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Journal of Algebra 311 (2007) 492–510

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Minimal free resolutions and asymptotic behavior of multigraded regularity

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Received 11 September 2006

Available online 9 February 2007

Communicated by Luchezar L. Avramov

Abstract

Let S be a standard $\mathbb{N}^k$-graded polynomial ring over a field $\mathbf{k}$, let I be a multigraded homogeneous ideal of S, and let M be a finitely generated $\mathbb{Z}^k$-graded S-module. We prove that the resolution regularity, a multigraded variant of Castelnuovo–Mumford regularity, of $I^n M$ is asymptotically a linear function. This shows that the well-known $\mathbb{Z}$-graded phenomenon carries to the multigraded situation.

Keywords: Regularity; Multigraded regularity; Powers of ideal; Asymptotic behavior

1. Introduction

In this paper we investigate the asymptotic behavior of a multigraded variant of Castelnuovo–Mumford regularity, the resolution regularity.

Throughout the paper, $\mathbb{N}$ will denote the set of non-negative integers. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be the standard unit vectors of $\mathbb{Z}^k$ and let $\mathbf{0} = (0, \ldots, 0)$. Let $\mathbf{k}$ be a field and let S be a standard $\mathbb{N}^k$-graded polynomial ring over $\mathbf{k}$. That is, $S_{\mathbf{0}} = \mathbf{k}$ and S is generated over $S_{\mathbf{0}}$ by elements of $\bigoplus_{l=1}^k S_{\mathbf{e}_l}$. For $l = 1, \ldots, k$, suppose $S_{\mathbf{e}_l}$ is generated as a vector space over $S_{\mathbf{0}}$ by $\{x_{l,1}, \ldots, x_{l,N_l}\}$.

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1 The first author is partially supported by Louisiana Board of Regents Enhancement Grant.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.01.030

For simplicity, we shall use $\mathbf{x}_l$ to represent the elements $\{x_{l,1}, \dots, x_{l,N_l}\}$ and $(\mathbf{x}_l)$ to denote the S-ideal generated by $\{x_{l,1}, \dots, x_{l,N_l}\}$.

Definition 1.1. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^k} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^k$-graded S-module. Let

$$\mathbb{F}: 0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a minimal $\mathbb{Z}^k$-graded free resolution of M over S, where $F_i = \bigoplus_j S(-c_{ij}^{[1]}, \dots, -c_{ij}^{[k]})$ for $i = 1, \dots, p$. For each $1 \leqslant l \leqslant k$, set $c_i^{[l]} = \max_j \{c_{ij}^{[l]}\}$ for $i = 0, \dots, p$ and let

$$\text{res-reg}_l(M) = \max_i \bigl\{c_i^{[l]} - i\bigr\}.$$

The multigraded *resolution regularity* of M is defined to be the vector

$$\textbf{res-reg}(M) = \bigl(\text{res-reg}_1(M), \dots, \text{res-reg}_k(M)\bigr) \in \mathbb{Z}^k.$$

Example 1.2. Let $S = \mathbb{Q}[x_0, x_1, y_0, y_1]$, where $\deg x_i = \mathbf{e}_1 \in \mathbb{Z}^2$ and $\deg y_j = \mathbf{e}_2 \in \mathbb{Z}^2$ for all $0 \leqslant i, j \leqslant 1$. Then S is a standard $\mathbb{N}^2$-graded polynomial ring over $\mathbb{Q}$. Consider the ideal $I = (x_0^2, x_0y_1, x_1y_0, y_0^2)$ as a $\mathbb{Z}^2$-graded S-module. Using any computational algebra package (such as, CoCoA [5] or Macaulay 2 [15]), we obtain the following minimal $\mathbb{Z}^2$-graded free resolution of I over S:

$$0 \to S(-3,-3) \to \begin{array}{c} S(-3,-2)^2 \\ \bigoplus \\ S(-2,-3)^2 \end{array} \to \begin{array}{c} S(-1,-2) \\ \bigoplus \\ S(-2,-1) \\ \bigoplus \\ S(-3,-1) \\ \bigoplus \\ S(-2,-2)^2 \\ \bigoplus \\ S(-1,-3) \end{array} \to \begin{array}{c} S(-2,0) \\ \bigoplus \\ S(-1,-1)^2 \\ \bigoplus \\ S(0,-2) \end{array} \to I \to 0.$$

By definition, $\text{res-reg}_1(I) = 2$ and $\text{res-reg}_2(I) = 2$. Thus, $\textbf{res-reg}(I) = (2, 2) \in \mathbb{Z}^2$.

Remark 1.3. For a vector $\mathbf{n} \in \mathbb{Z}^k$ we shall use n_l to denote its lth coordinate. It can be seen that $\operatorname{Tor}_i^S(M, \mathbf{k}) = H_i(\mathbb{F} \otimes \mathbf{k})$. Thus, the resolution regularity of M can also be calculated as follows. For each $1 \leqslant l \leqslant k$,

$$\text{res-reg}_l(M) = \max\bigl\{n_l \bigm| \exists \mathbf{n} \in \mathbb{Z}^k,\ i \geqslant 0 \text{ so that } \operatorname{Tor}_i^S(M, \mathbf{k})_{\mathbf{n}+i\mathbf{e}_l} \neq 0\bigr\}.$$

We shall often make use of this observation.

The resolution regularity is a multigraded variant of the well-known Castelnuovo–Mumford regularity, developed via the theory of Hilbert functions and minimal free resolutions by Aramova, Crona and DeNegre [1] (for $\mathbb{Z}^2$-gradings), and Sidman and Van Tuyl [21] (for $\mathbb{Z}^k$-gradings in general). This complements the notion of *multigraded regularity*, another variant of Castelnuovo–Mumford regularity, studied by Hoffman and Wang [13] (for $\mathbb{Z}^2$-graded), and Maclagan and Smith [16] (for G-graded, where G is an Abelian group). Roughly speaking, the resolution regularity of M captures the maximal coordinates of minimal generating multidegrees of syzygy modules of M. In particular, it provides a crude bound for the generating degrees of M. Thus, resolution regularity can be viewed as a refinement of Castelnuovo–Mumford regularity. Resolution regularity, furthermore, shares important similarities with the original definition of Castelnuovo–Mumford regularity (cf. [12]).

Many authors [1,11–13,16–18,21,22] in recent years have tried to extend our knowledge of Castelnuovo–Mumford regularity to the multigraded situation. On the other hand, there has been a surge of interest in the asymptotic behavior of Castelnuovo–Mumford regularity of powers of an ideal in an $\mathbb{N}$-graded algebra (cf. [2,4,6–8,10,14,23,26]). It is known that if S is a standard $\mathbb{N}$-graded algebra (over a Noetherian ring A), I is a homogeneous ideal in S and M is a finitely generated $\mathbb{Z}$-graded S-module, then $\operatorname{reg}(I^n M)$ is asymptotically a linear function in n with slope $\leqslant d(I)$, where $d(I)$ is the maximal generating degree of I. The aim of this paper is to show that the $\mathbb{Z}$-graded phenomenon carries to the multigraded setting.

Suppose M is a $\mathbb{Z}^k$-graded S-module, minimally generated in degrees $\mathbf{d}_1(M), \ldots, \mathbf{d}_v(M)$, where $\mathbf{d}_i(M) = (d_{i,1}, \ldots, d_{i,k})$. Then, for each $1 \leqslant l \leqslant k$, we define

$$d^{[l]}(M) = \max\{d_{i,l} \mid i = 1, \ldots, v\}$$

to be the maximal lth coordinate among the minimal generating degrees of M. Our first main result is stated as follows.

Theorem 1.4. *(Theorem 4.1) Let S be a standard $\mathbb{N}^k$-graded polynomial ring over a field* $\mathbf{k}$ *and let M be a finitely generated $\mathbb{Z}^k$-graded S-module. Let I be a multigraded homogeneous ideal in S minimally generated in degrees $\mathbf{d}_1(I), \ldots, \mathbf{d}_v(I)$. Then,* $\mathbf{res\text{-}reg}(I^n M)$ *is asymptotically a linear function with slope vector at most $(d^{[1]}(I), \ldots, d^{[k]}(I))$ componentwise.*

The slope vector of $\mathbf{res\text{-}reg}(I^n M)$, in fact, can be described explicitly via the theory of *reductions*.

Definition 1.5. We say that a multigraded homogeneous ideal $J \subset I$ is an *M-reduction* of I if $I^n M = J I^{n-1} M$ for all $n \gg 0$. For each $l = 1, \ldots, k$, we define

$$\rho_M^{[l]}(I) = \min\bigl\{d^{[l]}(J) \bigm| J \text{ is an } M\text{-reduction of } I\bigr\}$$

(here, $d^{[l]}(J)$ is defined similarly to $d^{[l]}(I)$), and

$$\operatorname{beg}^{[l]}(M) = \min\bigl\{n_l \bigm| \exists \mathbf{n} \in \mathbb{Z}^k \text{ such that } M_{\mathbf{n}} \neq 0\bigr\}.$$

Theorem 1.6. *(Theorem 4.5) Let S be a standard $\mathbb{N}^k$-graded polynomial ring over a field* $\mathbf{k}$ *and let M be a finitely generated $\mathbb{Z}^k$-graded S-module. Let I be a multigraded homogeneous ideal in S. Then, there exists a vector $\mathbf{b} \geqslant (\operatorname{beg}^{[1]}(M), \ldots, \operatorname{beg}^{[k]}(M))$ such that for $n \gg 0$,*

$$\mathbf{res\text{-}reg}\bigl(I^n M\bigr) = n\bigl(\rho_M^{[1]}(I), \ldots, \rho_M^{[k]}(I)\bigr) + \mathbf{b}.$$

It is well known that in the $\mathbb{Z}$-graded case, Castelnuovo–Mumford regularity can be defined either via the shifts of a minimal free resolution or by local cohomology modules with respect to the homogeneous irrelevant ideal (cf. [9]). In the multigraded situation (i.e. when $k \geqslant 2$), the two approaches, using minimal free resolutions and local cohomology with respect to the irrelevant ideal, to define variants of regularity do not agree and thus give different invariants (see [12] for a discussion on the relationship between these two multigraded variants of regularity). The incomparability between these approaches makes it difficult to generalize $\mathbb{Z}$-graded results to the multigraded setting. Another conceptual difficulty is the simple fact that $\mathbb{Z}^k$ is not a totally ordered set. Thus, it is not easy to capture maximal coordinates of shifts in the minimal free resolution by a single vector.

Our method in this paper is inspired by a recent work of Trung and Wang [26], where the authors study Castelnuovo–Mumford regularity via *filter-regular sequences* (which originated from [19]). To start, we first define a multigraded variant of the notion of filter-regular sequences. We then need to establish the correspondence between resolution regularity and filter-regular sequences, which is no longer apparent after passing to the multigraded situation. To achieve this, we develop previous work of Römer [18] and Trung [25] to fuller generality to link resolution regularity, filter-regular sequences and local cohomology with respect to different graded ideals, via the theory of Koszul homology. Our main theorems are obtained by generalizing to the multigraded setting techniques of [26] in investigating filter-regular sequences. Furthermore, as a byproduct of our work, it seems possible to define multigraded regularity for $\mathbb{Z}^k$-graded modules over a standard $\mathbb{N}^k$-graded algebra over an arbitrary Noetherian ring A (not necessarily a field), even though, in this case, a finite minimal free resolution as in Definition 1.1 may not exist. This addresses a question Bernd Ulrich has asked us.

Our paper is structured as follows. In Section 2, we define a multigraded variant of the notion of filter-regular sequences (Definition 2.4), and generalize properties of filter-regular sequences from the $\mathbb{Z}$-graded case to the multigraded situation (Lemmas 2.6 and 2.7). In Section 3, we establish the correspondence between resolution regularity, filter-regular sequences and local cohomology with respect to different graded ideals (Theorem 3.5). Theorem 3.5 lays the groundwork for the rest of the paper. In Section 4, we prove our main theorems (Theorems 4.1 and 4.5). In Appendix A, making use of our work in Section 3, we propose an alternative definition for a multigraded variant of Castelnuovo–Mumford regularity when there might not exist finite minimal free resolutions (Definition A.1).

Now, we shall fix some notations and terminology. If $f \in M$ is a homogeneous element of multidegree $\mathbf{d} = (d_1, \dots, d_k)$, then we define

$$\deg_l(f) = d_l$$

to be the lth coordinate of the multidegree of f. Since coordinates of multidegrees will be discussed frequently in the paper, we point out that the operator $\bullet^{[l]}$ indicates that the maximal lth coordinate of involved multidegrees is being considered. For $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{Z}^k$ and $n \in Z$, by writing $(\mathbf{m}, n)$ we refer to $(m_1, \dots, m_k, n) \in \mathbb{Z}^{k+1}$.

2. Filter-regular sequences

In this section, we extend the notion of filter-regular sequences to the multigraded situation. More precisely, we define M-filter-regular sequences with respect to a given coordinate.

The first step in our proof of Theorem 1.4 is to express $\text{res-reg}_l(I^n M)$ in terms of invariants associated to M-filter-regular sequences with respect to the lth coordinate. To achieve this we show that, by a flat base extension, $\mathbf{x}_l$ can be taken to be an M-filter-regular sequence with respect to the lth coordinate for all $1 \leqslant l \leqslant k$.

Definition 2.1. Let $M = \bigoplus_{\mathbf{n}\in\mathbb{Z}^k} M_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^k$-graded S-module. For each $1 \leqslant l \leqslant k$, define

$$\text{end}_l(M) = \max\{n_l \mid \exists \mathbf{n} \in \mathbb{Z}^k\colon\ M_{\mathbf{n}} \neq 0\}.$$

Remark 2.2. In general, $\text{end}_l(M)$ may be infinite, and $\text{end}_l(M) = -\infty$ if and only if $M = 0$.

Example 2.3. Consider $S = \mathbf{k}[x, y, z]$ as a standard $\mathbb{N}^3$-graded polynomial ring, where $\deg(x) = (1, 0, 0)$, $\deg(y) = (0, 1, 0)$, and $\deg(z) = (0, 0, 1)$. Let $M = S/I$ where $I = (x^2y, xy^2, xyz, y^3, y^2z, yz^2)$. Then $\text{end}_1(M) = \infty$, $\text{end}_2(M) = 2$ and $\text{end}_3(M) = \infty$.

Definition 2.4. Fix an integer $1 \leqslant l \leqslant k$. A sequence $\mathbf{f} = \langle f_1, \dots, f_s\rangle$ of multigraded homogeneous elements in S is said to be an *M-filter-regular sequence with respect to the lth coordinate* if

(i) $(f_1, \dots, f_s)M \neq M$.
(ii) For every $1 \leqslant i \leqslant s$, we have

$$\text{end}_l\left(\frac{(f_1, \dots, f_{i-1})M :_M f_i}{(f_1, \dots, f_{i-1})M}\right) < \infty.$$

When $\mathbf{f} = \langle f_1, \dots, f_s\rangle$ is an M-filter-regular sequence with respect to the lth coordinate, we define

$$\mathfrak{a}_M^{[l]}(\mathbf{f}) = \max_{1\leqslant i\leqslant s}\left\{\text{end}_l\left(\frac{(f_1, \dots, f_{i-1})M :_M f_i}{(f_1, \dots, f_{i-1})M}\right)\right\}.$$

Example 2.5. Consider $S = \mathbf{k}[x, y, z]$ and $M = S/(x^2y, xy^2, xyz, y^3, y^2z, yz^2)$ as in Example 2.3. Then $yM \neq M$ and $\text{end}_2(0 :_M y) = 2$. Thus, y is M-filter-regular with respect to the second coordinate. Notice also that y is not M-filter-regular with respect to the first or third coordinate, while $\langle x, z\rangle$ is an M-filter-regular sequence with respect to the first, second, and third coordinates.

The following lemma gives a characterization for M-filter-regular sequences with respect to the lth coordinate via primes avoidance (see also [25, Lemma 2.1]).

Lemma 2.6. *A sequence $\mathbf{f} = \langle f_1, \dots, f_s\rangle$ of elements in S is an M-filter-regular sequence with respect to the lth coordinate if and only if for each $i = 1, \dots, s$, $f_i \notin P$ for any associated prime $P \not\supseteq (\mathbf{x}_l)$ of $M/(f_1, \dots, f_{i-1})M$.*

Proof. By replacing M with $M/(f_1, \dots, f_{i-1})M$, it suffices to prove the statement for $i = 1$. For simplicity, we shall write f for f_1.

Observe that $\mathrm{end}_l(0:_M f) < \infty$ if and only if there exists an integer m such that $(\mathbf{x}_l)^m(0:_M f) = 0$. In particular, this means that

$$0:_M f \subseteq \bigcup_{j=1}^{\infty}\big[0:_M (\mathbf{x}_l)^j\big] = H^0_{(\mathbf{x}_l)}(M).$$

Let $N = M/H^0_{(\mathbf{x}_l)}(M)$. Observe further that $\mathrm{Ass}(M)$ is the disjoint union of $\mathrm{Ass}(H^0_{(\mathbf{x}_l)}(M))$ and $\mathrm{Ass}(N)$ (cf. [3, Exercise 2.1.12]).

Consider an arbitrary $Q \in \mathrm{Ass}(M)$ and assume that $Q = \mathrm{ann}_S(h)$ for some $h \in M$. Then, $Q \supseteq (\mathbf{x}_l)$ if and only if $h(\mathbf{x}_l)^j = 0$ for some $j \geqslant 1$ (since Q is prime), that is, $h \in H^0_{(\mathbf{x}_l)}(M)$. This implies that $Q \in \mathrm{Ass}(H^0_{(\mathbf{x}_l)}(M))$. Thus, the set of associated primes of M which do not contain $(\mathbf{x}_l)$ is $\mathrm{Ass}(N)$. Hence, it remains to show that $0:_M f \subseteq H^0_{(\mathbf{x}_l)}(M)$ if and only if $f \notin P$ for any $P \in \mathrm{Ass}(N)$. Indeed, $0:_M f \subseteq H^0_{(\mathbf{x}_l)}(M)$ if and only if $0:_N f = 0$ in N, that is, f is a non-zerodivisor on N, and this is the case if and only if f does not belong to any associated prime of N. The lemma is proved. □

The next result allows us to assume that $\mathbf{x}_l$ is M-filter-regular with respect to the lth coordinate after a flat extension (see also [26, Lemma 1.2]).

Lemma 2.7. *For $i = 1, \ldots, N_l$, let $z_i = \sum_{j=1}^{N_l} u_{ij}x_{l,j}$, where $U = (u_{ij})_{1 \leqslant i,j \leqslant N_l}$ is a square matrix of indeterminates. Let $\mathbf{k}' = \mathbf{k}(U)$, $S' = S \otimes_{\mathbf{k}} \mathbf{k}'$ and $M' = M \otimes_{\mathbf{k}} \mathbf{k}'$. Then,*

(1) *The sequence $\mathbf{z} = \langle z_1, \ldots, z_{N_l}\rangle$ is an M'-filter-regular sequence with respect to the lth coordinate. Notice that $(z_1, \ldots, z_{N_l}) = (\mathbf{x}_l)S'$.*
(2) $\mathbf{res\text{-}reg}(M) = \mathbf{res\text{-}reg}(M')$ *where* $\mathbf{res\text{-}reg}(M')$ *is calculated over S'.*

Proof. To prove (1) we first observe that since the elements $z_1, \ldots, z_{N_l}$ are given by independent sets of indeterminates, by induction it suffices to show that z_1 is M'-filter-regular with respect to the lth coordinate. By Lemma 2.6, we need to show that $z_1 \notin P$ for any associated prime $P \not\supseteq (\mathbf{x}_l)S'$ of M'. Indeed, if $P \not\supseteq (\mathbf{x}_l)S'$ is an associated prime of M', then since $S \to S'$ is a flat extension, we must have $P = \wp S'$ for some associated prime $\wp \not\supseteq (\mathbf{x}_l)S$ of M. However, since $(\mathbf{x}_l)S = (x_{l,1}, \ldots, x_{l,N_l})S \not\subseteq \wp$, we must have $z_1 = u_{1,1}x_{l,1} + \cdots + u_{1,N_l}x_{l,N_l} \notin \wp S' = P$.

Now, let $\mathbb{F}$ be a minimal $\mathbb{Z}^k$-graded free resolution of M over S. To prove (2), we observe that since $S \to S'$ is a flat extension, $\mathbb{F} \otimes_S S'$ is a minimal $\mathbb{Z}^k$-graded free resolution of M' over S'. Thus, by definition, $\mathbf{res\text{-}reg}(M) = \mathbf{res\text{-}reg}(M')$. □

3. Koszul homology and a-invariant

The goal of this section is to relate the resolution regularity to invariants associated to filter-regular sequences. Our techniques are based upon investigating Koszul complexes and local cohomology modules with respect to different irrelevant ideals (given by different sets of variables).

For $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{Z}^k_{\geqslant 0}$ such that $u_l \leqslant N_l$ for all $l = 1, \ldots, k$, we shall denote by $\mathbb{K}(\mathbf{u})$ the Koszul complex with respect to $(x_{1,1}, \ldots, x_{1,u_1}, \ldots, x_{k,1}, \ldots, x_{k,u_k})$. Let $H_i(\mathbf{u}; M)$ denote the homology group $H_i(\mathbb{K}(\mathbf{u}) \otimes_S M)$. When $\mathbf{u} = (N_1, \ldots, N_{s-1}, r, 0, \ldots, 0)$, for simplicity, we shall write $\mathbb{K}^{\{s\}}(r)$ and $H_i^{\{s\}}(r; M)$ to denote $\mathbb{K}(\mathbf{u})$ and $H_i(\mathbf{u}; M)$, respectively.

Lemma 3.1. *For each s and $0 \leqslant r \leqslant N_s - 1$, let*

$$\tilde{H}_0^{\{s\}}(r; M) = [0 :_{M/(\mathbf{x}_1,\ldots,\mathbf{x}_{s-1},x_{s,1},x_{s,2},\ldots,x_{s,r})M} x_{s,r+1}].$$

Then we have the following exact sequence

$$\begin{aligned} \cdots \to H_i^{\{s\}}(r; M)(-\mathbf{e}_s) &\xrightarrow{x_{s,r+1}} H_i^{\{s\}}(r; M) \to H_i^{\{s\}}(r+1; M) \to H_{i-1}^{\{s\}}(r; M)(-\mathbf{e}_s) \\ \to \cdots \to H_1^{\{s\}}(r; M)(-\mathbf{e}_s) &\xrightarrow{x_{s,r+1}} H_1^{\{s\}}(r; M) \to H_1^{\{s\}}(r+1; M) \to \tilde{H}_0^{\{s\}}(r; M)(-\mathbf{e}_s) \to 0. \end{aligned}$$

Proof. Consider the following exact sequence of Koszul complexes

$$0 \to \mathbb{K}^{\{s\}}(r)(-\mathbf{e}_s) \xrightarrow{x_{s,r+1}} \mathbb{K}^{\{s\}}(r) \to \mathbb{K}^{\{s\}}(r+1) \to 0. \tag{3.1}$$

The conclusion is obtained by taking the long exact sequence of homology groups associated to (3.1); notice that the last term $\tilde{H}_0^{\{s\}}(r; M)(-\mathbf{e}_s)$ is given by the kernel of the map $H_0^{\{s\}}(r; M)(-\mathbf{e}_s) \xrightarrow{x_{s,r+1}} H_0^{\{s\}}(r; M)$. □

The following theorem allows us to relate resolution regularity to invariants associated to filter-regular sequences.

Theorem 3.2. *Let S be a standard $\mathbb{N}^k$-graded polynomial ring over a field* **k** *and let M be a finitely generated $\mathbb{Z}^k$-graded S-module. Suppose that $\mathbf{x}_l$ is an M-filter-regular sequence with respect to the lth coordinate. Then,*

$$\text{res-reg}_l(M) = \max\{\mathfrak{a}_M^{[l]}(\mathbf{x}_l), d^{[l]}(M)\}.$$

Proof. Without loss of generality, we shall prove the statement for $l = 1$.

For each $1 \leqslant s \leqslant k$ and $0 \leqslant r \leqslant N_s$, let $Q_{s,r} = N_1 + \cdots + N_{s-1} + r$ and define

$$v^{[s]}(r) = \max\{n_1 \mid \exists \mathbf{n} \in \mathbb{Z}^k\colon\ H_i^{\{s\}}(r; M)_{\mathbf{n}+i\mathbf{e}_1} \neq 0 \text{ for some } 1 \leqslant i \leqslant Q_{s,r}\}$$

(we make the convention that $v^{[1]}(0) = -\infty$). Observe that $\operatorname{Tor}_0^S(M, \mathbf{k})_{\mathbf{d}} \neq 0$ if and only if $\mathbf{d}$ is a generating degree of M. Thus,

$$d^{[1]}(M) = \max\{n_1 \mid \exists \mathbf{n} \in \mathbb{Z}^k\colon\ \operatorname{Tor}_0^S(M, \mathbf{k})_{\mathbf{n}} \neq 0\}.$$

Now, since $\mathbb{K}^{\{k\}}(N_k)$ gives a minimal $\mathbb{Z}^k$-graded free resolution of $\mathbf{k}$ over S, we have $H_i^{\{k\}}(N_k; M) = \operatorname{Tor}_i^S(M, \mathbf{k})$ for all $i \geqslant 1$. Hence,

$$\text{res-reg}_1(M) = \max\{v^{[k]}(N_k), d^{[1]}(M)\}.$$

It remains to show that

$$\max\{v^{[k]}(N_k), d^{[1]}(M)\} = \max\{\mathfrak{a}_M^{[1]}(\mathbf{x}_1), d^{[1]}(M)\}.$$

To this end, we proceed in the following steps.

(i) For any $1 \leqslant r \leqslant N_1$,

$$v^{[1]}(r) = \max_{1\leqslant i\leqslant r}\left\{\operatorname{end}_1\left(\frac{(x_{1,1},\dots,x_{1,i-1})M :_M x_{1,i}}{(x_{1,1},\dots,x_{1,i-1})M}\right)\right\}.$$

(ii) For any $s \geqslant 2$ and any $0 \leqslant r \leqslant N_s$,

$$\max\{v^{[s]}(r), d^{[1]}(M)\} = \max\{\mathfrak{a}_M^{[1]}(\mathbf{x}_1), d^{[1]}(M)\}.$$

For simplicity of notation, we shall denote

$$\operatorname{end}_1\left(\frac{(x_{1,1},\dots,x_{1,i-1})M :_M x_{1,i}}{(x_{1,1},\dots,x_{1,i-1})M}\right)$$

by s_i for $i = 1,\dots,N_1$. Then $\mathfrak{a}_M^{[1]}(\mathbf{x}_1) = \max\{s_1,\dots,s_{N_1}\}$.

We shall prove (i) by induction on r. By Lemma 3.1 we have

$$0 \to H_1^{\{1\}}(1; M) \to \tilde{H}_0^{\{1\}}(0; M)(-\mathbf{e}_1) \to 0.$$

Thus,

$$\begin{aligned} v^{[1]}(1) &= \max\{n_1 \mid \exists \mathbf{n} = (n_1,\dots,n_k)\colon\ H_1^{\{1\}}(1; M)_{\mathbf{n}+\mathbf{e}_1} \neq 0\} \\ &= \max\{n_1 \mid \exists \mathbf{n} = (n_1,\dots,n_k)\colon\ \tilde{H}_0^{\{1\}}(0; M)_{\mathbf{n}} \neq 0\} \\ &= \operatorname{end}_1(0 :_M x_{1,1}) = s_1. \end{aligned}$$

This proves the statement for $r = 1$.

Suppose that $r > 1$. By Lemma 3.1, we have

$$\cdots \to H_1^{\{1\}}(r; M) \to \tilde{H}_0^{\{1\}}(r-1; M)(-\mathbf{e}_1) \to 0.$$

This implies that $v^{[1]}(r) \geqslant s_r$. Observe that if $v^{[1]}(r-1) = -\infty$ then $v^{[1]}(r) \geqslant v^{[1]}(r-1)$, which implies that $v^{[1]}(r) \geqslant \max\{v^{[1]}(r-1), s_r\}$. Assume that $v^{[1]}(r-1) > -\infty$. That is, there exist an integer i and $\mathbf{n} = (n_1,\dots,n_k) \in \mathbb{Z}^k$ with $n_1 = v^{[1]}(r-1)$ such that $H^{\{1\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1} \neq 0$ and $H^{\{1\}}(r-1; M)_{\mathbf{n}+(i+1)\mathbf{e}_1} = 0$. By Lemma 3.1, we have

$$\cdots \to H_{i+1}^{\{1\}}(r; M)_{\mathbf{n}+(i+1)\mathbf{e}_1} \to H_i^{\{1\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1} \to 0.$$

This implies that $H_{i+1}^{\{1\}}(r; M)_{\mathbf{n}+(i+1)\mathbf{e}_1} \neq 0$. Thus, $v^{[1]}(r) \geqslant v^{[1]}(r-1)$, and therefore, we also have $v^{[1]}(r) \geqslant \max\{v^{[1]}(r-1), s_r\}$.

We will prove the other direction, that is, $v^{[1]}(r) \leqslant \max\{v^{[1]}(r-1), s_r\}$. Consider an arbitrary $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ with $n_1 > \max\{v^{[1]}(r-1), s_r\}$. For $i \geqslant 2$, by Lemma 3.1, we have

$$\cdots \to H_i^{\{1\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1} \to H_i^{\{1\}}(r; M)_{\mathbf{n}+i\mathbf{e}_1} \to H_{i-1}^{\{1\}}(r-1; M)_{\mathbf{n}+(i-1)\mathbf{e}_1} \to \cdots.$$

Since $n_1 > v^{[1]}(r-1)$, we have $H_{i-1}^{\{1\}}(r-1; M)_{\mathbf{n}+(i-1)\mathbf{e}_1} = H_i^{\{1\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1} = 0$. This implies that $H_i^{\{1\}}(r; M)_{\mathbf{n}+i\mathbf{e}_1} = 0$. For $i = 1$, by Lemma 3.1, we have

$$\cdots \to H_1^{\{1\}}(r-1; M)_{\mathbf{n}+\mathbf{e}_1} \to H_1^{\{1\}}(r; M)_{\mathbf{n}+\mathbf{e}_1} \to \tilde{H}_0^{\{1\}}(r-1; M)_{\mathbf{n}} \to 0. \tag{3.2}$$

Since $n_1 > v^{[1]}(r-1)$, we have $H_1^{\{1\}}(r-1; M)_{\mathbf{n}+\mathbf{e}_1} = 0$. Since $n_1 > s_r$, we have $\tilde{H}_0^{\{1\}}(r-1; M)_{\mathbf{n}} = 0$. Thus, (3.2) implies that $H_1^{\{1\}}(r; M)_{\mathbf{n}+\mathbf{e}_1} = 0$. Hence, $n_1 > v^{[1]}(r)$. This is true for any $n_1 > \max\{v^{[1]}(r-1), s_r\}$, so we must have $v^{[1]}(r) \leqslant \max\{v^{[1]}(r-1), s_r\}$.

We have shown that $v^{[1]}(r) = \max\{v^{[1]}(r-1), s_r\}$. By induction,

$$v^{[1]}(r-1) = \max\{s_1, \dots, s_{r-1}\}.$$

Thus, $v^{[1]}(r) = \max\{s_1, \dots, s_r\}$ and (i) is proved.

We shall prove (ii) by using double induction on s and on r. By part (i), we have $v^{[2]}(0) = v^{[1]}(N_1) = \max\{s_1, \dots, s_{N_1}\} = \mathfrak{a}_M^{[1]}(\mathbf{x}_1)$. Therefore, the statement is true for $s = 2$ and $r = 0$. Assume that either $s > 2$ or $r > 0$. It can be seen that $v^{[s]}(0) = v^{[s-1]}(N_{s-1})$. Thus, we can assume that $r > 0$ (and $s \geqslant 2$).

Let $N = M/(\mathbf{x}_1, \dots, \mathbf{x}_{s-1}, x_{s,1}, x_{s,2}, \dots, x_{s,r-1})M$. Observe first that $N_{\mathbf{m}} \neq 0$ for some $\mathbf{m}$ if and only if m_1 equals the first coordinate of a generating degree of M (i.e., $m_1 = d_{i,1}(M)$ for some i). This implies that

$$\operatorname{end}_1(0 :_N x_{s,r}) \leqslant \operatorname{end}_1(N) = d^{[1]}(M). \tag{3.3}$$

Consider an arbitrary $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ with $n_1 > \max\{v^{[s]}(r-1), d^{[1]}(M)\}$. By Lemma 3.1, we have

$$H_i^{\{s\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1} \to H_i^{\{s\}}(r; M)_{\mathbf{n}+i\mathbf{e}_1} \to H_{i-1}^{\{s\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1-\mathbf{e}_s}, \tag{3.4}$$

$$H_1^{\{s\}}(r-1; M)_{\mathbf{n}+\mathbf{e}_1} \to H_1^{\{s\}}(r; M)_{\mathbf{n}+\mathbf{e}_1} \to \tilde{H}_0^{\{s\}}(r-1; M)_{\mathbf{n}+\mathbf{e}_1-\mathbf{e}_s} \to 0. \tag{3.5}$$

Since $n_1 > \max\{v^{[s]}(r-1), d^{[1]}(M)\}$, it follows from (3.3) and (3.5) that

$$H_1^{\{s\}}(r; M)_{\mathbf{n}+\mathbf{e}_1} = 0.$$

Since $n_1 > v^{[s]}(r-1)$, we have $H_i^{\{s\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1} = H_{i-1}^{\{s\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1-\mathbf{e}_s} = 0$. Thus, (3.4) implies that $H_i^{\{s\}}(r; M)_{\mathbf{n}+i\mathbf{e}_1} = 0$ for all $i \geqslant 2$. Hence,

$$v^{[s]}(r) \leqslant \max\bigl\{v^{[s]}(r-1), d^{[1]}(M)\bigr\}. \tag{3.6}$$

Observe now that if $v^{[s]}(r-1) \leqslant d^{[1]}(M)$ then (3.6) implies that

$$\max\{v^{[s]}(r), d^{[1]}(M)\} = d^{[1]}(M) = \max\{v^{[s]}(r-1), d^{[1]}(M)\}.$$

On the other hand, if $v^{[s]}(r-1) > d^{[1]}(M)$ then there exist an integer $i \geqslant 1$ and $\mathbf{n} \in \mathbb{Z}^k$ with $n_1 = v^{[s]}(r-1)$ such that $H_i^{\{s\}}(r-1; M)_{\mathbf{n}+i\mathbf{e}_1} \neq 0$. In this case, if $H_i^{\{s\}}(r; M)_{\mathbf{m}+i\mathbf{e}_1} = 0$ for any $\mathbf{m} \in \mathbb{Z}^k$ with $m_1 \geqslant v^{[s]}(r-1)$, then by Lemma 3.1 we have

$$0 \to \bigoplus_{m_1=n_1} H_i^{\{s\}}(r-1; M)_{\mathbf{m}+i\mathbf{e}_1-\mathbf{e}_s} \xrightarrow{x_{s,r}} \bigoplus_{m_1=n_1} H_i^{\{s\}}(r-1; M)_{\mathbf{m}+i\mathbf{e}_1} \to 0.$$

This implies that

$$\bigoplus_{m_1=n_1} H_i^{\{s\}}(r-1; M)_{\mathbf{m}+i\mathbf{e}_1} = x_{s,r}\left(\bigoplus_{m_1=n_1} H_i^{\{s\}}(r-1; M)_{\mathbf{m}+i\mathbf{e}_1}\right),$$

which is a contradiction by Nakayama's lemma. Therefore, there exists, in this case, $\mathbf{m}_0 = (m_{01}, \ldots, m_{0k}) \in \mathbb{Z}^k$ with $m_{01} = v^{[s]}(r-1)$ such that $H_i^{\{s\}}(r; M)_{\mathbf{m}_0+i\mathbf{e}_1} \neq 0$. That is, $v^{[s]}(r) \geqslant v^{[s]}(r-1) > d^{[1]}(M)$. We, hence, have $\max\{v^{[s]}(r), d^{[1]}(M)\} = \max\{v^{[s]}(r-1), d^{[1]}(M)\}$. The conclusion now follows by induction. □

Example 3.3. Consider $S = \mathbf{k}[x, y, z]$ and $M = S/(x^2y, xy^2, xyz, y^3, y^2z, yz^2)$ as in Example 2.3. Notice that $\dim M = 2$. It can be seen that $\mathfrak{a}_M^{[1]}(\langle x, z\rangle) = 1$, $\mathfrak{a}_M^{[2]}(\langle x, z\rangle) = 2$, and $\mathfrak{a}_M^{[3]}(\langle x, z\rangle) = 1$ while $d^{[1]}(M) = d^{[2]}(M) = d^{[3]}(M) = 0$. Theorem 3.2 implies that $\mathbf{res\text{-}reg}(M) = (1, 2, 1)$.

The following lemma (see [25, Lemma 2.3]) exhibits the behavior of modding out by a filter-regular element with respect a given coordinate.

Lemma 3.4. *Let $g \in S_{\mathbf{e}_l}$ be an M-filter-regular element with respect to the lth coordinate and let $(\mathbf{x}'_l)$ be the image of the ideal $(\mathbf{x}_l)$ in S/gS. Then for all $i \geqslant 0$,*

$$\begin{aligned} \operatorname{end}_l\big(H^{i+1}_{(\mathbf{x}_l)}(M)\big) + 1 &\leqslant \operatorname{end}_l\big(H^i_{(\mathbf{x}'_l)}(M/gM)\big) \\ &\leqslant \max\big\{\operatorname{end}_l\big(H^i_{(\mathbf{x}_l)}(M)\big), \operatorname{end}_l\big(H^{i+1}_{(\mathbf{x}_l)}(M)\big) + 1\big\}. \end{aligned}$$

Proof. It follows from the proof of Lemma 2.6 that $(0:_M g) \subseteq \bigcup_{n=0}^{\infty}[0:_M (\mathbf{x}_l)^n]$. That is, $(0:_M g)$ is annihilated by some power of $(\mathbf{x}_l)$. This implies that

$$H^i_{(\mathbf{x}_l)}(0:_M g) = 0 \quad \forall i \geqslant 1. \tag{3.7}$$

Consider the exact sequence

$$0 \to (0:_M g) \to M \to M/(0:_M g) \to 0.$$

(3.7) implies that $H^i_{(\mathbf{x}_l)}(M) = H^i_{(\mathbf{x}_l)}(M/(0:_M g))$ for all $i \geqslant 1$. Now, consider the long exact sequence of cohomology groups associated to the exact sequence

$$0 \to M/(0:_M g)(-\mathbf{e}_l) \xrightarrow{g} M \to M/gM \to 0$$

we get

$$H^i_{(\mathbf{x}_l)}(M)_\mathbf{n} \to H^i_{(\mathbf{x}'_l)}(M/gM)_\mathbf{n} \to H^{i+1}_{(\mathbf{x}_l)}(M)_{\mathbf{n}-\mathbf{e}_l} \xrightarrow{g} H^{i+1}_{(\mathbf{x}_l)}(M)_\mathbf{n} \tag{3.8}$$

for any $i \geqslant 0$ and $\mathbf{n} \in \mathbb{Z}^k$.

By the definition of the function end_l, there exists $\mathbf{m} \in \mathbb{Z}^k$ with $m_l = \operatorname{end}_l(H^{i+1}_{(\mathbf{x}_l)}(M)) + 1$ such that $H^{i+1}_{(\mathbf{x}_l)}(M)_\mathbf{m} = 0$ and $H^{i+1}_{(\mathbf{x}_l)}(M)_{\mathbf{m}-\mathbf{e}_l} \neq 0$. (3.8) then implies that $H^i_{(\mathbf{x}'_l)}(M/gM)_\mathbf{m} \neq 0$. Thus,

$$\operatorname{end}_l\big(H^i_{(\mathbf{x}'_l)}(M/gM)\big) \geqslant \operatorname{end}_l\big(H^{i+1}_{(\mathbf{x}_l)}(M)\big) + 1.$$

On the other hand, for any $\mathbf{n} \in \mathbb{Z}^k$ such that

$$n_l > \max\big\{\operatorname{end}_l\big(H^i_{(\mathbf{x}_l)}(M)\big), \operatorname{end}_l\big(H^{i+1}_{(\mathbf{x}_l)}(M)\big) + 1\big\},$$

we have $H^i_{(\mathbf{x}_l)}(M)_\mathbf{n} = H^{i+1}_{(\mathbf{x}_l)}(M)_{\mathbf{n}-\mathbf{e}_l} = 0$. (3.8) now implies that

$$\operatorname{end}_l\big(H^i_{(\mathbf{x}'_l)}(M/gM)\big) \leqslant \max\big\{\operatorname{end}_l\big(H^i_{(\mathbf{x}_l)}(M)\big), \operatorname{end}_l\big(H^{i+1}_{(\mathbf{x}_l)}(M)\big) + 1\big\}.$$

The lemma is proved. □

The next theorem lays the groundwork for the rest of the paper, allowing us to prove our main theorems in Section 4.

Theorem 3.5. *Let S be a standard $\mathbb{N}^k$-graded polynomial ring over a field $\mathbf{k}$ and let M be a finitely generated $\mathbb{Z}^k$-graded S-module. Suppose that $\mathbf{x}_l$ is an M-filter-regular sequence with respect to the lth coordinate. Then,*

(1) $$\mathfrak{a}^{[l]}_M(\mathbf{x}_l) = \max\Big\{\operatorname{end}_l\big(H^i_{(\mathbf{x}_l)}(M)\big) + i \;\Big|\; 0 \leqslant i \leqslant N_l - 1\Big\}.$$

(2) $$\mathfrak{a}^{[l]}_M(\mathbf{x}_l) = \max\left\{\operatorname{end}_l\left(\frac{(x_{l,1},\dots,x_{l,i})M :_M (\mathbf{x}_l)}{(x_{l,1},\dots,x_{l,i})M}\right) \;\middle|\; 0 \leqslant i \leqslant N_l - 1\right\}.$$

(3) $$\operatorname{res-reg}_l(M) = \max\left\{\operatorname{end}_l\left(\frac{(x_{l,1},\dots,x_{l,i})M :_M (\mathbf{x}_l)}{(x_{l,1},\dots,x_{l,i})M}\right) \;\middle|\; 0 \leqslant i \leqslant N_l\right\}.$$

Proof. We prove (1) using induction on N_l. Since $x_{l,1}$ is an M-filter-regular element with respect to the lth coordinate, as in the proof of Lemma 2.6, we have $(0 :_M x_{l,1}) \subseteq \bigcup_{n=0}^{\infty}[0 :_M (\mathbf{x}_l)^n] = H^0_{(\mathbf{x}_l)}(M)$. Thus, $\mathfrak{a}^{[l]}_M(x_{l,1}) \leqslant \operatorname{end}_l(H^0_{(\mathbf{x}_l)}(M))$. On the other hand, there exists $\mathbf{n} \in \mathbb{Z}^k$ with $n_l = \operatorname{end}_l(H^0_{(\mathbf{x}_l)}(M))$ such that $H^0_{(\mathbf{x}_l)}(M)_\mathbf{n} \neq 0$. Furthermore, $x_{l,1}H^0_{(\mathbf{x}_l)}(M)_\mathbf{n} \subseteq$

$H^0_{(\mathbf{x}_l)}(M)_{\mathbf{n}+\mathbf{e}_l} = 0$, which implies that $H^0_{(\mathbf{x}_l)}(M)_{\mathbf{n}} \subseteq (0 :_M x_{l,1})_{\mathbf{n}}$. Therefore, $\mathfrak{a}^{[l]}_M(x_{l,1}) \geqslant \operatorname{end}_l(H^0_{(\mathbf{x}_l)}(M))$. Hence, $\mathfrak{a}^{[l]}_M(x_{l,1}) = \operatorname{end}_l(H^0_{(\mathbf{x}_l)}(M))$ and the statement is true for $N_l = 1$.

Assume that $N_l > 1$. Let $\mathbf{x}'_l$ denote the sequence of images of $x_{l,2}, \ldots, x_{l,N_l}$ in $S/x_{l,1}S$ and let $N = M/x_{l,1}M$. Then $\mathbf{x}'_l$ gives an N-filter-regular sequence with respect to the lth coordinate. By induction and using Lemma 3.4, we have

$$\begin{aligned}
&\max\bigl\{\operatorname{end}_l\bigl(H^i_{(\mathbf{x}_l)}(M)\bigr) + i \bigm| i = 1, \ldots, N_l - 1\bigr\} \\
&\quad\leqslant \max\bigl\{\operatorname{end}_l\bigl(H^{i-1}_{(\mathbf{x}'_l)}(N)\bigr) + i - 1 \bigm| i = 1, \ldots, N_l - 1\bigr\} = \mathfrak{a}^{[l]}_N\bigl(\mathbf{x}'_l\bigr) \\
&\quad\leqslant \max\bigl\{\operatorname{end}_l\bigl(H^j_{(\mathbf{x}_l)}(M)\bigr) + j \bigm| j = 0, \ldots, N_l - 1\bigr\}.
\end{aligned}$$

This, together with the fact that $\mathfrak{a}^{[l]}_M(\mathbf{x}_l) = \max\{\mathfrak{a}^{[l]}_M(x_{l,1}), \mathfrak{a}^{[l]}_N(\mathbf{x}'_l)\}$, implies that

$$\mathfrak{a}^{[l]}_M(\mathbf{x}_l) = \max\bigl\{\operatorname{end}_l\bigl(H^i_{(\mathbf{x}_l)}(M)\bigr) + i \bigm| i = 0, \ldots, N_l - 1\bigr\}.$$

(1) is proved.

To prove (2), we first successively apply Lemma 3.4 to get

$$\begin{aligned}
\operatorname{end}_l\bigl(H^i_{(\mathbf{x}_l)}(M)\bigr) + i &\leqslant \operatorname{end}_l\bigl(H^0_{(\mathbf{x}_l)S_i}\bigl(M/(x_{l,1}, \ldots, x_{l,i-1})M\bigr)\bigr) \\
&\leqslant \max\bigl\{\operatorname{end}_l\bigl(H^j_{(\mathbf{x}_l)}(M)\bigr) + j \bigm| j = 0, \ldots, i\bigr\}
\end{aligned}$$

for any $i \geqslant 0$, where $S_i = S/(x_{l,1}, \ldots, x_{l,i-1})S$. It follows that for any $t \leqslant N_l$,

$$\max_{0\leqslant i\leqslant t}\bigl\{\operatorname{end}_l\bigl(H^i_{(\mathbf{x}_l)}(M)\bigr) + i\bigr\} = \max_{0\leqslant i\leqslant t}\bigl\{\operatorname{end}_l\bigl(H^0_{(\mathbf{x}_l)S_i}\bigl(M/(x_{l,1}, \ldots, x_{l,i-1})M\bigr)\bigr)\bigr\}. \tag{3.9}$$

Fix i and consider an arbitrary $\mathbf{m} \in \mathbb{Z}^k$ such that

$$m_l = \operatorname{end}_l\bigl(H^0_{(\mathbf{x}_l)S_i}\bigl(M/(x_{l,1}, \ldots, x_{l,i-1})M\bigr)\bigr).$$

Then, $(\mathbf{x}_l)H^0_{(\mathbf{x}_l)S_i}(M/(x_{l,1}, \ldots, x_{l,i-1})M)_{\mathbf{m}} \subseteq H^0_{(\mathbf{x}_l)S_i}(M/(x_{l,1}, \ldots, x_{l,i-1})M)_{\mathbf{m}+\mathbf{e}_l} = 0$. Moreover, $H^0_{(\mathbf{x}_l)S_i}(M/(x_{l,1}, \ldots, x_{l,i-1})M) = \bigcup_{n=0}^{\infty}(\frac{(x_{l,1},\ldots,x_{l,i-1})M :_M (\mathbf{x}_l)}{(x_{l,1},\ldots,x_{l,i-1})M})$. Thus, we have

$$\begin{aligned}
H^0_{(\mathbf{x}_l)S_i}\bigl(M/(x_{l,1}, \ldots, x_{l,i-1})M\bigr)_{\mathbf{m}} &\subseteq \frac{(x_{l,1}, \ldots, x_{l,i-1})M :_M (\mathbf{x}_l)}{(x_{l,1}, \ldots, x_{l,i-1})M} \\
&\subseteq H^0_{(\mathbf{x}_l)S_i}\bigl(M/(x_{l,1}, \ldots, x_{l,i-1})M\bigr).
\end{aligned}$$

This implies that

$$\operatorname{end}_l\left(\frac{(x_{l,1}, \ldots, x_{l,i-1})M :_M (\mathbf{x}_l)}{(x_{l,1}, \ldots, x_{l,i-1})M}\right) = m_l. \tag{3.10}$$

(2) now follows from part (1), (3.9) and (3.10).

In view of Theorem 3.2 and part (2), to prove (3) we only need to show that

$$\mathrm{end}_l\big(M/(\mathbf{x}_l)M\big) = d^{[l]}(M).$$

This is indeed true since $\bigoplus_{n_l=a}[M/(\mathbf{x}_l)M]_{(n_1,\dots,n_k)} \neq 0$ if and only if a equals to the lth coordinate of a generating degree of M (i.e. $a = d_{i,l}(M)$ for some i). (3) is proved. □

4. Asymptotic behavior of resolution regularity

In this section, we prove our main theorems. Having Theorem 3.5, our arguments now are direct generalization to the $\mathbb{Z}^k$-graded situation of techniques used in [26].

Our first main result establishes the asymptotic linearity of $\mathbf{res\text{-}reg}(I^n M)$.

Theorem 4.1. *Let S be a standard $\mathbb{N}^k$-graded polynomial ring over a field $\mathbf{k}$ and let M be a finitely generated $\mathbb{Z}^k$-graded S-module. Let I be a multigraded homogeneous ideal in S minimally generated in degrees $\mathbf{d}_1(I), \dots, \mathbf{d}_v(I)$. Then, $\mathbf{res\text{-}reg}(I^n M)$ is asymptotically a linear function with slope vector at most $(d^{[1]}(I), \dots, d^{[k]}(I))$ componentwise.*

Proof. Observe first that if for each $1 \leqslant l \leqslant k$, $\mathrm{res\text{-}reg}_l(I^n M)$ is asymptotically a linear function $a_l n + b_l$, then $\mathbf{res\text{-}reg}(I^n M)$ is asymptotically the linear function $(a_1, \dots, a_k)n + (b_1, \dots, b_k)$. Thus, it suffices to show that for each $1 \leqslant l \leqslant k$, $\mathrm{res\text{-}reg}_l(M)$ is asymptotically indeed a linear function with slope at most $d^{[l]}(I)$.

Assume that $\{F_1, \dots, F_v\}$ is a minimal set of generators for I, where $\deg F_i = \mathbf{d}_i(I)$. Let $\mathcal{M} = \bigoplus_{n \geqslant 0} I^n M$ be the Rees module of M with respect to I. Then $\mathcal{M}$ is a finitely generated $\mathbb{Z}^{k+1}$-graded module over the Rees algebra of I, $\mathcal{R} = \bigoplus_{n \geqslant 0} I^n t^n$. Observe further that there is a natural surjection of $\mathbb{Z}^{k+1}$-graded algebras $R = \mathbf{k}[\mathbf{X}_1, \dots, \mathbf{X}_k, Y_1, \dots, Y_v] \to \mathcal{R}$ given by $\mathbf{X}_i \mapsto \mathbf{x}_i$ and $Y_s \mapsto F_s t$, where $\deg X_{i,j} = (\mathbf{e}_i, 0) \in \mathbb{Z}^{k+1}$ and $\deg Y_s = (\mathbf{d}_s(I), 1) \in \mathbb{Z}^{k+1}$ for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant N_i$ and $1 \leqslant s \leqslant v$.

Under this surjection, $\mathcal{M}$ can be viewed as a finitely generated $\mathbb{Z}^{k+1}$-graded R-module. Moreover, $\mathcal{M}_n = I^n M$. The conclusion now follows from Theorem 4.2 below. □

Let $\mathbf{X}_i$ represent the set of indeterminates $\{X_{i,1}, \dots, X_{i,N_i}\}$ for $i = 1, \dots, k$. Let $\mathbf{d}_1, \dots, \mathbf{d}_w \in \mathbb{Z}^k$, where $\mathbf{d}_i = (d_{i,1}, \dots, d_{i,k})$, be non-negative vectors (i.e. $d_{i,j} \geqslant 0$ for all i and j). For each $1 \leqslant l \leqslant k$, let

$$d^{[l]} = \max\{d_{i,l} \mid 1 \leqslant i \leqslant w\} \geqslant 0.$$

Theorem 4.2. *Let $R = \mathbf{k}[\mathbf{X}_1, \dots, \mathbf{X}_k, Y_1, \dots, Y_w]$ be an $\mathbb{N}^{k+1}$-graded polynomial ring where $\deg X_{i,j} = (\mathbf{e}_i, 0) \in \mathbb{Z}^{k+1}$ for all $i = 1, \dots, k$ and $j = 1, \dots, N_i$, and $\deg Y_s = (\mathbf{d}_s, 1) \in \mathbb{Z}^{k+1}$ for all $s = 1, \dots, w$. Let $\mathcal{M} = \bigoplus_{\mathbf{k} \in \mathbb{Z}^{k+1}} \mathcal{M}_{\mathbf{k}}$ be a finitely generated $\mathbb{Z}^{k+1}$-graded R-module. For each $n \in \mathbb{Z}$, let*

$$\mathcal{M}_n = \bigoplus_{\mathbf{m} \in \mathbb{Z}^k} \mathcal{M}_{(\mathbf{m},n)}.$$

Then for each $1 \leqslant l \leqslant k$, $\mathrm{res\text{-}reg}_l(\mathcal{M}_n)$ is asymptotically a linear function with slope at most $d^{[l]}$.

Proof. Considering the flat base extension $\mathbf{k} \to \mathbf{k}' = \mathbf{k}(U)$ where $U = (u_{ij})_{1 \leqslant i,j \leqslant N_l}$ is a square matrix of indeterminates. By Lemma 2.7 and a change of variables, we may assume that $\mathbf{x}_l$ is an $\mathcal{M}_n$-filter-regular sequence with respect to the lth coordinate for any $n \in \mathbb{Z}$. By Theorem 3.5, we have

$$\text{res-reg}_l(\mathcal{M}_n) = \max\left\{\text{end}_l\left(\frac{(x_{l,1}, \ldots, x_{l,i})\mathcal{M}_n :_{\mathcal{M}_n} (\mathbf{x}_l)}{(x_{l,1}, \ldots, x_{l,i})\mathcal{M}_n}\right) \,\Big|\, 0 \leqslant i \leqslant N_l\right\}. \tag{4.1}$$

It is easy to see that

$$\frac{(x_{l,1}, \ldots, x_{l,i})\mathcal{M}_n :_{\mathcal{M}_n} (\mathbf{x}_l)}{(x_{l,1}, \ldots, x_{l,i})\mathcal{M}_n} = \left[\frac{(x_{l,1}, \ldots, x_{l,i})\mathcal{M} :_{\mathcal{M}} (\mathbf{x}_l)}{(x_{l,1}, \ldots, x_{l,i})\mathcal{M}}\right]_n.$$

Moreover, we can consider

$$\frac{(x_{l,1}, \ldots, x_{l,i})\mathcal{M} :_{\mathcal{M}} (\mathbf{x}_l)}{(x_{l,1}, \ldots, x_{l,i})\mathcal{M}}$$

as a $\mathbb{Z}^k$-graded module over the polynomial ring $\mathbf{k}[\mathbf{X}_1, \ldots, \widehat{\mathbf{X}_l}, \ldots, \mathbf{X}_k, Y_1, \ldots, Y_w]$, where $\widehat{\mathbf{X}_l}$ indicates that $\mathbf{X}_l$ is removed. It remains to show that

$$\text{end}_l\left(\left[\frac{(x_{l,1}, \ldots, x_{l,i})\mathcal{M} :_{\mathcal{M}} (\mathbf{x}_l)}{(x_{l,1}, \ldots, x_{l,i})\mathcal{M}}\right]_n\right)$$

is asymptotically a linear function with slope $\leqslant d^{[l]}$ for each $i = 0, \ldots, N_l$. Indeed, this follows from a more general statement of our next result, Theorem 4.3. □

Theorem 4.3. *Let $0 \leqslant t \leqslant k$ and let $1 \leqslant l_1 < \cdots < l_t \leqslant k$. Let*

$$R = \mathbf{k}[\mathbf{X}_{l_1}, \ldots, \mathbf{X}_{l_t}, Y_1, \ldots, Y_w]$$

be a $\mathbb{N}^{k+1}$-graded polynomial ring where $\deg X_{l_i,j} = (\mathbf{e}_{l_i}, 0) \in \mathbb{Z}^{k+1}$ for all $i = 1, \ldots, t$ and $j = 1, \ldots, N_{l_i}$, and $\deg Y_s = (\mathbf{d}_s, 1) \in \mathbb{Z}^{k+1}$ for all $s = 1, \ldots, w$. Let $\mathcal{M} = \bigoplus_{\mathbf{k} \in \mathbb{Z}^{k+1}} \mathcal{M}_{\mathbf{k}}$ be a finitely generated $\mathbb{Z}^{k+1}$-graded R-module. For each $n \in \mathbb{Z}$, let

$$\mathcal{M}_n = \bigoplus_{\mathbf{m} \in \mathbb{Z}^k} \mathcal{M}_{(\mathbf{m},n)}.$$

Then for each $1 \leqslant l \leqslant k$, $\text{end}_l(\mathcal{M}_n)$ is asymptotically a linear function with slope at most $d^{[l]}$.

Proof. We use induction on w. Suppose $w = 0$. Let $h = d^{[k+1]}(\mathcal{M})$ be the maximum of the $(k+1)$st coordinate of minimal generating degrees of $\mathcal{M}$. Then, it is easy to see that for $n \gg 0$, specifically for $n > h$, we must have $\mathcal{M}_n = 0$. Thus, $\text{end}_l(\mathcal{M}_n)$ is asymptotically the zero function.

Assume now that $w \geqslant 1$. Consider the exact sequence of $\mathbb{Z}^{k+1}$-graded S-modules

$$0 \to [0 :_{\mathcal{M}} Y_w]_{(\mathbf{m},n)} \to \mathcal{M}_{(\mathbf{m},n)} \xrightarrow{Y_w} \mathcal{M}_{(\mathbf{m}+\mathbf{d}_w,n+1)} \to [\mathcal{M}/Y_w\mathcal{M}]_{(\mathbf{m}+\mathbf{d}_w,n+1)} \to 0.$$

Since $0:_{\mathcal{M}} Y_w$ and $\mathcal{M}/Y_w\mathcal{M}$ can be viewed as $\mathbb{Z}^{k+1}$-graded modules over the ring $\mathbf{k}[\mathbf{X}_{l_1},\dots,\mathbf{X}_{l_t},Y_1,\dots,Y_{w-1}]$, by induction $\mathrm{end}_l([0:_{\mathcal{M}} Y_w]_n)$ and $\mathrm{end}_l([\mathcal{M}/Y_w\mathcal{M}]_n)$ are asymptotically linear functions with slopes at most $d^{[l]}$. As a consequence, we have

$$\mathrm{end}_l\big([0:_{\mathcal{M}} Y_w]_n\big)+d^{[l]}\geqslant \mathrm{end}_l\big([0:_{\mathcal{M}} Y_w]_{n+1}\big)\quad\text{and}\tag{4.2}$$

$$\mathrm{end}_l\big([\mathcal{M}/Y_w\mathcal{M}]_n\big)+d^{[l]}\geqslant \mathrm{end}_l\big([\mathcal{M}/Y_w\mathcal{M}]_{n+1}\big)\tag{4.3}$$

for all $n\gg 0$.

It is clear that $\mathrm{end}_l(\mathcal{M}_n)\geqslant \mathrm{end}_l([0:_{\mathcal{M}} Y_w]_n)$ for all $n\geqslant 0$. If $\mathrm{end}_l(\mathcal{M}_n)=\mathrm{end}_l([0:_{\mathcal{M}} Y_w]_n)$ for $n\gg 0$ then the statement follows by induction. It remains to consider the case that there exists an infinite sequence of integers n for which $\mathrm{end}_l(\mathcal{M}_n)>\mathrm{end}_l([0:_{\mathcal{M}} Y_w]_n)$.

Let m be such an integer. Making use of the inequalities (4.2) and (4.3), we obtain

$$\mathrm{end}_l(\mathcal{M}_{m+1})=\max\big\{\mathrm{end}_l(\mathcal{M}_m)+d^{[l]},\mathrm{end}_l\big([\mathcal{M}/Y_w\mathcal{M}]_{m+1}\big)\big\}$$

(since we can take m to be bigger than the last coordinate of all generators in a minimal system of generators for $\mathcal{M}$). By taking m large enough, we further have

$$\mathrm{end}_l(\mathcal{M}_m)+d^{[l]}\geqslant \mathrm{end}_l\big([\mathcal{M}/Y_w\mathcal{M}]_m\big)+d^{[l]}\geqslant \mathrm{end}_l\big([\mathcal{M}/Y_w\mathcal{M}]_{m+1}\big).$$

Thus, $\mathrm{end}_l(\mathcal{M}_{m+1})=\mathrm{end}_l(\mathcal{M}_m)+d^{[l]}>\mathrm{end}_l([0:_{\mathcal{M}} Y_w]_m)+d^{[l]}\geqslant \mathrm{end}_l([0:_{\mathcal{M}} Y_w]_{m+1})$.

We have shown that for $m\gg 0$, if $\mathrm{end}_l(\mathcal{M}_m)>\mathrm{end}_l([0:_{\mathcal{M}} Y_w]_m)$ then $\mathrm{end}_l(\mathcal{M}_{m+1})>\mathrm{end}_l([0:_{\mathcal{M}} Y_w]_{m+1})$. By repeating this argument we can show that $\mathrm{end}_l(\mathcal{M}_n)>\mathrm{end}_l([0:_{\mathcal{M}} Y_w]_n)$ for all $n\gg 0$. This, by a similar argument as above, implies that $\mathrm{end}_l(\mathcal{M}_{n+1})=\mathrm{end}_l(\mathcal{M}_n)+d^{[l]}$ for all $n\gg 0$. Hence, $\mathrm{end}_l(\mathcal{M}_n)$ is also asymptotically a linear polynomial with slope $d^{[l]}$ in this case. The assertion, and hence, Theorem 4.2 and subsequently Theorem 4.1, is proved. □

The rest of this section is to explicitly describe the slope vector of the asymptotically linear function $\mathbf{res\text{-}reg}(I^nM)$.

Recall from Definition 1.5 that a multigraded homogeneous ideal $J\subset I$ is an *M-reduction* of I if $I^nM=JI^{n-1}M$ for all $n\gg 0$. For each $l=1,\dots,k$,

$$\rho_M^{[l]}(I)=\min\big\{d^{[l]}(J)\;\big|\;J\text{ is an }M\text{-reduction of }I\big\},$$

and

$$\mathrm{beg}^{[l]}(M)=\min\big\{n_l\;\big|\;\exists\mathbf{n}\in\mathbb{Z}^k\text{ such that }M_{\mathbf{n}}\neq 0\big\}.$$

It can further be seen that if $I^{n_0}M=JI^{n_0-1}M$ then $I^nM=JI^{n-1}M$ for all $n\geqslant n_0$ and J is an M-reduction of I.

Lemma 4.4. *For any $n\geqslant 0$, we have*

$$d^{[l]}\big(I^nM\big)\geqslant n\rho_M^{[l]}(I)+\mathrm{beg}^{[l]}(M).$$

Proof. Write I as the sum of two ideals, $I = J + K$, in which J is generated by multigraded homogeneous elements f of I such that $\deg_l(f) < \rho_M^{[l]}(I)$ and K is generated by multigraded homogeneous elements g of I such that $\deg_l(g) \geqslant \rho_M^{[l]}(I)$. Then,

$$I^n M = (J+K)I^{n-1}M = JI^{n-1}M + K(J+K)^{n-1}M = JI^{n-1}M + K^n M.$$

It can be seen that $K^n M$ is generated by multigraded homogeneous elements h for which $\deg_l(h) \geqslant n\rho_M^{[l]}(I) + \mathrm{beg}^{[l]}(M)$.

If for some $n_0 \geqslant 0$, $d^{[l]}(I^{n_0}M) < n_0\rho_M^{[l]}(I) + \mathrm{beg}^{[l]}(M)$ then $K^{n_0}M = 0$ and $I^{n_0}M = JI^{n_0-1}M$. This implies that $I^n M = JI^{n-1}M$ for any $n \geqslant n_0$ and J is an M-reduction of I. However, by the construction of J, we have $d^{[l]}(J) < \rho_M^{[l]}(I)$, which is a contradiction to the definition of $\rho_M^{[l]}(I)$. The statement is proved. □

The next theorem is a refinement of Theorem 4.1 and gives an explicit description for the slope vector of the asymptotically linear function $\mathbf{res\text{-}reg}(I^n M)$.

Theorem 4.5. *Let S be a standard $\mathbb{N}^k$-graded polynomial ring over a field $\mathbf{k}$ and let M be a finitely generated $\mathbb{Z}^k$-graded S-module. Let I be a multigraded homogeneous ideal in S. Then, there exists a vector $\mathbf{b} \geqslant (\mathrm{beg}^{[1]}(M), \ldots, \mathrm{beg}^{[k]}(M))$ such that for $n \gg 0$,*

$$\mathbf{res\text{-}reg}\big(I^n M\big) = n\big(\rho_M^{[1]}(I), \ldots, \rho_M^{[k]}(I)\big) + \mathbf{b}.$$

Proof. Suppose $1 \leqslant l \leqslant k$. Let a multigraded homogeneous ideal J be an M-reduction of I such that $d^{[l]}(J) = \rho_M^{[l]}(I)$. Let $\mathcal{M}$ be the Rees module of M with respect to I and let $\mathcal{R}$ be the Rees algebra of I. As before, $\mathcal{M}$ is a finitely generated $\mathbb{Z}^{k+1}$-graded module over the Rees algebra of I, $\mathcal{R} = \bigoplus_{n\geqslant 0} I^n t^n$. Let $S[Jt]$ be the Rees algebra of J. Then, since J is an M-reduction of I, $S[Jt] \to \mathcal{R}$ is a finite map. It follows that $\mathcal{M}$ is a finitely generated $\mathbb{Z}^{k+1}$-graded module over $S[Jt]$.

Assume that J is minimally generated by $G_1, \ldots, G_u$ of degrees $\mathbf{d}_1(J), \ldots, \mathbf{d}_u(J)$. Then, there is a natural surjection $R = \mathbf{k}[\mathbf{X}_1, \ldots, \mathbf{X}_k, Y_1, \ldots, Y_u] \to S[Jt]$ given by $\mathbf{X}_i \mapsto \mathbf{x}_i$ and $Y_s \mapsto G_s t$, where $\deg X_{i,j} = (\mathbf{e}_i, 0) \in \mathbb{Z}^{k+1}$ and $\deg Y_s = (\mathbf{d}_s(J), 1) \in \mathbb{Z}^{k+1}$ for all $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant N_i$ and $1 \leqslant s \leqslant u$. Under this surjection, $\mathcal{M}$ can be viewed as a finitely generated $\mathbb{Z}^{k+1}$-graded module over R. By Theorem 4.2, for each $l = 1, \ldots, k$, $\text{res-reg}_l(I^n M) = \text{res-reg}_l(\mathcal{M}_n)$ is asymptotically a linear function with slope at most $d^{[l]}(J)$. Suppose, $\text{res-reg}_l(I^n M) = a_l n + b_l$ for $n \gg 0$, in which $a_l \leqslant d^{[l]}(J)$.

By Lemma 4.4, we have

$$\text{res-reg}_l\big(I^n M\big) \geqslant d^{[l]}\big(I^n M\big) \geqslant n\rho_M^{[l]}(I) + \mathrm{beg}^{[l]}(M) \quad \forall \mathbf{n} \geqslant 0. \tag{4.4}$$

This implies that $a_l \geqslant \rho_M^{[l]}(I)$. Together with the fact that $a_l \leqslant d^{[l]}(J) = \rho_M^{[l]}(I)$, we now have $a_l = \rho_M^{[l]}(I)$. (4.4) also implies that $b_l \geqslant \mathrm{beg}^{[l]}(M)$. The theorem is proved. □

Example 4.6. Let $S = \mathbb{Q}[a,b,c]$ be a standard $\mathbb{N}^3$-graded polynomial ring, where $\deg a = \mathbf{e}_1$, $\deg b = \mathbf{e}_2$ and $\deg c = \mathbf{e}_3$. Consider a homogeneous ideal $I = (a,b,c)$ and a $\mathbb{Z}^3$-graded module $M = S/(a^3, a^2b, a^2c, abc)$. Observe that $J = (b,c)$ is a minimal M-reduction of I.

Thus, $(\rho_M^{[1]}(I), \rho_M^{[2]}(I), \rho_M^{[3]}(I)) = (0,1,1) \neq (1,1,1) = (d^{[1]}(I), d^{[2]}(I), d^{[3]}(I))$. Moreover, $(\mathrm{beg}^{[1]}(M), \mathrm{beg}^{[2]}(M), \mathrm{beg}^{[3]}(M)) = \mathbf{0}$. Theorem 4.5 says that there exists $\mathbf{b} \geqslant \mathbf{0}$ so that for $n \gg 0$ we have

$$\textbf{res-reg}\big(I^n M\big) = n(0,1,1) + \mathbf{b}.$$

Using CoCoA [5], we can also calculate $\textbf{res-reg}(IM) = (1,1,1)$, $\textbf{res-reg}(I^2M) = (2,2,2) \neq 2(0,1,1) + (1,0,0)$, while $\textbf{res-reg}(I^3M) = 3(0,1,1) + (1,0,0)$, $\textbf{res-reg}(I^4M) = 4(0,1,1) + (1,0,0)$, and $\textbf{res-reg}(I^5M) = 5(0,1,1) + (1,0,0)$. It seems that in this example, $\textbf{res-reg}(I^nM) = n(0,0,1) + (1,0,0)$ for $n \geqslant 3$.

Remark 4.7. It would be interesting to find the vector $\mathbf{b}$ and a lower bound n_0 so that in Theorem 4.5,

$$\textbf{res-reg}\big(I^n M\big) = n\big(\rho_M^{[1]}(I), \dots, \rho_M^{[k]}(I)\big) + \mathbf{b}$$

for all $n \geqslant n_0$. Even in the $\mathbb{Z}$-graded case, this is still open.

Acknowledgments

This project started when hurricane Katrina was about to hit New Orleans, where the two authors were at the time. The collaboration was then made possible with the help of many people from various places. We thank the University of Missouri–Columbia and the University of Arkansas for their hospitalities. Especially, we also express our thanks to Ian Aberbach, Jean Chan, Dale Cutkosky, Mark Johnson, Hema Srinivasan and Bernd Ulrich. We thank an anonymous referee for a careful reading of the paper.

Appendix A

During the preparation of this paper, Bernd Ulrich asked us whether multigraded regularity can be defined even when a *finite* minimal free resolution as in Definition 1.1 may not exist. Ulrich's question has helped us better understand resolution regularity. We would like to thank him for his question. We shall also propose an alternative definition for multigraded regularity even when finite minimal free resolutions may not exist.

Let A be a Noetherian ring (commutative with identity). Let T be a standard $\mathbb{Z}^k$-graded algebra over A; that is, $T_{\mathbf{0}} = A$ and T is generated over $T_{\mathbf{0}}$ by elements of $\bigoplus_{l=1}^k T_{\mathbf{e}_l}$. Let $N = \bigoplus_{\mathbf{n}\in\mathbb{Z}^k} N_{\mathbf{n}}$ be a finitely generated $\mathbb{Z}^k$-graded module over T. It is well known that, in general, a finite minimal free resolution of N over T may not exist. Thus, Definition 1.1 will not work in this case. We shall introduce an alternative approach to define $\textbf{reg}(N)$. This approach results from our Theorem 3.5. Our new notion, $\textbf{reg}(N)$, still bounds (but does not capture) the supremum of coordinates (with suitable shifts) in a minimal free resolution of N.

As before, for each $l = 1, \dots, k$, assume that $T_{\mathbf{e}_l}$ is generated as a module over $T_{\mathbf{0}}$ by $\mathbf{x}_l = \{x_{l,1}, \dots, x_{l,N_l}\}$. Write $T = \bigoplus_{m\in N} T_m$, where $T_m = \bigoplus_{\mathbf{n}\in\mathbb{N}^k, n_l=m} T_{\mathbf{n}}$. Then, T can be viewed as a standard $\mathbb{N}$-graded algebra over $T_0 = \bigoplus_{\mathbf{n}\in\mathbb{N}^k, n_l=0} T_{\mathbf{n}}$. Under this new grading of T, $(\mathbf{x}_l)$ is the homogeneous irrelevant ideal. We can also view N as a $\mathbb{Z}$-graded T-module with the grading given by $N = \bigoplus_{m\in\mathbb{Z}} N_m$, where $N_m = \bigoplus_{\mathbf{n}\in\mathbb{Z}^k, n_l=m} N_{\mathbf{n}}$. This induces a natural $\mathbb{Z}$-graded structure on local cohomology modules $H^i_{(\mathbf{x}_l)}(N)$ for all $i \geqslant 0$. Under this $\mathbb{Z}$-graded structure (depend on l), the usual a-invariants are well defined and have been much studied (cf. [20,24]). We shall use the superscript $\bullet^{[l]}$ to indicate that these invariants depend on l. That is, we define

$$a_i^{[l]}(N) = \begin{cases} -\infty & \text{if } H^i_{(\mathbf{x}_l)}(N) = 0, \\ \max\{m \mid H^i_{(\mathbf{x}_l)}(N)_m \neq 0\} & \text{if } H^i_{(\mathbf{x}_l)}(N) \neq 0. \end{cases}$$

Definition A.1. For each $l = 1, \dots, k$, let

$$\operatorname{reg}_l(N) = \max\{a_i^{[l]}(N) + i \mid i \geqslant 0\}.$$

We define the multigraded (resolution) *regularity* of N to be the vector

$$\mathbf{reg}(N) = (\operatorname{reg}_1(N), \dots, \operatorname{reg}_k(N)) \in \mathbb{Z}^k.$$

Observe that since $H^i_{(\mathbf{x}_l)}(N) = 0$ for all $i \gg 0$, $\mathbf{reg}(N)$ is well defined.

Remark A.2. In our current setting, N-filter regular sequences with respect to the lth coordinate, and invariants $\mathfrak{a}_N^{[l]}(\bullet)$ and $\operatorname{end}_l(\bullet)$ can be defined as before. By applying a similar line of arguments as that of Theorem 3.5(1), we have

$$\operatorname{reg}_l(N) = \max\{\mathfrak{a}_N^{[l]}(\mathbf{x}_l), \operatorname{end}_l(H^{N_l}_{(\mathbf{x}_l)}(N)) + N_l\}.$$

This, together with a similar line of arguments as that of Theorem 3.5(2), implies that

$$\begin{aligned} \operatorname{reg}_l(N) &= \max\{\mathfrak{a}_N^{[l]}(\mathbf{x}_l), d^{[l]}(N)\} && \text{(A.1)} \\ &= \max\left\{\operatorname{end}_l\left(\frac{(x_{l,1}, \dots, x_{l,i})N :_N (\mathbf{x}_l)}{(x_{l,1}, \dots, x_{l,i})N}\right) \,\middle|\, 0 \leqslant i \leqslant N_l\right\}. && \text{(A.2)} \end{aligned}$$

Let $S \to T$ be the natural surjection from a standard $\mathbb{N}^k$-graded polynomial ring over A to T. Then N is a $\mathbb{Z}^k$-graded S-module. Let $\mathbb{G}: \cdots \to G_p \to \cdots \to G_0 \to N \to 0$ be a (infinite) minimal free resolution of N over S, where $G_i = \bigoplus_j S(-c_{ij}^{[1]}, \dots, -c_{ij}^{[k]})$ for all $i \geqslant 0$. For each $1 \leqslant l \leqslant k$ and $i \geqslant 0$, set $c_i^{[l]} = \max_j\{c_{ij}^{[l]}\}$, and let

$$r_l = \sup_i\{c_i^{[l]} - i\}.$$

Let $\mathbf{r} = (r_1, \dots, r_k)$. By definition, $\mathbf{r}$ captures maximal coordinates (with suitable shifts) of shifts in a minimal free resolution of N. By applying a similar line of arguments as that of Theorem 3.2, we can show that

$$r_l \leqslant \max\{\mathfrak{a}_N^{[l]}(\mathbf{x}_l), d^{[l]}(N)\}. \qquad \text{(A.3)}$$

Notice that we do not get the equality in (A.3) due to the lack of Nakayama's lemma when A is not necessarily a field.

Now, by virtue of (A.1) and (A.3), $\operatorname{reg}_l(N)$ then gives a bound on the l-coordinate (with a suitable shift) in a minimal free resolution of N.

Remark A.3. Let I be a multigraded homogeneous ideal in T. Techniques in Section 4 can now be applied, together with (A.2), to show that $\mathbf{reg}(I^n N)$ again is asymptotically a linear function.

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