{for } y = 0,\ 0 \leq x \leq 1 \\ &(\text{resp. } h(x, y) = (1 - y)(1 - e^{-6y}) < 1,\ \text{for } x = 0,\ 0 \leq y \leq 1). \end{aligned} \tag{2.23}$$

Finally, suppose that

$$x + y = 1, \quad x, y \geq 0.$$

Then from (2.11) we get

$$h(x, y) = (1 - y)(1 - e^{-6y}) < 1, \quad \text{for } x + y = 1,\ x, y \geq 0. \tag{2.24}$$

From (2.8)–(2.11) and (2.22)–(2.24) we get that

$$y_2 + y_1 < 1. \tag{2.25}$$

Similarly, we can prove that

$$x_2 + x_1 < 1. \tag{2.26}$$

So, relations (2.25) and (2.26) imply that (2.7) are satisfied for $n = 2$. Working inductively we can prove that (2.7) are true for all $n = 3, 4, \ldots$. Then it is obvious that (2.2) are satisfied. This completes the proof of the proposition. □

In what follows we need the following lemma.

Lemma 2.1. *Consider the functions*

$$g(x) = (1-2x)(1-e^{-Bx}), \qquad h(x) = (1-2x)(1-e^{-4x}),$$
$$H(x) = -2(1-e^{-Ag(x)}) + Ae^{-Ag(x)}(1-2g(x)), \tag{2.27}$$
$$F(x) = -2(1-e^{-4h(x)}) + 4e^{-4h(x)}(1-2h(x)),$$

where

$$1 < A \le 4, \quad 1 < B \le 4. \tag{2.28}$$

Then the following statements are true.

(i)

$$g'(x) > h'(x), \quad \text{for } 0.19 < x < 0.373. \tag{2.29}$$

(ii)

$$H(x) > F(x), \quad \text{for } 0.19 < x < 0.373. \tag{2.30}$$

(iii) *Equation*

$$h'(x) = 0 \quad (\text{resp. } F(x) = 0)$$

has a unique solution c (resp. d), such that

$$c \in (0.19, 0.373), \quad (\text{resp. } d \in (c, 0.373)). \tag{2.31}$$

(iv)

$$h'(x)F(x) < 1, \quad \text{for } c < x < d. \tag{2.32}$$

Proof. (i) From (2.27) we get

$$g'(x) = -2(1-e^{-Bx}) + Be^{-Bx}(1-2x), \qquad h'(x) = -2(1-e^{-4x}) + 4e^{-4x}(1-2x). \tag{2.33}$$

We set

$$R(B) = -2(1-e^{-Bx}) + Be^{-Bx}(1-2x), \quad 1 < B \le 4, \tag{2.34}$$

then

$$\frac{dR}{dB} = e^{-Bx}[Bx(2x-1) - 4x + 1]. \tag{2.35}$$

If $\frac{dR}{dB} = 0$ then $B = \frac{1-4x}{x(1-2x)}$. We set

$$\Phi(x) = \frac{1-4x}{x(1-2x)}. \tag{2.36}$$

It is easy to prove that $\Phi(x)$ is a decreasing function for $0 < x < 0.5$ and so

$$\Phi(x) \le \Phi(0.19) = \frac{2400}{1178}, \quad \text{for } 0.19 \le x \le 0.373. \tag{2.37}$$

From (2.35)–(2.37) and since $0.19 \le x \le 0.373$ we get that $R(B)$ is a decreasing function if $\frac{2400}{1178} < B \le 4$, which means that

$$R\left(\frac{2400}{1178}\right) > R(B) \ge R(4), \quad \text{for } \frac{2400}{1178} < B \le 4,\ 0.19 \le x \le 0.373, \tag{2.38}$$

and obviously

$$R\left(\frac{2400}{1178}\right) > R(4), \quad \text{for } 0.19 \le x \le 0.373. \tag{2.39}$$

Now, suppose that

$$1 < B \le \frac{2400}{1178}. \tag{2.40}$$

Since $\Phi(x)$ is a decreasing function for $0 < x < 0.5$ we have

$$\Phi(x) \leq \Phi\left(\frac{5-\sqrt{17}}{4}\right) = 1, \quad \frac{5-\sqrt{17}}{4} \leq x \leq 0.373. \tag{2.41}$$

From (2.35), (2.36), (2.40) and (2.41) we get that $R(B)$ is a decreasing function and so from (2.39)

$$R(1) > R(B) \geq R\left(\frac{2400}{1178}\right) > R(4), \quad \text{for } 1 < B \leq \frac{2400}{1178}, \ \frac{5-\sqrt{17}}{4} \leq x \leq 0.373, \tag{2.42}$$

and obviously

$$R(1) > R\left(\frac{2400}{1178}\right), \quad \text{for } \frac{5-\sqrt{17}}{4} \leq x \leq 0.373. \tag{2.43}$$

Finally, suppose that (2.40) and

$$0.19 \leq x < \frac{5-\sqrt{17}}{4} \tag{2.44}$$

hold. Since $\Phi(x)$ is a decreasing function $0 < x < 0.5$ we have

$$1 < \Phi(x) \leq \frac{2400}{1178}, \quad 0.19 \leq x < \frac{5-\sqrt{17}}{4}. \tag{2.45}$$

From (2.35), (2.36), (2.40) and (2.45) we have that for every x, such that (2.44) holds, there exists a $B_0(x)$ such that

$$\frac{dR}{dB} > 0, \quad \text{for } 1 < B < B_0(x) \quad \text{and} \quad \frac{dR}{dB} < 0, \quad \text{for } B_0(x) < B \leq \frac{2400}{1178}. \tag{2.46}$$

We claim that

$$R(1) > R(4), \quad \text{for } 0.19 \leq x < \frac{5-\sqrt{17}}{4}. \tag{2.47}$$

From (2.34) and after some calculations, in order to prove (2.47) it is sufficient to prove that

$$e^{3x} + \frac{8x-6}{3-2x} > 0, \qquad 0.19 \leq x < \frac{5-\sqrt{17}}{4},$$

which is true, since if we set

$$w(x) = e^{3x} + \frac{8x-6}{3-2x},$$

it is easy to prove that $w(x)$ is an increasing function for every x, such that (2.44) holds, and $w(0.19) > 0$.

Relations (2.39), (2.46) and (2.47) imply that

$$R(B) \geq \min\left\{R(1), R\left(\frac{2400}{1178}\right)\right\} > R(4), \quad \text{for } 1 < B \leq \frac{2400}{1178}, \ 0.19 \leq x < \frac{5-\sqrt{17}}{4}. \tag{2.48}$$

From (2.33), (2.34), (2.38), (2.42) and (2.48) we have that relation (2.29) is true. This completes the proof of statement (i).

(ii) Since

$$(1-2x)(1-e^{-Bx})$$

is an increasing function with respect to B, for $1 < B \leq 4$ and $0.19 \leq x \leq 0.373$ we have from (2.27)

$$0 < g(x) \leq h(x), \qquad 1 < B \leq 4, \qquad 0.19 \leq x \leq 0.373. \tag{2.49}$$

We claim that

$$0.19 \leq h(x) \leq 0.373, \qquad 0.19 \leq x \leq 0.373. \tag{2.50}$$

From (2.33) we get

$$h'(0.19) = 0.0951456, \qquad h'(0.373) = -1.32163,$$

and

$$h''(x) = -16e^{-4x} - 16e^{-4x}(1-2x) < 0, \quad \text{for } 0.19 \leq x \leq 0.373. \tag{2.51}$$

So, equation

$$h'(x) = 0$$

has a unique solution in the interval (0.19, 0.373). Using Newton's Method we can see that this solution is $c = 0.198015$ with the precision set to six decimal places.

Therefore, and after some calculations, we have

$$0.19 \leq \min\{h(0.19), h(0.373)\} \leq h(x) \leq h(0.198015) \leq 0.373, \quad \text{for } 0.19 \leq x \leq 0.373,$$

which implies that (2.50) is true.

Moreover, we consider the function

$$K(x) = -2(1 - e^{-Ax}) + Ae^{-Ax}(1 - 2x). \tag{2.52}$$

It is easy to prove that $K(x)$ is a decreasing function for $1 < A \leq 4$ and $0 < x < 0.5$, and so from (2.27), (2.49), (2.50) and (2.52) we get that

$$H(x) = K(g(x)) \geq K(h(x)), \qquad 1 < A \leq 4,\ 0.19 \leq x \leq 0.373. \tag{2.53}$$

From (2.27), (2.28), (2.50), (2.52) and using (2.29) and (2.33), where in stand of B and x we set A and $h(x)$ respectively, we have

$$\begin{aligned} K(h(x)) &= -2(1 - e^{-Ah(x)}) + Ae^{-Ah(x)}(1 - 2h(x)) \\ &> -2(1 - e^{-4h(x)}) + 4e^{-4h(x)}(1 - 2h(x)) = F(x), \quad 1 < A \leq 4,\ 0.19 \leq x \leq 0.373. \end{aligned} \tag{2.54}$$

Relations (2.53) and (2.54) imply that (2.30) is true. This completes the proof of statement (ii).

(iii) From statement (ii) equation $h'(x) = 0$ has a unique solution $c = 0.198015$ and obviously, $c \in (0.19, 0.3730)$.

From (2.27) we get

$$\begin{aligned} F'(x) &= -16e^{-4h(x)}h'(x) - 16e^{-4h(x)}h'(x)(1 - 2h(x)), \\ F''(x) &= 96e^{-4h(x)}(h'(x))^2 - 16e^{-4h(x)}h''(x) + 64e^{-4h(x)}(h'(x))^2(1 - 2h(x)) - 16e^{-4h(x)}h''(x)(1 - 2h(x)). \end{aligned} \tag{2.55}$$

Since c is the unique solution of equation $h'(x) = 0$ from (2.31) and (2.51) we have that

$$h'(x) < h'(c) = 0, \quad \text{for } c < x \leq 0.373. \tag{2.56}$$

In addition, from (2.31), (2.50), (2.55) and (2.56) we get that

$$F'(x) > 0, \quad \text{for } c < x \leq 0.373. \tag{2.57}$$

Obviously, from (2.50), (2.51) and (2.55)

$$F''(x) > 0, \quad \text{for } 0.19 \leq x \leq 0.373. \tag{2.58}$$

From (2.27) we get

$$F(c) = F(0.198015) = -1.10486, \quad F(0.373) = 0.0133519,$$

and so from (2.57) equation

$$F(x) = 0$$

has a unique solution in the interval $(c, 0.373)$. Using Newton's Method, we can prove that this solution is $d = 0.372132$ with the precision set to six decimal places. Obviously, $d \in (c, 0.3730)$. This completes the proof of statement (iii).

(iv) We consider the function

$$Q(x) = h'(x)F(x), \quad c \leq x \leq d, \tag{2.59}$$

then

$$\begin{aligned} Q'(x) &= h''(x)F(x) + h'(x)F'(x), \\ Q''(x) &= h'''(x)F(x) + 2h''(x)F'(x) + h'(x)F''(x), \quad c \leq x \leq d, \end{aligned} \tag{2.60}$$

where from (2.31) and (2.51)

$$h'''(x) = 96e^{-4x} + 64e^{-4x}(1 - 2x) > 0, \quad \text{for } c \leq x \leq d. \tag{2.61}$$

Since d is the unique solution of equation $F(x) = 0$, from (2.31) and (2.57) we get that

$$F(x) < F(d) = 0, \quad c \leq x \leq d. \tag{2.62}$$

Relations (2.31), (2.51), (2.56)–(2.58), (2.60), (2.61), (2.62) imply that

$$Q''(x) < 0, \quad c \le x \le d. \tag{2.63}$$

Since from statement (iii) and (2.59)

$$Q(c) = Q(d) = 0 \tag{2.64}$$

then from (2.63) equation

$$Q'(x) = 0$$

has a unique solution in the interval (c, d). Using Newton's Method, we can prove that this solution is $x = 0.294221$, with the precision set to five decimal places.

Therefore, since from (2.27), (2.33) and (2.59)

$$Q(0.294221) = 0.707952,$$

and (2.64) holds, we have that (2.32) is true. This completes the proof of statement (iv) and the proof of the lemma. □

In the following proposition we study the existence of a unique positive equilibrium for system (1.1).

Proposition 2.2. *Consider the system of algebraic equations*

$$\begin{aligned} x &= (1-2y)(1-\mathrm{e}^{-Ay}) \\ y &= (1-2x)(1-\mathrm{e}^{-Bx}), \quad x, y \in (0, 0.5). \end{aligned} \tag{2.65}$$

Then the following statements are true:

(i) *If*

$$0 < A \le 1, \qquad 0 < B \le 1 \tag{2.66}$$

the system (2.65) *has a unique nonnegative solution* $(\bar{x}, \bar{y}) = (0, 0)$.

(ii) *If* (2.28) *holds, then system* (2.65) *has a unique positive solution* $(\bar{x}, \bar{y})$, $\bar{x}, \bar{y} \in (0, 0.5)$.

Proof. (i) We consider the functions

$$E(y) = (1-2y)(1-\mathrm{e}^{-Ay}) - y, \qquad K(x) = (1-2x)(1-\mathrm{e}^{-Bx}) - x, \quad x, y \in (0, 0.5). \tag{2.67}$$

Then from (2.66) and (2.67) we get

$$\begin{aligned} E'(y) &= -2(1-\mathrm{e}^{-Ay}) + A\mathrm{e}^{-Ay}(1-2y) - 1 < 0, \\ K'(x) &= -2(1-\mathrm{e}^{-Bx}) + B\mathrm{e}^{-Bx}(1-2x) - 1 < 0 \end{aligned}$$

which imply that

$$E(y) \le E(0) = 0, \qquad K(x) \le K(0) = 0. \tag{2.68}$$

Therefore, from (2.65), (2.67) and (2.68) we take that $(\bar{x}, \bar{y}) = (0, 0)$ is the unique nonnegative solution for system (2.65). This completes the proof of statement (i).

(ii) Suppose that (2.28) holds. We set

$$G(x) = (1-2g(x))(1-\mathrm{e}^{-Ag(x)}) - x, \quad x \in [0, 0.5], \tag{2.69}$$

where $g(x)$ was defined in (2.27). From (2.28) we have that

$$0 < g(x) < (1-2x)Bx = -2Bx^2 + Bx \le \frac{B}{8} \le \frac{1}{2}, \quad x \in (0, 0.5). \tag{2.70}$$

Furthermore, from (2.28) and (2.33) we get that

$$g''(x) = -4B\mathrm{e}^{-Bx} - B^2\mathrm{e}^{-Bx}(1-2x) < 0, \quad 0 < x < 0.5, \tag{2.71}$$

and

$$g'(0) = B > 0, \qquad g'(0.5) = -2(1-\mathrm{e}^{-\frac{B}{2}}) < 0, \tag{2.72}$$

and so $g'(x)$ has a unique solution $x_0 \in (0, 0.5)$ such that

$$g'(x) > 0, \quad \text{for } 0 < x < x_0 \quad \text{and} \quad g'(x) < 0, \quad \text{for } x_0 < x < 0.5. \tag{2.73}$$

In addition, from (2.27) and (2.69) we get

$$G(0) = 0, \qquad G(0.5) = -0.5, \tag{2.74}$$

and

$$G'(x) = -2g'(x)(1 - e^{-Ag(x)}) + Ag'(x)e^{-Ag(x)}(1 - 2g(x)) - 1 = g'(x)H(x) - 1. \tag{2.75}$$

Using (2.27), (2.28), (2.72) and (2.75)

$$G'(0) = AB - 1 > 0, \qquad G'(0.5) = -2A(1 - e^{-\frac{B}{2}}) - 1 < 0. \tag{2.76}$$

Relations (2.74) and (2.76) imply that equation $G(x) = 0$ has a solution $\bar{x} \in (0, 0.5)$. Then from (2.70)

$$0 < g(\bar{x}) < 0.5,$$

and so $(\bar{x}, \bar{y})$ is a solution of system (2.65), such that $\bar{x}, \bar{y} \in (0, 0.5)$.

We prove now that $(\bar{x}, \bar{y})$ is the unique solution of system (2.65).

From (2.27) we have that

$$H'(x) = -Ag'(x)e^{-Ag(x)}\left(4 + A(1 - 2g(x))\right), \tag{2.77}$$

and so from (2.28), (2.70) and (2.73) we get

$$H'(x) < 0, \quad \text{for } 0 < x < x_0 \quad \text{and} \quad H'(x) > 0, \quad \text{for } x_0 < x < 0.5. \tag{2.78}$$

First, suppose that

$$H(x) \geq 0, \quad \text{for } 0 < x < 0.5. \tag{2.79}$$

From (2.71), (2.73), (2.78) and (2.79) we have that $g'(x),\ H(x)$ are positive and decreasing functions for $0 < x < x_0$. So from (2.75), $G'(x)$ is a decreasing function for $0 < x < x_0$.

In addition, from (2.73), (2.75) and (2.79)

$$G'(x) < 0, \quad \text{for } x_0 < x < 0.5.$$

Therefore, from (2.76) we get that there exists a unique $x' \in (0, x_0)$, such that

$$G'(x) > 0, \quad \text{for } 0 < x < x' \quad \text{and} \quad G'(x) < 0, \quad \text{for } x' < x < 0.5.$$

Thus, $\bar{x}$ is the unique solution of equation $G(x) = 0$, such that $\bar{x} \in (0, 0.5)$ and so $(\bar{x}, \bar{y})$ is the unique solution of system (2.65), such that $\bar{x}, \bar{y} \in (0, 0.5)$.

Now, suppose that there exists an $x \in (0, 0.5)$, such that $H(x) < 0$. Then from (2.78) and since from (2.27) and (2.28)

$$H(0) = H(0.5) = A > 0 \tag{2.80}$$

we get that there exist exactly two real numbers x_1, x_2, such that

$$H(x_1) = H(x_2) = 0, \quad 0 < x_1 < x_0 < x_2 < 0.5. \tag{2.81}$$

From (2.78), (2.80), (2.81) we have that

$$H(x) > 0, \quad \text{for } 0 < x < x_1 \text{ or } x_2 < x < 0.5 \quad \text{and} \quad H(x) < 0, \quad \text{for } x_1 < x < x_2. \tag{2.82}$$

From (2.71), (2.73), (2.78) and (2.82), and since $x_1 < x_0$, we have that $g'(x),\ H(x)$ are positive and decreasing functions for $0 < x < x_1$. So from (2.75), $G'(x)$ is a decreasing function for $0 < x < x_1$, and since from (2.75), (2.76) and (2.81)

$$G'(0) > 0, \quad G'(x_1) = -1 < 0,$$

we have that there exists an $x'' \in (0, x_1)$, such that

$$G'(x) > 0, \quad \text{for } 0 < x < x'' \quad \text{and} \quad G'(x) < 0, \quad \text{for } x'' < x < x_1. \tag{2.83}$$

In addition, using (2.73), (2.75) and (2.82) and since $x_1 < x_0$

$$G'(x) < 0, \quad \text{for } x_1 < x < x_0. \tag{2.84}$$

We, also, claim that

$$G'(x) < 0, \quad \text{for } x_0 < x < x_2. \tag{2.85}$$

First, we prove that

$$c \leq x_0, \tag{2.86}$$

and

$$x_2 \leq d, \tag{2.87}$$

where c, d were defined in statement (iii) of Lemma 2.1.

Since x_0 is the unique solution of equation $g'(x) = 0$, for $0 < x < 0.5$, and (2.72) holds, in order to prove (2.86) it is sufficient to prove that

$$g'(c) > 0, \quad \text{for any } 1 < B \leq 4. \tag{2.88}$$

From statements (i) and (iii) of Lemma 2.1 we have

$$g'(c) > h'(c) = 0, \quad 1 < B \leq 4$$

and so (2.88) is true.

In addition, from (2.81) and (2.82), in order to prove (2.87), it is sufficient to prove that

$$x_0 < d \tag{2.89}$$

and

$$H(d) > 0 \quad \text{for any } 1 < A \leq 4,\ 1 < B \leq 4. \tag{2.90}$$

From (2.38), (2.42) and (2.43) we get

$$R(B) < R(1), \quad \text{for } 1 < B \leq 4 \quad \text{and} \quad \frac{5-\sqrt{17}}{4} \leq x \leq 0.373. \tag{2.91}$$

Using (2.33), (2.34) and (2.91) we get

$$g'(x) < -2(1-e^{-x}) + e^{-x}(1-2x), \quad 1 < B \leq 4 \text{ and } \frac{5-\sqrt{17}}{4} \leq x \leq 0.373,$$

and since from the proof of statement (iii) of Lemma 2.1, $d = 0.372132$, we have

$$g'(d) < -2(1-e^{-d}) + e^{-d}(1-2d) = -0.445204 < 0. \tag{2.92}$$

Relations (2.73) and (2.92) imply that (2.89) is true.

In addition, from statements (ii) and (iii) of Lemma 2.1 we have

$$H(d) > F(d) = 0,\ 1 < A \leq 4, \quad 1 < B \leq 4,$$

and so (2.90) is true.

From relations (2.73), (2.82), statements (i), (ii) of Lemma 2.1 and since $0.19 < x_1 < x_0 < x_2 < 0.373$, we get that

$$g'(x)H(x) < h'(x)F(x), \quad \text{for } x_0 < x < x_2. \tag{2.93}$$

Using statement (iv) of Lemma 2.1, relations (2.75), (2.86), (2.87), (2.93), and since $x_0 < x_2$, we get that our claim (2.85) is true.

Finally, from (2.73), (2.75), (2.82) and since $x_0 < x_2$ we get that

$$G'(x) < 0 \quad \text{for } x_2 < x < 0.5. \tag{2.94}$$

Relations (2.83)–(2.85) and (2.94) imply that $\bar{x}$ is the unique solution of equation $G(x) = 0$, such that $\bar{x} \in (0, 0.5)$, and so $(\bar{x}, \bar{y})$ is the unique positive solution of system (2.65) such that $\bar{x},\ \bar{y} \in (0, 0.5)$. This completes the proof of the proposition. □

In the last proposition we study the convergence of the positive solutions of system of difference equations (1.2).

Proposition 2.3. *Consider system (1.2). Let (x_n, y_n) be a solution of (1.2) such that (1.3) are satisfied. Then the following statements are true:*

(i) *If (2.66) are satisfied, the solution (x_n, y_n) tends to the zero equilibrium $(0, 0)$ of (1.2) as $n \to \infty$.*

(ii) *Suppose that (2.28) are satisfied and there exists an $m \in N$ such that for $n \geq m$ either*

$$x_n < \bar{x}, \qquad y_n < \bar{y} \tag{2.95}$$

or

$$x_n \geq \bar{x}, \qquad y_n \geq \bar{y}. \tag{2.96}$$

hold. Then (x_n, y_n) tends to the unique positive equilibrium $(\bar{x}, \bar{y})$ of (1.2) as $n \to \infty$.

Proof. (i) From (1.2), (1.3), (2.2), (2.7) and (2.66) we get

$$x_{n+1} < (1-y_n)(1-e^{-Ay_n}), \qquad y_{n+1} < (1-x_n)(1-e^{-Bx_n}), \quad n = 0, 1, \ldots.$$

Then from (1.3), (2.2), (2.7), (2.66) and Lemma 2.1 of [4] we have

$$x_{n+1} < y_n, \qquad y_{n+1} < x_n, \quad n = 0, 1, \ldots \tag{2.97}$$

which imply that $x_{2n}, x_{2n+1}, y_{2n}, y_{2n+1}$ are decreasing sequences. Therefore, there exist

$$\lim_{n\to\infty} x_{2n} = l_0, \quad \lim_{n\to\infty} x_{2n+1} = l_1, \quad \lim_{n\to\infty} y_{2n} = m_0, \quad \lim_{n\to\infty} y_{2n+1} = m_1. \tag{2.98}$$

Using (2.2), (2.7) and (2.98) we get

$$0 \le l_0, l_1, m_0, m_1 < 1. \tag{2.99}$$

Relations (2.97) and (2.98) imply that

$$l_1 \le m_0, \quad l_0 \le m_1, \quad m_0 \le l_1, \quad m_1 \le l_0$$

and so

$$l_1 = m_0, \quad l_0 = m_1. \tag{2.100}$$

In addition, from (1.2), (2.98) and (2.100) we get

$$l_1 = (1-l_0-l_1)(1-e^{-Al_1}), \qquad l_0 = (1-l_0-l_1)(1-e^{-Al_0}). \tag{2.101}$$

First, suppose that $l_0 = 0$ (resp. $l_1 = 0$), then from (2.101) we get $l_1 = (1-l_1)(1-e^{-Al_1})$ (resp. $l_0 = (1-l_0)(1-e^{-Al_0})$) and so using (2.66), (2.99) and Lemma 2.1 of [4] we get $l_1 = 0$ (resp. $l_0 = 0$).

Now, suppose that

$$l_0 \ne 0, \qquad l_1 \ne 0 \tag{2.102}$$

then from (2.101) we get that

$$\frac{1-e^{-Al_1}}{l_1} = \frac{1-e^{-Al_0}}{l_0}. \tag{2.103}$$

If

$$f(x) = \frac{1-e^{-cx}}{x}, \quad c > 0,\ x > 0, \tag{2.104}$$

then since $e^{cx} > 1 + cx$ we get

$$f'(x) = \frac{(cx+1)e^{-cx} - 1}{x^2} < 0$$

and so f is a decreasing function. Therefore, from (2.66), (2.99), (2.102) and (2.103), we have $l_1 = l_0$ and from (2.100) we get that $m_1 = m_0$. Hence, there exist the $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$. From statement (i) of Proposition 2.2 we get that $l_0 = l_1 = 0$ which contradicts to (2.102). Hence, from (2.100) $l_0 = l_1 = m_0 = m_1 = 0$ and so $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. This completes the proof of statement (i).

(ii) Suppose that (2.95) are satisfied. Using (1.2), (2.95) and since from statement (ii) of Proposition 2.2, $\bar{x}, \bar{y}, \in (0, 0.5)$, we obtain that

$$x_{n+1} \ge (1-2\bar{y})(1-e^{-Ay_n}), \qquad y_{n+1} \ge (1-2\bar{x})(1-e^{-Bx_n}), \quad n = m+1, m+2 \ldots.$$

Then using Lemma 2.7 of [4], (2.2) and (2.95), it follows that

$$x_{n+1} \ge (1-2\bar{y})(1-e^{-A\bar{y}})\frac{(1-e^{-Ay_n})}{(1-e^{-A\bar{y}})} = \bar{x}\frac{(1-e^{-Ay_n})}{(1-e^{-A\bar{y}})} \ge \frac{\bar{x}}{\bar{y}} y_n. \tag{2.105}$$

$$y_{n+1} \ge (1-2\bar{x})(1-e^{-B\bar{x}})\frac{(1-e^{-Bx_n})}{(1-e^{-B\bar{x}})} = \bar{y}\frac{(1-e^{-Bx_n})}{(1-e^{-B\bar{x}})} \ge \frac{\bar{y}}{\bar{x}} x_n. \tag{2.106}$$

Then relations (2.105) and (2.106) imply that

$$x_{n+1} \ge x_{n-1}, \qquad y_{n+1} \ge y_{n-1}, \quad n = 1, 2, \ldots \tag{2.107}$$

Therefore, (2.98) are satisfied. From (2.2), (2.7), (2.98) and (2.107) we get

$$0 < l_0, l_1, m_0, m_1 \le 1.$$

Using (2.98) and (2.105), (2.106) and (2.107) we take

$$l_0 \geq \frac{\bar{x}}{\bar{y}} m_1, \qquad l_1 \geq \frac{\bar{x}}{\bar{y}} m_0, \qquad m_0 \geq \frac{\bar{y}}{\bar{x}} l_1, \qquad m_1 \geq \frac{\bar{y}}{\bar{x}} l_0$$

which imply that

$$l_0 = \frac{\bar{x}}{\bar{y}} m_1, \qquad l_1 = \frac{\bar{x}}{\bar{y}} m_0. \tag{2.108}$$

In addition, from (1.2) and (2.98) we get

$$m_1 = (1 - l_0 - l_1)(1 - e^{-Bl_0}), \qquad m_0 = (1 - l_1 - l_0)(1 - e^{-Bl_1}). \tag{2.109}$$

So relations (2.108) and (2.109) imply that

$$\begin{aligned} m_1 &= \left(1 - (m_0 + m_1)\frac{\bar{x}}{\bar{y}}\right)(1 - e^{-Bm_1\frac{\bar{x}}{\bar{y}}}) \\ m_0 &= \left(1 - (m_0 + m_1)\frac{\bar{x}}{\bar{y}}\right)(1 - e^{-Bm_0\frac{\bar{x}}{\bar{y}}}). \end{aligned} \tag{2.110}$$

Then from (2.110) we get

$$\frac{1 - e^{-Bm_1\frac{\bar{x}}{\bar{y}}}}{m_1} = \frac{1 - e^{-Bm_0\frac{\bar{x}}{\bar{y}}}}{m_0}. \tag{2.111}$$

Since function f, defined in (2.104), is a decreasing function, from (2.111) we take $m_1 = m_0$. Then from (2.108) it is obvious that $l_1 = l_0$. Since from the statement (ii) of Proposition 2.2 $(\bar{x}, \bar{y})$ is the unique positive equilibrium of (1.2), the proof of this statement is completed. This completes the proof of the proposition. □

Finally, we give the following open problem.

Open problems. Study the asymptotic behavior of the positive solutions of the system of difference equations

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} y_{n-j}\right)(1 - e^{-Ay_n}), \qquad y_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)(1 - e^{-Bx_n}), \quad k \in \{3, 4, \ldots\}.$$

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