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Explicit Estimates on the Fundamental Solution of Higher-Order Parabolic Equations with Measurable Coefficients

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INTRODUCTION

Let H be a self-adjoint uniformly elliptic differential operator of order $2m$ defined on a domain $\Omega \subset \mathbf{R}^N$ and let $K(t, x, y)$ be the fundamental solution of the associated parabolic equation

$$\frac{\partial u}{\partial t} = -Hu, \qquad t > 0.$$

Pointwise estimates on $K(t, x, y)$ typically have the form

$$|K(t, x, y)| \leqslant c_1 t^{-N/2m} \exp\left\{ -c_2 \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}} + c_3 t \right\} \tag{1}$$

for some positive constants c_i and all $t > 0$ and $x, y \in \Omega$. Such estimates for uniformly elliptic operators have been proved under a variety of local regularity assumptions [14–17] and more recently for operators with measurable coefficients provided $2m > N$ [8]. This has been shown to be true in the limit case $2m = N$ [3, 11] while there are examples that show that (1) is not in general true when $2m < N$ and H has measurable coefficients [10]. In [6] estimates are obtained for singular or degenerate operators with measurable coefficients, with the Euclidean distance being replaced by one that reflects the behavior of the coefficients near the singularities. We refer to [9] for a detailed account of much of the recent progress on the spectral theory of higher-order elliptic operators.

The problem of sharpening such bounds was addressed recently. The sharpness of estimates is checked by comparison against short time asymptotics of $K(t, x, y)$. These were proved by Tintarev [19] who showed, following work of Evgrafov and Postnikov [12], that if $\Omega = \mathbf{R}^N$, if the coefficients $\{a_{\alpha\beta}\}$ are smooth and if the symbol $a(x, \xi)$ satisfies the strong

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convexity condition (see the definition in Section 2) then the fundamental solution $K(t, x, y)$ has short-time asymptotics which, modulo subexponential terms, are given by

$$K(t, x, y) \sim t^{-N/2m} \exp\left\{-\sigma_m \frac{d(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}}\right\}, \tag{2}$$

provided x and y are close enough. Here $\sigma_m = (2m-1)(2m)^{-2m/(2m-1)} \sin(\frac{\pi}{4m-2})$ and $d(x, y)$ is a Finsler metric induced by the principal symbol $a(x, \xi)$ of H. It has length element $ds = p(x, dx)$ given by

$$p(x, \xi) = \sup_{\substack{\eta \in \mathbf{R}^N \\ \eta \neq 0}} \frac{\langle \xi, \eta \rangle}{a(x, \eta)^{1/2m}}. \tag{3}$$

Once ds is defined one can then define lengths of paths and hence the distance $d(x, y)$ between any two points x and y. An equivalent definition of $d(x, y)$ and one that is more relevant to our approach is [1, Lemma 1.3]

$$d(x, y) = \sup\{\phi(y) - \phi(x) \mid \phi \in \mathrm{Lip}(\Omega),\ a(z, \nabla\phi(z)) \leqslant 1 \text{ a.e. } z \in \Omega\}.$$

It immediately follows from (3) that the metric is Riemannian if the operator is second-order or, more generally, if its principal symbol is the mth power of a polynomial of degree two.

Hence it was with an eye on (2) that the quest for sharp bounds began. In [4] it was proved that if H is homogeneous with measurable coefficients acting on a domain $\Omega \subset \mathbf{R}^N$ and $(-\Delta)^m \leqslant H \leqslant (1+\delta)(-\Delta)^m$, then one can take in (1)

$$c_2 = \sigma_m - O(\delta) \qquad (\text{small } \delta) \tag{4}$$

provided $2m > N$. That was improved in [5] at the cost of a strong local regularity assumption. It was shown that if the coefficients lie in $W^{m,\infty}(\Omega)$ and the strong convexity condition is satisfied then one has the estimate

$$|K(t, x, y)| \leqslant c_{\varepsilon, M} t^{-N/2m} \exp\left\{-(\sigma_m - \varepsilon) \frac{d_M(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} + ct\right\} \tag{5}$$

for all $\varepsilon > 0$ and $M > 0$, where $d_M(x, y)$ is a family of Finsler-type metrics that approximate $d(x, y)$ as $M \to \infty$. Sufficient conditions under which the ratio $d_M(x, y)/d(x, y)$ converges to one uniformly in x, y have been established in [7], allowing one to replace $d_M(x, y)$ by $d(x, y)$ in (5). We point out that one cannot have $\varepsilon = 0$ in (5), because of the subexponential terms mentioned above.

Our primary aim in this paper is to extend the above results to operators with measurable coefficients. The bounds obtained are expressed in terms of the Finsler-type metrics $d_M(\cdot, \cdot)$ that were used in [5]; however, it is of particular interest that the new estimates have a certain difference from the earlier ones. In fact, the exponential constant σ_m is now perturbed by a term of order $O(d)$ (for d small) where d is the distance of the principal coefficients from the Sobolev space $W^{m-1,\infty}(\Omega)$ in the uniform norm; hence, in particular, $d=0$ when $2m=2$. This indicates a qualitative difference between the cases $2m=2$ and $2m>2$, suggesting that, possibly, this extra term really exists and cannot be removed from the estimate. If this is indeed true it would provide a striking contrast with the second-order case, where local regularity assumptions do not affect pointwise estimates on $K(t, x, y)$.

If on the other hand that term is removable, then it appears that a fundamentally new approach is needed in order to demonstrate this. This is because of Proposition 8 where we present an example which shows that it is not possible to remove that term employing the method of this paper. We point out on the other hand that we are not aware of any other method for obtaining sharp estimates for operators of this type, even if the coefficients are assumed to be smooth.

As a corollary of our main theorem we obtain (5) for operators whose principal coefficients lie in the closure of $W^{m-1,\infty}(\Omega)$ in the uniform norm. This is substantially larger than the space $W^{m,\infty}(\Omega)$ which was used in [5]; see the two examples following the main theorem.

The proof of our main estimate uses Davies' exponential perturbation technique and relies heavily on an inequality of Evgrafov and Postnikov [12]. Apart from the main theorem as such, a technical improvement of this work upon [5] is the introduction of the notion of the *essential order*. This clarifies and simplifies substantially many parts of the paper and, more important, it can also be applied to operators acting on manifolds.

2. PRELIMINARIES

Given a multi-index $\alpha=(\alpha_1, ..., \alpha_N)$ and a vector $u=(u_1, ..., u_N)$, we use the standard notation D^α and u^α for the differential operator $(\partial/\partial_1)^{\alpha_1}\cdots(\partial/\partial_N)^{\alpha_N}$ and the number $u_1^{\alpha_1}\cdots u_N^{\alpha_N}$ correspondingly. We also write $\alpha!=\alpha_1!\cdots\alpha_N!$. For $k\geqslant 0$ we denote by $\nabla^k f$ the vector $(D^\alpha f)_{|\alpha|=k}$. Given two multi-indices α and γ with $\gamma\leqslant\alpha$, we also set $c_\gamma^\alpha=\alpha!/\gamma!(\alpha-\gamma)!$ and $c_\alpha^{|\alpha|}=|\alpha|!/\alpha!$. Given a quadratic form $\Gamma(\cdot)$, we denote by $\Gamma(\cdot, \cdot)$ the associated sesquilinear form obtained by polarization. Finally we let $\hat{f}$ denote the Fourier transform of a function f, $\hat{f}(\xi)=(2\pi)^{-N/2}\int e^{i\xi\cdot x}f(x)\,dx$.

Let $\Omega \subset \mathbf{R}^N$ be open and connected and let m be an integer such that $2m > N$. Let $\{a_{\alpha\beta}(x)\}_{|\alpha|, |\beta| \leqslant m}$ be a self-adjoint matrix with entries in $L^\infty(\Omega)$. We assume that the entries corresponding to $|\alpha+\beta| = 2m$ are real-valued but do not require this for the lower-order entries. We point out that there is a standard way of relaxing the condition $a_{\alpha\beta} \in L^\infty(\Omega)$ when $|\alpha+\beta| < 2m$ and replace it with appropriate L^p-type conditions. Our assumption however is that the entries are bounded.

We define the quadratic form $Q(\cdot)$ by

$$Q(f) = \int_\Omega \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} a_{\alpha\beta}(x)\, D^\alpha f(x)\, D^\beta \bar{f}(x)\, dx$$

and assume that Gårding's inequality

$$Q(f) \geqslant c\, \|(-\Delta)^m f\|_2^2 \tag{6}$$

is valid for all $f \in C_c^\infty(\Omega)$, or, equivalently, for all f in the Sobolev space $W_0^{m,2}(\Omega) =: \mathrm{Dom}(Q)$. The form Q is then closed and we define the operator H to be the self-adjoint operator associated to it, so that formally it is given by

$$Hf(x) = \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} (-1)^{|\alpha|}\, D^\alpha \{a_{\alpha\beta}(x)\, D^\beta f(x)\}. \tag{7}$$

We say that H satisfies Dirichlet boundary conditions since this agrees with the corresponding classical notion when enough regularity is assumed on the coefficients and the boundary.

We next associate to H a family of Finsler-type metrics $d_M(x, y)$ where M is a large parameter. Let

$$a(x, \xi) = \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha\beta}(x)\, \xi^{\alpha+\beta}, \qquad x \in \Omega, \qquad \xi \in \mathbf{R}^N,$$

be the principal symbol of H, which we assume satisfies

$$c^{-1}\, |\xi|^{2m} \leqslant a(x, \xi) \leqslant c\, |\xi|^{2m}, \tag{8}$$

for some constant $c \geqslant 1$ and all $x \in \Omega$, $\xi \in \mathbf{R}^N$. We define the class of real-valued functions

$$\mathscr{E}_m(\Omega) = \{\phi \in C^m_{\mathbf{R}}(\Omega) \mid \|D^\alpha \phi\|_\infty < +\infty,\ 0 \leqslant |\alpha| \leqslant m\}$$

and we will be particularly interested in the subclasses

$$\mathscr{E}_{a,M} = \{\phi \in \mathscr{E}_m(\Omega) \mid a(x, \nabla\phi(x)) \leqslant 1,\, x \in \Omega;\ \|D^\gamma \phi\|_\infty \leqslant M,\, 2 \leqslant |\gamma| \leqslant m\}$$

and

$$\mathscr{E}_a = \bigcup_{M > 1} \mathscr{E}_{a,M}.$$

The distance $d_M(x, y)$ is defined as

$$d_M(x, y) = \sup\{\phi(y) - \phi(x) \mid \phi \in \mathscr{E}_{a,M}\}. \tag{9}$$

We now write the principal symbol of H in a different way, namely we define the functions a_γ, $|\gamma| = 2m$, by

$$a(x, \xi) = \sum_{|\gamma| = 2m} c_\gamma^{2m}\, a_\gamma(x)\, \xi^\gamma$$

(so $a_{\alpha\beta} \neq a_{\alpha+\beta}$ in general). For each $x \in \Omega$ we then define a sesquilinear form $\Gamma(x, \cdot, \cdot)$ on $\mathbf{C}^\nu$, $\nu = \nu(N, m)$ being the number of multi-indices of length m, by

$$\Gamma(x, p, q) = \sum_{\substack{|\alpha| = m \\ |\beta| = m}} a_{\alpha+\beta}(x)\, p_\alpha \overline{q_\beta}, \qquad \text{all} \quad p = (p_\alpha), \quad q = (q_\beta) \in \mathbf{C}^\nu, \tag{10}$$

and let $\Gamma(x, \cdot)$ denote the corresponding quadratic form, $\Gamma(x, p) = \Gamma(x, p, p)$, $x \in \Omega$, $p \in \mathbf{C}^\nu$.

Definition. The symbol $a(x, \xi)$ is *strongly convex* if the form Γ is non-negative for all $x \in \Omega$.

Strong convexity was a basic assumption in [19], as well as in [12] where it was first introduced.

Now let $A_{pr} = \{a_{\alpha\beta}\}_{|\alpha| = |\beta| = m}$ be the principal coefficient matrix of H and

$$d = \mathrm{dist}_{L^\infty}(A_{pr}, W^{m-1,\infty}(\Omega)) := \sup_{\substack{|\alpha| = m \\ |\beta| = m}} \inf_{u \in W^{m-1,\infty}} \|a_{\alpha\beta} - u\|_\infty. \tag{11}$$

The primary aim of this paper is the following theorem:

THEOREM 1. *Let $2m > N$ and assume that the principal symbol $a(x, \xi)$ of H is strongly convex. Then for d small, for all $\varepsilon \in (0, 1)$ and all M large there exists $c_\varepsilon, c_{\varepsilon, M} < \infty$ such that*

$$|K(t, x, y)| \leqslant c_\varepsilon t^{-N/2m} \exp\{-(\sigma_m - cd - \varepsilon)\, d_M(x, y)^{2m/(2m-1)}\, t^{-1/(2m-1)} + c_{\varepsilon, M} t\} \tag{12}$$

for all $x, y \in \Omega$ and $t > 0$.

The proof makes use of Davies' perturbation technique. Given $\phi \in \mathscr{E}_m(\Omega)$ the (multiplication) operator e^ϕ leaves $W_0^{m,2}(\Omega)$ invariant so that one can define the non-symmetric sesquilinear form Q_ϕ by

$$\begin{aligned} Q_\phi(f, g) &= Q(e^\phi f, e^{-\phi} g) \\ &= \int_\Omega \sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} a_{\alpha\beta}(x)\, D^\alpha[e^\phi f(x)]\, D^\beta[e^{-\phi} \bar{g}(x)]\, dx \end{aligned} \tag{13}$$

for all $f, g \in \mathrm{Dom}(Q_\phi) := W_0^{m,2}(\Omega)$. We denote by H_ϕ the (non-symmetric) operator associated with Q_ϕ, so that

$$H_\phi f = e^{-\phi} H e^\phi f \tag{14}$$

for all $f \in \mathrm{Dom}(H_\phi) = \{f \in L^2 \mid e^\phi f \in \mathrm{Dom}(H)\}$. Expanding $Q_\phi(f)$ we see that $Q(f)$ and $Q_\phi(f)$ have the same highest-order terms and a simple argument, combined with Gårding's inequality (6), shows that

$$|Q(f) - Q_\phi(f)| < \varepsilon Q(f) + c_\varepsilon\{\|\phi\|_{W^{m,\infty}} + \|\phi\|_{W^{m,\infty}}^{2m}\}\, \|f\|_2^2, \tag{15}$$

for all $0 < \varepsilon < 1$ and $f \in C_c^\infty(\Omega)$. See [5] for a detailed proof.

The following basic proposition was proved in [4]. It provides an explicit estimate on $K(t, x, y)$ in terms of the bottom of the spectrum of the auxiliary operator $\mathrm{Re}\, H_\phi = (H_\phi + H_\phi^*)/2$.

PROPOSITION 2. *Assume $2m > N$. Let $\phi \in \mathscr{E}_m(\Omega)$ and let the constant $k > 0$ be such that*

$$\mathrm{Re}\,\langle H_\phi f, f\rangle \geqslant -k\, \|f\|_2^2 \tag{16}$$

for all $f \in C_c^\infty(\Omega)$. Then for any $\delta \in (0, 1)$ there exists a constant c_δ such that

$$|K(t, x, y)| \leqslant c_\delta t^{-N/2m} \exp\{\phi(x) - \phi(y) + (1 + \delta)\, kt\} \tag{17}$$

for all $x, y \in \Omega$ and all $t > 0$.

Proof. This is Proposition 2.5 of [4].

Remark. Let $\tilde{\mathscr{E}}_m$ be the class of functions ϕ which satisfy all the conditions in order to lie in $\mathscr{E}_m$ except that ϕ itself is not bounded but only locally bounded. If $\phi \in \tilde{\mathscr{E}}_m$ then $e^{\phi} f$ does not belong to $W_0^{m,2}(\Omega)$, but $\operatorname{Re} Q_\phi(f)$ can still be defined as the RHS of (13), which is well defined as it contains derivatives of ϕ but not ϕ itself. Then Proposition 2 is still valid. This is shown by approximating Ω by an increasing and exhausting sequence of compact subdomains (Ω_n), apply Proposition 2 to the operator H_n that acts on $L^2(\Omega_n)$ and then let $n \to \infty$. We refer to Section 3 of [4] where this has been carried out in detail.

3. MAIN ESTIMATES

From now on we shall be considering forms $Q_{\lambda\phi}$ where $\lambda > 0$ and $\phi \in \mathscr{E}_m$. Our aim is to establish an effective lower bound on $\operatorname{Re} Q_{\lambda\phi}$ so as to apply Proposition 2. Using Leibniz's rule to expand $D^\alpha(e^{\lambda\phi} f)$ we see that $Q_{\lambda\phi}(f)$ is a linear combination of terms of the form

$$\lambda^s \int_\Omega b_{s\gamma\delta}(x)\, D^\gamma f\, D^\delta \bar{f}\, dx,$$

where $b_{s\gamma\delta}$ are bounded functions. We define the *essential order* of such a term to be the number $s + |\gamma + \delta|$, a non-negative integer smaller than or equal to $2m$. We denote by $\mathscr{L}_m$ the space of linear combinations of terms whose essential order is smaller than $2m$. We shall see later that such terms are in a certain sense negligible. We also point out the estimate

$$\left| \lambda^s \int b\, D^\gamma f\, D^\delta \bar{f}\, dx \right| < c\, \|b\|_\infty \{ Q(f) + \lambda^{2m} \|f\|_2^2 \}, \tag{18}$$

which is valid for all terms of essential order $2m$. This can be seen by transfering to the Fourier space and using estimates of the form $|\lambda^s \xi^\beta| < c(\lambda^{s+|\beta|} + |\xi|^{s+|\beta|})$ together with Gårding's inequality (6).

Let

$$Q_{1,\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\substack{\gamma \leqslant \alpha \\ \delta \leqslant \beta}} a_{\alpha\beta} c_\gamma^\alpha c_\delta^\beta (\lambda \nabla\phi)^\gamma (-\lambda\nabla\phi)^\delta D^{\alpha-\gamma} f\, D^{\beta-\delta} \bar{f}\, dx.$$

We have

LEMMA 3. *The difference $Q_{\lambda\phi}(f) - Q_{1,\lambda\phi}(f)$ lies in $\mathscr{L}_m$.*

Proof. It is readily seen that any terms in $Q_{\lambda\phi}(f)$ which arise from pairs (α, β) with $|\alpha+\beta| < 2m$ lie in $\mathscr{L}_m$. So we restrict our attention to highest-order terms. Defining the functions

$$P_{\gamma,\lambda\phi}(x) = e^{-\lambda\phi(x)}[D^\gamma e^{\lambda\phi(x)}], \qquad (\gamma \text{ a multi-index})$$

it follows that

$$Q_{\lambda\phi}(f) = \int_\Omega \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma\leqslant\alpha\\\delta\leqslant\beta}} c_\gamma^\alpha c_\delta^\beta P_{\gamma,\lambda\phi} P_{\delta,-\lambda\phi}\, D^{\alpha-\gamma} f\, D^{\beta-\delta}\bar{f}\, dx + T_1(f)$$

for a form $T_1(\cdot) \in \mathscr{L}_m$. Using induction on $|\gamma|$ one sees that $P_{\gamma,\lambda\phi}(x)$ is a polynomial of degree $|\gamma|$ in λ whose highest-degree term is $\lambda^{|\gamma|}(\nabla\phi)^\gamma$; the result then follows. ∎

We next introduce some auxiliary notation. Let $k_m = [\sin(\pi/(4m-2))]^{-2m+1}$ and define

$$a(x, \xi, \eta) = \sum_{\substack{|\alpha|=m\\|\beta|=m}} a_{\alpha\beta}(x)\, \xi^\alpha \bar{\eta}^\beta, \qquad x \in \Omega, \qquad \xi, \eta \in \mathbf{C}^N$$

and

$$S(x, \zeta; \xi, \eta) = \operatorname{Re} a(x, \xi - i\zeta, \eta + i\zeta) + k_m a(x, \zeta)$$

for $x \in \Omega$, $\zeta \in \mathbf{R}^N$ and $\xi, \eta \in \mathbf{C}^N$.

PROPOSITION 4. *We have*

$$\operatorname{Re} Q_{1,\lambda\phi}(f) + k_m \int a(x, \lambda\nabla\phi(x))\, |f|^2\, dx \tag{19}$$

$$= (2\pi)^{-N} \iiint_{\Omega\times\mathbf{R}^N\times\mathbf{R}^N} S(x, \lambda\nabla\phi; \xi, \eta)\, e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\, \overline{\hat{f}(\eta)}\, dx\, d\xi\, d\eta \tag{20}$$

for all $\phi \in \mathscr{E}_m$, $\lambda > 0$ and $f \in C_c^\infty(\Omega)$.

Proof. Using the relation $D^\gamma f(x) = (2\pi)^{-N/2} \int_{\mathbf{R}^N} (i\xi)^\gamma e^{i\xi \cdot x} \hat{f}(\xi)\, d\xi$ we get

$$Q_{1,\lambda\phi}(f) = (2\pi)^{-N} \iiint_{\Omega \times \mathbf{R}^N \times \mathbf{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} \sum_{\substack{\gamma \leqslant \alpha \\ \delta \leqslant \beta}} c_\gamma^\alpha c_\delta^\beta$$

$$\times (-i\lambda\nabla\phi)^\gamma (-i\lambda\nabla\phi)^\delta \xi^{\alpha-\gamma} \eta^{\beta-\delta} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx$$

$$= (2\pi)^{-N} \iiint_{\Omega \times \mathbf{R}^N \times \mathbf{R}^N} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha\beta} (\xi - i\lambda\nabla\phi)^\alpha (\eta - i\lambda\nabla\phi)^\beta$$

$$\times e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx$$

$$= (2\pi)^{-N} \iiint_{\Omega \times \mathbf{R}^N \times \mathbf{R}^N} a(x, \xi - i\lambda\nabla\phi(x), \eta + i\lambda\nabla\phi(x))$$

$$\times e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx.$$

Hence, using Lemma 3 of [5] to pass the Re operator through the triple integral (for this we make use of the assumption that the highest-order coefficients $a_{\alpha\beta}$ are real-valued) we obtain

$$\operatorname{Re} Q_{1,\lambda\phi}(f) + k_m \int_\Omega a(x, \lambda\nabla\phi(x))\, |f|^2\, dx$$

$$= (2\pi)^{-N} \iiint_{\Omega \times \mathbf{R}^N \times \mathbf{R}^N} [\operatorname{Re} a(x, \xi - i\lambda\nabla\phi(x), \eta + i\lambda\nabla\phi(x))$$

$$+ k_m a(x, \lambda\nabla\phi)]\, e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx$$

$$= (2\pi)^{-N} \iiint_{\Omega \times \mathbf{R}^N \times \mathbf{R}^N} S(x, \lambda\nabla\phi; \xi, \eta)\, e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx$$

as required. ∎

We now make use of a result which has been used toward proving asymptotic estimates for the fundamental solution of constant coefficient parabolic equations on $\mathbf{R}^N$. Let $\theta_m = \pi/(4m-2)$. It is shown in [12, Theorem 2.1] that there exist positive real numbers $w_0, w_1, ..., w_{m-2}$ such that for all $x \in \Omega$ and $\xi \in \mathbf{R}^N$ we have

$$S(x, \zeta; \xi, \xi) = \sum_{s=0}^{m-2} w_s \Gamma(x, p_{\xi,\zeta}^{(s)}), \tag{21}$$

where Γ is the quadratic form associated to the principal symbol of H and $p^{(s)}_{\xi,\zeta}$ is the vector in $\mathbf{R}^{\nu}$ defined for all $\xi, \zeta \in \mathbf{R}^N$ by requiring that

$$\sum_{|\alpha|=m} p^{(s)}_{\xi,\zeta,\alpha}\, u^{\alpha} = (\sin\theta_m)^{-s-2}\,(\xi\cdot u)^{m-s-2}\,(\zeta\cdot u)^{s}$$

$$\times\{(\sin\theta_m)^2\,(\xi\cdot u)^2 - (\cos\theta_m)^2\,(\zeta\cdot u)^2\}$$

for all $u \in \mathbf{R}^N$. To simplify the notation we define the form $\tilde{\Gamma}$ on $\mathbf{C}^{m-1} \otimes \mathbf{C}^{\nu} \simeq \mathbf{C}^{\nu(m-1)}$ by

$$\tilde{\Gamma}(x, u, v) = \sum_{s=0}^{m-2} w_s \Gamma(x, u^{(s)}, v^{(s)}) = \sum_{s=0}^{m-2} \sum_{\substack{|\alpha|=m\\|\beta|=m}} w_s a_{\alpha+\beta} \Gamma(x, u^{(s)}_{\alpha}, v^{(s)}_{\beta})$$

for all $u = (u^{(s)}_{\alpha}) \in \mathbf{C}^{\nu(m-1)}$. From the strong convexity of the symbol we conclude that $\tilde{\Gamma}(x, \cdot)$ is positive semi-definite for all $x \in \Omega$.

The above considerations lead us to consider two auxiliary elliptic differential operators $S_{\lambda\phi}$ and $\Gamma_{\lambda\phi}$ on $L^2(\Omega)$. For c large enough so that the square-root can be taken we define them by

$$\mathrm{Dom}(S_{\lambda\phi}+c)^{1/2} = \mathrm{Dom}(\Gamma_{\lambda\phi})^{1/2} = W^{m,2}_0(\Omega), \tag{22}$$

and

$$\langle S_{\lambda\phi} f, f\rangle = (2\pi)^{-N} \iiint S(x, \lambda\nabla\phi; \xi, \eta)\, e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\,\overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx, \tag{23}$$

$$\langle \Gamma_{\lambda\phi} f, f\rangle = (2\pi)^{-N} \iiint \tilde{\Gamma}(x, p_{\xi,\lambda\nabla\phi}, p_{\eta,\lambda\nabla\phi})\, e^{i(\xi-\eta)\cdot x} \hat{f}(\xi)\,\overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx, \tag{24}$$

where $p_{\xi,\lambda\nabla\phi} = (p^{(s)}_{\xi,\lambda\nabla\phi})^{m-2}_{s=0} \in \mathbf{C}^{\nu(m-1)}$ is as in (21). Note that the operator $\Gamma_{\lambda\phi}$ is positive semi-definite since letting $u(x) = \int_{\mathbf{R}^N} p_{\xi,\lambda\nabla\phi} e^{i\xi\cdot x} \hat{f}(\xi)\, d\xi$ (the vector-valued integral being defined in the standard way) we have

$$\langle \Gamma_{\lambda\phi} f, f\rangle = \int_{\Omega} \tilde{\Gamma}(x, u(x), u(x))\, dx \geqslant 0 \tag{25}$$

for all $f \in C^{\infty}_c(\Omega)$.

Now, although (21) tells us that $S(x, \zeta; \xi, \xi) = \sum_{s=0}^{m-2} w_s \Gamma(x, p^{(s)}_{\xi,\zeta}, p^{(s)}_{\xi,\zeta})$, simple examples such us the operator $f \mapsto f^{(4)}$ on $\mathbf{R}$ show that

$$S(x, \zeta; \xi, \eta) \neq \sum_{s=0}^{m-2} w_s \Gamma(x, p^{(s)}_{\xi,\zeta}, p^{(s)}_{\eta,\zeta})$$

in general when $\xi \neq \eta$. This prevents us from deducing that $S_{\lambda\phi}$ is positive semi-definite and brings in the distance d defined in (11).

LEMMA 5. *There exists a constant c such that*

$$\langle S_{\lambda\phi} f, f \rangle \geqslant -cd\{Q(f) + \lambda^{2m} \|f\|^2\} + T_2(f), \qquad \textit{all} \quad f \in C_c^\infty(\Omega), \qquad (26)$$

where T_2 *is a form in* $\mathscr{L}_m$.

Proof. We shall need the following

Claim. For any three multi-indices γ, δ and κ and for any bounded function $b(x)$ that satisfies $\|D^{\kappa_1} b\|_\infty < \infty$, $\kappa_1 \leqslant \kappa$, we have

$$\iiint_{\Omega \times \mathbf{R}^N \times \mathbf{R}^N} b(x)\, \xi^\gamma \eta^{\delta+\kappa} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx$$

$$= \iiint_{\Omega \times \mathbf{R}^N \times \mathbf{R}^N} b(x)\, \xi^{\gamma+\kappa} \eta^{\delta} e^{i(\xi-\eta)\cdot x} \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta\, dx + R(f)$$

for all $f \in C_c^\infty$, where $R(\cdot)$ is a quadratic form satisfying

$$|R(f)| \leqslant c\{\sup_{0<\kappa_1\leqslant\kappa} \|D^{\kappa_1} b\|_\infty\}(\|f\|_2^2 + \|\nabla^{|\gamma+\delta+\kappa|-1} f\|_2^2). \qquad (27)$$

Proof of claim. We use the relation $D^\gamma f(x) = (2\pi)^{-N/2} \int_{\mathbf{R}^N} (i\xi)^\gamma e^{i\xi\cdot x} \hat{f}(\xi)\, d\xi$ and apply integration by parts $|\kappa|$ times.

Now let $\hat{A} = \{\hat{a}_{\alpha\beta}(x)\}_{|\alpha|=|\beta|=m} \in W^{m-1,\infty}$ be such that $\|A_{pr} - \hat{A}\|_\infty \leqslant 2d$. Using a hat to indicate the quantities associated with the operator $\hat{H}$ which has coefficient matrix $\hat{A}(x)$ we have

$$\langle S_{\lambda\phi} f, f \rangle = \{\langle S_{\lambda\phi} f, f \rangle - \langle \hat{S}_{\lambda\phi} f, f \rangle\} + \{\langle \hat{S}_{\lambda\phi} f, f \rangle - \langle \hat{\Gamma}_{\lambda\phi} f, f \rangle\} + \langle \hat{\Gamma}_{\lambda\phi} f, f \rangle.$$

The first of the three terms can be estimated by $cd\{Q(f) + \lambda^{2m} \|f\|_2^2\}$ by (18). For the second term we observe that $\hat{S}_{\lambda\phi}$ and $\hat{\Gamma}_{\lambda\phi}$ have kernels which are polynomials of degree m in ξ and η and whose values coincide for $\xi = \eta$ by (21). Moreover, terms where both ξ and η appear in degree m clearly coincide even if $\xi \neq \eta$. Since the coefficients of $\hat{H}$ lie in $W^{m-1,\infty}(\Omega)$, we can therefore apply the claim and thus deduce that the second term lies in $\mathscr{L}_m$. Finally, the third term is non-negative by (25). Inequality (26) then follows. ∎

PROPOSITION 6. *For any $\phi \in \mathscr{E}_{a,M}$ and $\lambda > 0$ we have*

$$\operatorname{Re} Q_{\lambda\phi}(f) \geqslant -(k_m + cd)\,\lambda^{2m}\,\|f\|_2^2 + T(f) \tag{28}$$

for a form $T \in \mathscr{L}_m$ and all $f \in C_c^\infty(\Omega)$.

Proof. Combining Lemma 3, Proposition 4, and (23) yields

$$\operatorname{Re} Q_{\lambda\phi}(f) + k_m \lambda^{2m} \int_\Omega a(x, \nabla\phi(x))\,|f|^2\,dx = \langle S_{\lambda\phi} f, f\rangle + T_3(f)$$

for some lower-order form $T_3 \in \mathscr{L}_m$ and all $f \in C_c^\infty(\Omega)$. The fact that $\phi \in \mathscr{E}_{a,M}$ further implies that $a(x, \nabla\phi(x)) \leqslant 1$ for all $x \in \Omega$. Hence, using also Lemma 5, we have for $\phi \in \mathscr{E}_{a,M}$

$$\operatorname{Re} Q_{\lambda\phi}(f) + k_m \lambda^{2m}\,\|f\|_2^2 \geqslant -cd[Q(f) + \lambda^{2m}\,\|f\|_2^2] + T_4(f). \tag{29}$$

Now, from (15) we have

$$|Q(f) - Q_{\lambda\phi}(f)| < \varepsilon Q(f) + c_\varepsilon \{\|\lambda\phi\|_{W^{m,\infty}} + \|\lambda\phi\|_{W^{m,\infty}}^{2m}\}\,\|f\|_2^2, \tag{30}$$

But we have already seen in the proof of Lemma 3 that the terms of $Q_{\lambda\phi}(f)$ that are of degree $2m$ in λ only involve the first derivatives of ϕ. Hence estimates on them do not involve M, and (30) can also be written as

$$|Q(f) - Q_{\lambda\phi}(f)| < \varepsilon Q(f) + c_\varepsilon \{c_M(\lambda + \lambda^{2m-1}) + c\lambda^{2m}\}\,\|f\|_2^2. \tag{31}$$

Taking $\varepsilon = 1/2$ implies

$$Q(f) < 2 \operatorname{Re} Q_{\lambda\phi}(f) + \{c_M(\lambda + \lambda^{2m-1}) + c\lambda^{2m}\}\,\|f\|_2^2, \tag{32}$$

and therefore (29) yields

$$(1 + cd) \operatorname{Re} Q_{\lambda\phi}(f) \geqslant -(k_m + cd)\,\lambda^{2m}\,\|f\|_2^2 + T(f), \tag{33}$$

where T is a form in $\mathscr{L}_m$. Finally, we can discard the term $cd \operatorname{Re} Q_{\lambda\phi}(f)$, since (28) is certainly true when $\operatorname{Re} Q_{\lambda\phi}(f)$ is positive. The result then follows. ∎

We need a last lemma before proving Theorem 1.

LEMMA 7. *Any form $T \in \mathscr{L}_m$ satisfies*

$$|T(f)| < \varepsilon Q(f) + c\varepsilon^{-2m+1}(1+\lambda^{2m-1}) \|f\|_2^2$$

for all $\varepsilon \in (0, 1)$, $\lambda > 0$, and $f \in C_c^\infty(\Omega)$.

Proof. By definition, $T(f)$ is a finite linear combination of expressions of the form

$$I = \lambda^s \int_\Omega b(x)\, D^\gamma f(x)\, D^\delta \bar{f}(x)\, dx,$$

where $s + |\gamma + \delta| < 2m$. Letting $\mu = \lambda^{s/(2m - |\gamma+\delta|)}$ we have

$$|I| < c\, \|\mu^{m-|\gamma|}\, D^\gamma f\|_2\, \|\mu^{m-|\delta|}\, D^\delta f\|_2.$$

Let $\theta = |\gamma + \delta|$ and assume first that neither γ nor δ has length equal to m. Then, using also Gårdings inequality (6),

$$\begin{aligned}
\|\mu^{m-|\gamma|} D^\gamma f\|_2^2 &= \int_{\mathbf{R}^N} \mu^{2m-2|\gamma|} \xi^{2\gamma}\, |\hat{f}(\xi)|^2\, d\xi \\
&\leqslant \int_{\mathbf{R}^N} \{\varepsilon^{(m-|\gamma|)/(2m-\theta)}\, |\xi|^{2m} + c\varepsilon^{-|\gamma|/(2m-\theta)} \mu^{2m}\}\, |\hat{f}(\xi)|^2\, d\xi \\
&\leqslant \varepsilon^{(m-|\gamma|)/(2m-\theta)} Q(f) + c\varepsilon^{-|\gamma|/(2m-\theta)} \mu^{2m}\, \|f\|_2^2.
\end{aligned}$$

We write a similar inequality for $\|\mu^{m-|\delta|}\, D^\delta f\|_2^2$ and after estimating the cross terms by the diagonal ones we obtain

$$\begin{aligned}
|I| &< c\varepsilon Q(f) + c_1 \varepsilon^{-\theta/(2m-\theta)} \mu^{2m}\, \|f\|_2^2 \\
&\leqslant c\varepsilon Q(f) + c_1 \varepsilon^{-2m+1} (1 + \lambda^{2m-1})\, \|f\|_2^2
\end{aligned}$$

for all $0 < \varepsilon < 1$ and $\lambda > 0$ as required. If either $|\gamma| = m$ or $|\delta| = m$ then a slighty modified argument works. We omit the details. ∎

Proof of Theorem 1. Let $\phi \in \mathscr{E}_{a,M}$ be given. Combining Proposition 6 with Lemma 7 and recalling (32) we obtain

$$\begin{aligned}
\operatorname{Re} Q_{\lambda\phi}(f) &\geqslant -(k_m + cd)\, \lambda^{2m}\, \|f\|_2^2 - \varepsilon Q(f) - c_{\varepsilon, M}(1 + \lambda^{2m-1})\, \|f\|_2^2 \\
&\geqslant -(k_m + cd + \varepsilon)\, \lambda^{2m}\, \|f\|_2^2 - c\varepsilon \operatorname{Re} Q_{\lambda\phi}(f) \\
&\quad - c_{\varepsilon, M}(\lambda^{2m-1} + 1)\, \|f\|_2^2,
\end{aligned}$$

for all $\lambda > 0$, $\varepsilon \in (0, 1)$, and $f \in C_c^\infty(\Omega)$. Discarding $c\varepsilon \operatorname{Re} Q_{\lambda\phi}(f)$ as in the proof of Proposition 6 yields

$$\operatorname{Re} Q_{\lambda\phi}(f) \geqslant -\{\lambda^{2m}(k_m + cd + \varepsilon) + c_{\varepsilon, M}(\lambda^{2m-1} + 1)\} \, \|f\|_2^2.$$

We now apply Proposition 2 obtaining

$$|K(t, x, y)| < c_\varepsilon t^{-N/2m} \exp\{\lambda[\phi(y) - \phi(x)] + (1+\varepsilon) \times \{(k_m + cd + \varepsilon)\, \lambda^{2m} + c_{\varepsilon, M}(\lambda^{2m-1} + 1)\}\; t\}.$$

for all $\varepsilon \in (0, 1)$. Optimizing over $\phi \in \mathscr{E}_{a, M}$ yields

$$|K(t, x, y)| < c_\varepsilon t^{-N/2m} \exp\{-\lambda d_M(x, y) + (1+\varepsilon) \times \{(k_m + cd + \varepsilon)\, \lambda^{2m} + c_{\varepsilon, M}(\lambda^{2m-1} + 1)\}\; t\}.$$

Finally choosing

$$\lambda = [d_M(x, y)/(2m[k_m + cd]\, t)]^{1/(2m-1)}$$

we have

$$\begin{aligned} &-\lambda d_M(x, y) + (k_m + cd)\, \lambda^{2m} t \\ &\quad = -\{(2m-1)(2m)^{-2m/(2m-1)}\, (k_m + cd)^{-1/(2m-1)}\} \, \frac{d_M(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} \\ &\quad = -[\sigma_m - O(d)]\, \frac{d_M(x, y)^{2m/(2m-1)}}{t^{1/(2m-1)}} \end{aligned}$$

while the term $c_{\varepsilon, M} d_M(x, y)$ that also appears in the exponential can be estimated by $\varepsilon[d_M(x, y)]^{2m/(2m-1)}\, t^{-1/(2m-1)} + c_{\varepsilon, M} t$. This concludes the proof. ∎

Examples.

1. Let $\Omega = \mathbf{R}^N$ and assume that the highest order coefficients of H are uniformly continuous. Then $\operatorname{dist}(A_{pr}, W^{m-1, \infty}) = 0$, as can be seen by convoluting with a compactly supported approximate identity.

2. Suppose that $a_{\alpha\beta} \in C(\bar{\Omega})$ for all principal coefficients. By Tychonof's theorem each $a_{\alpha\beta}$ can be extended to a function $\hat{a}_{\alpha\beta} \in C_c(\mathbf{R}^N)$. That function is uniformly continuous and therefore, as in Example 1, it can be uniformly approximated by functions in $W^{m-1, \infty}(\Omega)$. Hence $\operatorname{dist}(A_{pr}, W^{m-1, \infty}) = 0$ in this case.

Hence the actual constant σ_m is obtained for a class of functions that is substantially larger than $W^{m,\infty}(\Omega)$ which, for example, is contained in C^{m-1}.

A counterexample

As mentioned above, an interesting feature of Theorem 1 is the appearence of the term $O(d)$ in the exponential term, and the question is immediately posed whether that term can be removed. We shall construct an example that shows that with the method employed in this paper it is impossible to eliminate that term. Namely, we shall see that inequality (28) is not valid for $d=0$. This suggests strongly that the term actually appears in $K(t, x, y)$ but that remains to be proved. If that turns out to be true it would provide a contrast to the second-order case where estimates do not depend on the local regularity of the coefficients.

PROPOSITION 8. *There exists a fourth order operator H, a function $\phi \in \tilde{\mathscr{E}}_a$ and a positive number c such that given $\lambda > 0$ we can find $h \in W^{2,2}(\mathbf{R})$ with $\|h\|_2 = 1$ and such that*

$$\operatorname{Re} Q_{\lambda\phi}(h) < -(8+c)\,\lambda^4 + T(h), \tag{34}$$

for all $\lambda > 0$ and for a form $T \in \mathscr{L}_2$.

Note. The fact that $\phi \in \tilde{\mathscr{E}}_a$ instead of $\phi \in \mathscr{E}_a$ is not a problem, because of the remark at the end of Section 2. Alternatively, one can modify ϕ outside a large interval so that it lies in $\mathscr{E}_a$, with (34) still being valid.

Proof. Let $\alpha \in (0, 1)$ and let H be the operator on $L^2(-\infty, \infty)$ with quadratic form

$$Q(f) = \int_{-\infty}^{\infty} a(x)\, |f''|^2\, dx,$$

where

$$a(x) = \begin{cases} \alpha, & x < 0, \\ 1, & x > 0. \end{cases}$$

Given $\phi \in \mathscr{E}_2$ and $\lambda > 0$ and setting $\psi = \phi'$ a direct calculation gives

$$\operatorname{Re} Q_{\lambda\phi}(f) = \int_{-\infty}^{\infty} a\, |f'' + 3\lambda^2\psi^2 f|^2\, dx$$
$$- 4\lambda^2 \operatorname{Re} \int_{-\infty}^{\infty} a\psi^2 (f'\bar{f})'\, dx - 8\lambda^4 \int_{-\infty}^{\infty} a\psi^4\, |f|^2\, dx + T_1(f),$$

where $T_1(\cdot) \in \mathcal{L}_2$. Applying integration by parts to the second integral we obtain two terms: the one is of lower essential order while the other involves the first derivative of $a(x)$, thus producing a δ-type expression at the point $x=0$. We obtain

$$\operatorname{Re} Q_{\lambda\phi}(f) = \int_{-\infty}^{\infty} a\,|f'' + 3\lambda^2\psi^2 f|^2\,dx + 4(1-\alpha)\,\lambda^2\psi^2(0)\operatorname{Re}[f'(0)\overline{f(0)}]$$

$$-8\lambda^4 \int_{-\infty}^{\infty} a\psi^4\,|f|^2\,dx + T_2(f),$$

where $T_2 \in \mathcal{L}_2$. Letting $f_\lambda = \lambda^{-1/2} f(\lambda^{-1}x)$ and carrying out a change of variables yields

$$\frac{1}{\lambda^4}\operatorname{Re} Q_{\lambda\phi}(f) = \int_{-\infty}^{\infty} a\,|f''_\lambda + 3\psi^2 f_\lambda|^2\,dx + 4(1-\alpha)\,\psi^2(0)\operatorname{Re}[f'_\lambda(0)\overline{f_\lambda(0)}]$$

$$-8\int_{-\infty}^{\infty} a\psi^4\,|f_\lambda|^2\,dx + \frac{1}{\lambda^4}T_2(f), \qquad \text{all} \quad f \in W^{2,2}(\mathbf{R}). \tag{35}$$

Now define

$$K_\alpha(f) = \int_{-\infty}^{\infty} |a^{1/2} f'' + 3f|^2\,dx + 4(1-\alpha)\operatorname{Re}[f'(0)\overline{f(0)}] - 8\,\|f\|_2^2.$$

Note that (i) K_α is independent of λ and (ii) $K_\alpha(f_\lambda)$ is the RHS of (35) when ψ is formally replaced by the function

$$\psi_0(x) = \begin{cases} \alpha^{-1/4}, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Claim. In order to prove the proposition it is enough to find a function $g \in W^{2,2}(\mathbf{R})$ and a constant $c > 0$ such that that

$$K_\alpha(g) < -(8+c)\,\|g\|_2^2.$$

Proof of claim. Let $\varepsilon > 0$ be a small parameter and define ϕ_ε via $\psi_\varepsilon(x) = \phi'_\varepsilon(x)$ and

$$\psi_\varepsilon(x) = \begin{cases} \alpha^{-1/4}, & x \leq -\varepsilon \\ v_\varepsilon(x), & -\varepsilon \leq x \leq 0 \\ 1, & x \geq 0, \end{cases}$$

where the function $v_\varepsilon(x)$ takes values in the interval $[1, \alpha^{-1/4}]$ and is chosen so that we have C^1-matching at the points $x = -\varepsilon$ and $x = 0$. Hence ϕ_ε has bounded and continuous first and second derivatives, it satisfies $a(\phi'_\varepsilon)^4 \leqslant 1$ pointwise and therefore satisfies all the conditions necessary to lie in $\mathscr{E}_{a,M}$ for some $M = M(\varepsilon)$ except boundedness. Hence $\varphi_\varepsilon \in \tilde{\mathscr{E}}_a$.

Now, it follows from (35) that we have

$$\lim_{\varepsilon \to 0} \left\{ \frac{1}{\lambda^4} \operatorname{Re} Q_{\lambda\phi_\varepsilon}(f) - \frac{1}{\lambda^4} T_2(f) \right\} = K_\alpha(f_\lambda), \qquad f \in W^{2,2}(\mathbf{R}).$$

Hence, if g is a function such that

$$K_\alpha(g) < -(8+c) \, \|g\|_2^2, \qquad (c > 0)$$

then for $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} \frac{1}{\lambda^4} \operatorname{Re} Q_{\lambda\phi_\varepsilon}(g_{\lambda^{-1}}) &< -\left(8 + \frac{c}{2}\right) \|g\|_2^2 + \frac{1}{\lambda^4} T_2(g_{\lambda^{-1}}) \\ &= -\left(8 + \frac{c}{2}\right) \|g_{\lambda^{-1}}\|_2^2 + \frac{1}{\lambda^4} T_2(g_{\lambda^{-1}}). \end{aligned}$$

Hence (34) is proved.

Hence in order to prove the proposition it is enough to show that there exists $g \in W^{2,2}(\mathbf{R})$ and $c > 0$ such that

$$K_\alpha(g) < -(8+c) \, \|g\|_2^2$$

or, equivalently, that there exists g such that

$$\begin{aligned} \hat{K}_\alpha(g) := \int_{-\infty}^0 |\alpha^{1/2} g'' + 3g|^2 \, dx + \int_0^\infty |g'' + 3g|^2 \, dx \\ + 4(1-\alpha) \operatorname{Re}[g'(0) \, \overline{g(0)}] < 0. \end{aligned} \tag{36}$$

Let

$$\begin{aligned} \hat{K}_0(f) &= \lim_{\alpha \to 0} \hat{K}_\alpha(f) \\ &= 9 \int_{-\infty}^0 |f|^2 \, dx + \int_0^\infty |f'' + 3f|^2 \, dx + 4 \operatorname{Re}[f'(0) \, \overline{f(0)}]. \end{aligned}$$

We shall construct a function $g \in W^{2,2}(\mathbf{R})$ such that $\hat{K}_0(g) < 0$. Continuity then will imply that $\hat{K}_\alpha(g) < 0$ for small enough α, proving (36) for such α and thus completing the proof of the proposition. ▮

Let ζ be a non-negative smooth function with support in $(-1, 1)$ and such that $\int \zeta(\xi)\, d\xi = 1$. For $\delta > 0$ define $\zeta_\delta(\xi) = \delta^{-1}\zeta(x/\delta)$ and

$$\tilde{g}_\delta(x) = \int_{-\infty}^{\infty} \cos(x\xi)\, \zeta_\delta(\xi - \sqrt{3})\, d\xi.$$

Then

$$\begin{aligned} |\tilde{g}_\delta''(x) + 3\tilde{g}_\delta(x)| &< \left| \int_{-\infty}^{\infty} (-\xi^2 + 3)\, e^{i\xi x} \zeta_\delta(\xi - \sqrt{3})\, d\xi \right| \\ &= (2\pi)^{N/2}\, |\mathscr{F}^{-1} w_\delta|, \end{aligned}$$

where $\mathscr{F}$ denotes the Fourier transform and $w_\delta(\xi) = (-\xi^2 + 3)\, \zeta_\delta(\xi - \sqrt{3})$. Hence $\| \tilde{g}_\delta'' + 3\tilde{g}_\delta \|_2 \leqslant (2\pi)^{N/2}\, \|w_\delta\|_2$. But it is not difficult to see that $\|w_\delta\|_2^2 = O(\delta)$ and we thus deduce that

$$\| \tilde{g}_\delta'' + 3\tilde{g}_\delta \|_2^2 = O(\delta) \qquad \text{as} \quad \delta \to 0.$$

Moreover an application of the dominated convergence theorem shows that for any $x \in \mathbf{R}$

$$\begin{aligned} \tilde{g}_\delta(x) &= \cos(\sqrt{3}x) + O(\delta), \\ \tilde{g}_\delta'(x) &= -\sqrt{3}\sin(\sqrt{3}x) + O(\delta), \qquad \text{as} \quad \delta \to 0. \end{aligned}$$

We define

$$g_\delta(x) = \tilde{g}_\delta \left(\frac{x + \pi/4}{\sqrt{3}} \right)$$

and conclude that

$$g_\delta(0) = \frac{\sqrt{2}}{2} + O(\delta),$$

$$g_\delta'(0) = -\frac{\sqrt{6}}{2} + O(\delta),$$

$$\int_{-\infty}^{\infty} |g_\delta'' + 3g_\delta|^2\, dx = O(\delta).$$

Furthermore, it is clear than one can construct functions $(u_\delta) \subset W^{2,2}(-\infty, 0)$ such that

$$\begin{aligned} u_\delta(0) &= g_\delta(0), \\ u'_\delta(0) &= g'_\delta(0), \\ \int_{-\infty}^{0} |u_\delta|^2\, dx &= O(\delta). \end{aligned} \tag{37}$$

Defining the functions

$$h_\delta(x) = \begin{cases} u_\delta(x), & x \leqslant 0 \\ g_\delta(x), & x \geqslant 0, \end{cases} \tag{38}$$

it follows from (37) and (38) that $h_\delta \in W^{2,2}(\mathbf{R})$. Furthermore (37) shows that

$$\hat{K}_0(h_\delta) < 0$$

for δ sufficiently small, thus completing the proof of the proposition. ∎

One is tempted to think that this might be an example of an operator where the term *cd* cannot be removed from the heat kernel estimate (11). It turns out that this is not the case. For simplicity of later calculations we switch from the parameter α to $\beta = \alpha^{-1/4}$ and for $\lambda > 0$ we let $g(x, y; \lambda)$ denote the Greens's function of the equation

$$(af'')'' + 4\lambda^4 f = h$$

where, we recall,

$$a(x) = \begin{cases} \beta^{-4}, & x < 0, \\ 1, & x > 0. \end{cases}$$

We also set

$$B(\beta) = \frac{4}{(1 + \beta + \beta^2 + \beta^3)}.$$

Elementary but lengthy calculations allow one to compute $g(x, y; \lambda)$ explicitly and one sees that for $y < x$ the following hold:

if $0<y<x$ then

$$g(x, y; \lambda)=\frac{1}{8\lambda^3} e^{\lambda(y-x)}[\cos(\lambda(x-y))+\sin(\lambda(x-y))]$$

$$+\frac{B(\beta)}{32\lambda^3} e^{-\lambda(x+y)}\{(3\beta^3-5\beta^2+3\beta-1)\cos(\lambda x)\cos(\lambda y)$$

$$-(\beta^3+\beta^2-3\beta+1)\cos(\lambda x)\sin(\lambda y)$$

$$-(\beta^3+\beta^2-3\beta+1)\sin(\lambda x)\cos(\lambda y)$$

$$+(\beta^3+\beta^2+\beta-3)\sin(\lambda x)\sin(\lambda y)\};$$

if $y<0<x$ then

$$g(x, y; \lambda)=\frac{B(\beta)}{8\lambda^3} e^{\lambda(\beta y-x)}\{(\beta^3-\beta^2+\beta)\cos(\lambda x)\cos(\lambda\beta y)$$

$$-\beta^3\cos(\lambda x)\sin(\lambda\beta y)+\beta\sin(\lambda x)\cos(\lambda\beta y)$$

$$+\beta^2\sin(\lambda x)\sin(\lambda\beta y)\}; \tag{39}$$

and if $y<x<0$ then

$$g(x, y; \lambda)=\frac{\beta}{8\lambda^3} e^{\lambda\beta(y-x)}[\cos(\beta\lambda(x-y))+\sin(\beta\lambda(x-y))]$$

$$+\frac{B(\beta)}{32\lambda^3} e^{\beta\lambda(x+y)}\{(-\beta^4+3\beta^3-5\beta^2+3\beta)\cos(\lambda\beta x)\cos(\lambda\beta y)$$

$$+(\beta^4-3\beta^3+\beta^2+\beta)\cos(\lambda\beta x)\sin(\lambda\beta y)$$

$$+(\beta^4-3\beta^3+\beta^2+\beta)\sin(\lambda\beta x)\cos(\lambda\beta y)$$

$$+(-3\beta^4+\beta^3+\beta^2+\beta)\sin(\lambda\beta x)\sin(\lambda\beta y)\}. \tag{40}$$

Of course, symmetry extends the definition to the case $y>x$. Taking in particular $\beta=1$ we obtain the Green's function for the equation $f^{(4)}+4\lambda^4 f=h$, which reads simply

$$g_0(x, y; \lambda)=\tfrac{1}{8}e^{-\lambda|x-y|}(\cos\lambda|x-y|+\sin\lambda|x-y|).$$

It is not difficult to deduce from the above formulae that the short time asymptotics of the heat kernel of H do not deviate from the standard form of Tintarev's asymptotics. We will show this for x and y on either side of the singularity at zero, the other cases being simpler. Let $g_0(x, y; \lambda)$ and $K_0(t, x, y)$ be the Green's function and heat kernel for the equation $f^{(4)}+4\lambda^4 f=h$ and the operator $f\mapsto f^{(4)}$ respectively.

Let

$$g_0^{(1)}(x, y; \lambda) = \frac{e^{\lambda(y-x)}}{8\lambda^3} \cos(\lambda x) \cos(\lambda y),$$

$$g_0^{(2)}(x, y; \lambda) = \frac{e^{\lambda(y-x)}}{8\lambda^3} \cos(\lambda x) \sin(\lambda y)$$

$$g_0^{(3)}(x, y; \lambda) = \frac{e^{\lambda(y-x)}}{8\lambda^3} \sin(\lambda x) \cos(\lambda y),$$

$$g_0^{(4)}(x, y; \lambda) = \frac{e^{\lambda(y-x)}}{8\lambda^3} \sin(\lambda x) \sin(\lambda y)$$

so that $g_0(x, y; \lambda) = \sum_{k=1}^4 g_0^{(k)}(x, y; \lambda)$.

It follows from (40) that for $y < 0 < x$ we have

$$g(x, y; \lambda) = B(\beta)\{(\beta^3 - \beta^2 + \beta)\, g_0^{(1)}(x, \beta y; \lambda) - \beta^3 g_0^{(2)}(x, \beta y; \lambda) + \beta g_0^{(3)}(x, \beta y; \lambda) + \beta^2 g_0^{(2)}(x, \beta y; \lambda)\}.$$

From this we deduce that the short time asymptotics for $K(t, x, y)$ involve an exponential expression which is the same as for $K_0(t, x, \beta y)$. Hence that expression is

$$\exp\left[-\sigma_2 \frac{(x-\beta y)^{4/3}}{t^{1/3}}\right],$$

the same as in Tintarev's asymptotics for smooth coefficients since for $y < 0 < x$, $x - \beta y$ is the Finsler distance (induced by H) between x and y.

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Note added in proof. Recently (*Bull. Austr. Math. Soc.* **61** (2000), 189–200) and within a more general framework N. Dungey obtained heat kernel estimates for operators that are mth powers of second order operators. His bounds involve the exact constant σ_m (without the $O(d)$) and the distance $d(x, y)$ instead of $d_M(x, y)$.

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