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THE FIRING SQUAD SYNCHRONIZATION PROBLEM FOR GRAPHS

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Abstract. In this paper, we give a solution of the Firing Squad Synchronization Problem for graphs. The synchronization times of solutions which have been obtained are proportional to the number of nodes of a graph. The synchronization time of our solution is proportional to the radius r_G of a graph G ($3r_G+1$ or $3r_G$ time units), where r_G is the longest distance between the general and any other node of G. This synchronization time is minimum for an infinite number of graphs.

1. Introduction

The problem of synchronizing a finite (but arbitrarily long) one-dimensional array of finite automata, known as the firing squad synchronization problem, was proposed by Myhill in 1957 and Moore [10]. This problem was solved by Goto [2], Waksman [15], and Balzer [1], and they obtained the minimum synchronization time $2n-2$ for an n-element array. The problem was generalized in many different ways by Moore and Longdon [11], Herman [3, 4], Rosenstiehl [13, 14], and Kobayashi [8, 9].

This paper deals with the firing squad synchronization problem for graphs, which was studied by Rosenstiehl [13, 14], Kobayashi [5, 6, 7], and Romani [12]. Given a graph with a specified node and an finite automaton, we consider a network in which a copy of the finite automaton is placed on every node of the graph and these finite automata on the nodes are connected along every edge of the graph. The state of each finite automaton at time $t+1$ depends on its own state and those of its neighbours at time t. The problem consists in defining the structure of the finite automaton so that the automaton on the specified node, called the general, can cause all finite automata to enter a particular state, called the firing state, exactly at the same time.

The synchronization times of solutions which have been obtained, are proportional to the number n of nodes of a graph ($4n-6$ in [13] and $2n$ in [5, 14]), except for Romani's improved results for some special class of graphs. In this paper we present a solution whose synchronization time is proportional to the radius r_G of a graph G (about $3r_G$), where r_G is the longest distance between the specified node (the general) and any other node of the graph G. Note that r_G is essentially different from and less than the number of nodes of the graph. The synchronization time of our solution is minimum for graphs whose generals are (informally speaking) at the center of the graphs. Our solution is based on the synchronization of a particular type of digraphs called 'quasi-circuit structures'.

2. Preliminaries

Throughout this paper, by d we denote some fixed positive integer. A *digraph structure of valence d*, or simply a *digraph structure*, is a 4-tuple $G = (X, U, x_g, d)$, where X is a finite set of *cells*, x_g a particular cell in X called the *general cell*, and U a finite set of *arcs* of the form (x, y, i) $(x, y \in X, 1 \leqslant i \leqslant d)$ satisfying the condition: for each pair of y and i there is at most one arc of the form (x, y, i). If there is an arc (x, y, i) in U, a cell x is called the [ith] *predecessor* of a cell y, [denoted by $y(i)$]. (See Fig. 1(b).) If for a cell y and an integer i $(1 \leqslant i \leqslant d)$, there is no arc (x, y, i) in U, we say that the ith predecessor $y(i)$ of y does not exist.

A sequence of cells $[x_0, \ldots, x_l]$ of a digraph structure G is a *path* of *length l* from a cell x to a cell y, where $x_0 = x$, $x_l = y$, $x_i \neq x_j$ for any i, j with $0 \leqslant i < j \leqslant l-1$, and x_{i-1} is a predecessor of x_i for each $i = 1, \ldots, l$. A path $[x_0, \ldots, x_l]$ is said to be *cyclic* if $x_0 = x_l$. The *distance* from a cell x to a cell y in G, denoted by $d_G(x, y)$, is the length of the shortest path from x to y. Especially, the distance from the general cell x_g to a cell x is denoted by $d_G(x)$. We denote $\max\{d_G(x) \mid x \text{ is a cell of } G\}$ by r_G, called the *radius* of G.

A digraph structure G is said to be *connected* if for any two cells x and y of G, there is at least one path from x to y. A digraph structure in which for any two cells x and y,

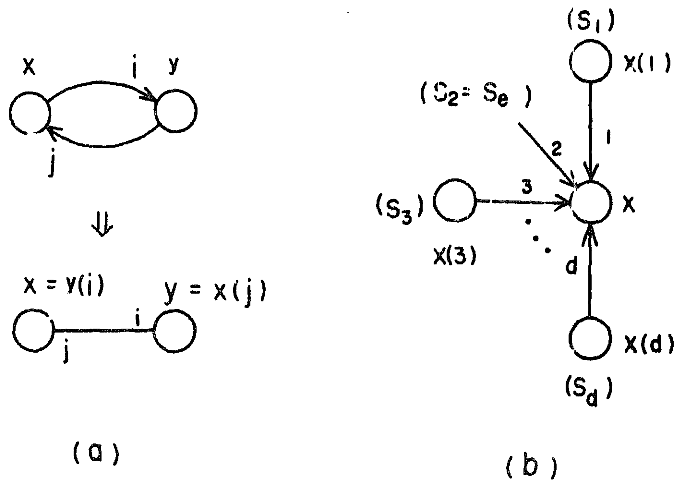


Fig. 1. A digraph structure and a graph structure.

if there is an arc (x, y, i), then there is an arc (y, x, j), is said to be *symmetric*. We call a connected symmetric digraph structure of valence d a *graph structure of valence d*, or simply a *graph structure*. We denote the class of all graph structures of valence d by $\mathscr{G}$. Usually, in a graph structure we call the [ith] predecessor of a cell x the [ith] *adjacent* cell of x, and in figures of graph structures we represent each two opposite arcs by one undirected arc as shown in Fig. 1(a).

An *automaton with d input terminals*, or simply an *automaton*, is a 6-tuple $M = (S, s_e, s_q, s_g, s_f, \lambda)$, where

(1) S is a finite set of *states*,

(2) s_e is an element not in S [the *external signal*],

(3) s_q, s_g, and s_f are particular distinct elements in S [the *quiescent state*, the *general state*, and the *firing state*, respectively], and

(4) λ is a *transition function* from $S \times (S \cup \{s_e\})^d$ into S such that $\lambda(s_q, s_1, \ldots d_d) = s_q$ if each of $s_1, \ldots, s_d$ is either s_q or s_e.

Let G be a digraph structure and let M be an automaton. We consider a network such that for each cell x of G, a copy of M is placed on x. In the network, the ith input terminal of the automaton on a cell x is connected with the output terminal of the automaton on the ith predecessor $x(i)$ of x if $x(i)$ exists and otherwise, open. Hereafter an automaton on a cell x will be also called a cell x. Formally the state of a cell x at time t, denoted by $s(x, t, G, M)$, is defined as follows. For $t = 0$, $s(x, 0, G, M)$ is s_g or s_q according as $x = x_g$ or not. For $t > 0$, $s(x, t, G, M)$ is $\lambda(s(x, t-1, G, M), s_1, \ldots, s_d)$, where s_i is $s(x(i), t-1, G, M)$ if the ith predecessor $x(i)$ of x exists and is s_e otherwise. (See Fig. 1(b).)

We say that a cell x of a digraph structure *G fires* at time t by an automaton M if $s(x, t', G, M) \neq s_f$ for any $t' < t$ and $s(x, t, G, M) = s_f$. We say that a digraph structure G *fires* at time t by an automaton M if all cells of G fires at time t by M. If there is such a time t, it is called the *synchronization time of M for G* and is denoted by $t(G, M)$.

Let $\mathscr{D}$ be a subclass of digraph structures. An automaton M is called a *solution of the firing squad synchronization problem for* $\mathscr{D}$, or simply a *solution for* $\mathscr{D}$, if each digraph structure in $\mathscr{D}$ fires by M.

3. A solution for $\mathscr{G}$

In this section, we give a solution for the class $\mathscr{G}$ of all graph structures, called the $(3r+1)$-*solution*. Its synchronization time for a graph structure G is $3r_G + 1$ time units. The principal idea is based on the synchronization of a particular type of digraph structures called *quasi-circuit structures* and the reduction of any graph structure to a quasi-circuit structure. In Section 3.1, we define quasi-circuit structures and observe that their synchronization is reduced to the synchronization of circuit structures. In Section 3.2, we give the reduction of a graph structure to a quasi-circuit structure. In Section 3.3, we consider the synchronization of circuit structures and finally give the $(3r+1)$-solution.

3.1. A quasi-circuit structure

A *quasi-circuit structure of length n* (and valence d) is a digraph structure $D_n = (X_D, U_D, x_g, d)$ such that each cell in X_D has at least one predecessor and X_D is partitioned into disjoint nonempty subsets $X_0, \ldots, X_{n-1}$ with $X_0 = \{x_g\}$ such that all predecessors of the cells in X_k are contained in X_{k-1} for $k = 0, \ldots, n-1$ where $X_{-1} = X_{n-1}$.

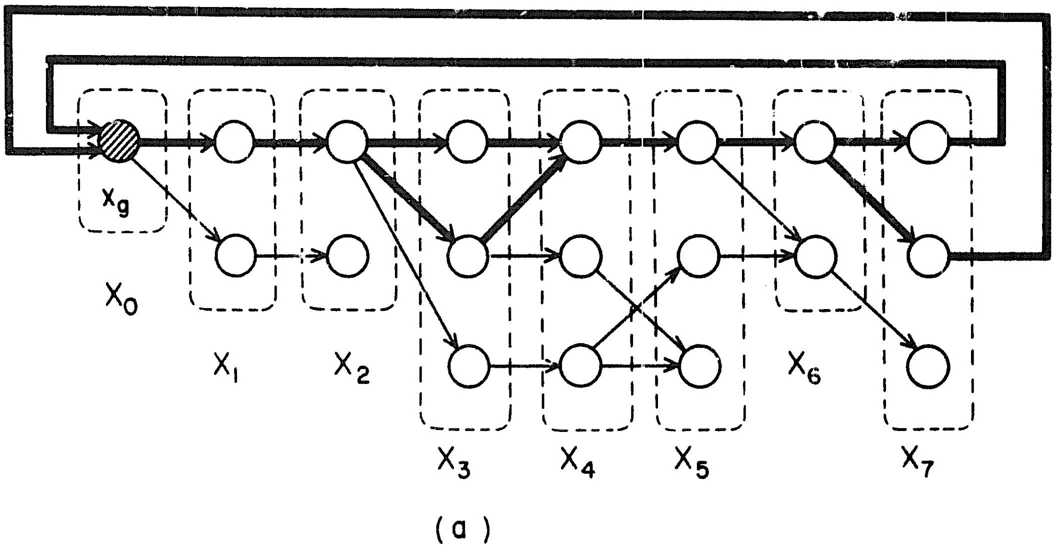
Example 1. In Fig. 2(a), we give a quasi-circuit structure of length 8. There are cyclic paths of length 8 (shown as bold arcs). Generally, in D_n there is at least one cyclic path, all cyclic paths have length n and pass through x_g, and for any cell x, the length of every path from x_g to x is equal to $d_{D_n}(x)$ ($<n$).

Next we define a *circuit structure of length n*, denoted by $C_n = (X_C, U_C, x_0, 1)$, as a digraph structure of valence 1 which consists of one cyclic path of length n. (See Fig. 2(b).)

Let $M = (S, s_e, s_q, s_g, s_f, \lambda)$ be an automaton with one input terminal. A *modified automaton* $M' = (S', s_e, s'_q, s'_g, s'_f, \lambda')$ *of* M is an automaton with d input terminal such that

(1) $S' = S$, $s'_q = s_q$, $s'_g = s_g$, and $s'_f = s_f$, and

(2) for any two states s and s' in S', if $(s_1, \ldots, s_d) \in (\{s'\} \cup \{s_e\})^d - \{s_e\}^d$, then $\lambda'(s, s_1, \ldots, s_d) = \lambda(s, s')$.



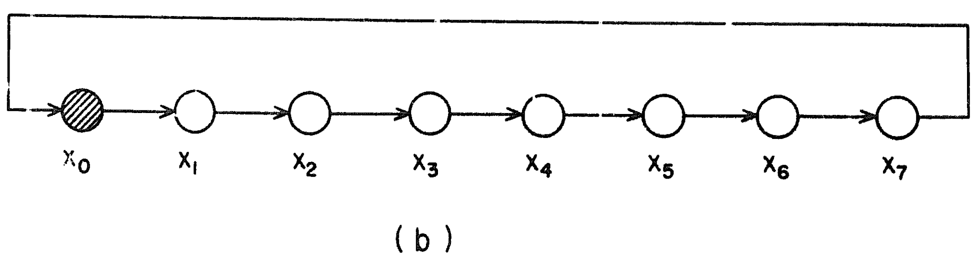


Fig. 2. (a) A quasi-circuit structure of length 8 and (b) a circuit structure of length 8.

Lemma 3.1. *Let M be an automaton with one imput terminal and M' a modified automaton of M. Let D_n and C_n be a quasi-circuit structure of length n and a circuit structure of length n respectively. Then for each cell x in X_k of D_n, $s(x, t, D_n, M') = s(x_k, t, C_n, M)$, where x_k is a cell of C_n with $d_{C_n}(x_k) = k$.*

Proof. It is proved by the induction on t. The lemma holds at time $t = 0$ since the only general cells x_g of D_n and x_0 of C_n are in the general state s_g and all other cells are in the quiescent state s_q at time 0. Suppose that the lemma holds at time t. Let x be a cell in X_k of D_n. Then $s(x, t+1, D_n, M') = \lambda'(s(x, t, D_n, M'), s_1, \ldots, s_d)$, where each of $s_1, \ldots, s_d$ is $s(x(i), t, D_n, M')$ or s_e. By the induction hypothesis, $s(x, t, D_n, M') = s(x_k, t, C_n, M)$ and for each $x(i)$ $s(x(i), t, D_n, M') = s(x_{k-1}, t, C_n, M)$ since $x(i)$ is contained in X_{k-1}. Since x has at least one predecessor, $(s_1, \ldots, s_d)$ is in $(\{s(x_{k-1}, t, C_n, M)\} \cup \{s_e\})^d - \{s_e\}^d$. Hence

$$s(x, t+1, D_n, M') = \lambda(s(x_k, t, C_n, M), s(x_{k-1}, t, C_n, M)) = s(x_k, t+1, C_n, M).$$

By Lemma 3.1, the synchronization problem of quasi-circuit structures is reduced to that of circuit structures. If we can construct an automaton M_R which reduces graph structures to quasi-circuit structures and an automaton M_c which synchronizes circuit structures, then we obtain a solution for the class $\mathscr{G}$ of graph structures which simulates M_R and then simulates the modified automaton of M_c.

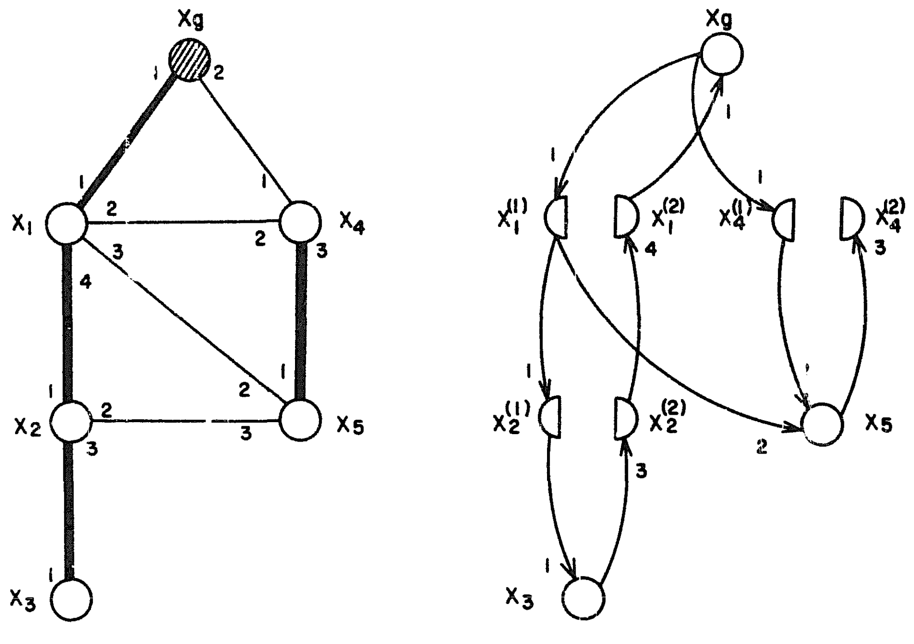
3.2. *Reduction*

In this section, we consider the reduction of a graph structure G to a quasi-circuit structure of length $2r_G$, where r_G is the radius of G. Before explaining the reduction, we give some definitions and notations.

In a graph structure G, a path $[x_0, \ldots, x_l]$ is called a *descending path* of a cell x if $x_0 = x$ and $d_G(x_{j-1}) < d_G(x_j)$ for $j = 1, \ldots, l$. A cell x is *terminal* if x has no descending path of length ≥ 1, that is x has no adjacent cell y with $d_G(y) > d_G(x)$. A descending path of x is *maximal* if it has the maximum length among all descending paths of x. We denote the length of a maximal descending path of x by $l_G(x)$. Note that for the general cell x_g, $l_G(x_g)$ is equal to the radius r_G and if a path $[x_0, \ldots, x_l]$ is a maximal descending path of x_0, then $[x_1, \ldots, x_l]$ is also a maximal descending path of x_1 with $l_G(x_1) = l_G(x_0) - 1$.

Given a graph structure G, we have the *reduced digraph structure G' of G* as follows. (See Fig. 3.)

First, we assign each cell x of G a d-vector $m(x) = (m_1(x), \ldots, m_d(x))$, where for each $i = 1, \ldots, d$, if $d_G(x(i)) < d_G(x)$, then $m_i(x) = 1$, if there exists a maximal descending path $[x_0, \ldots, x_l]$ of x such that $x_1 = x(i)$, then $m_i(x) = 2$, and else $m_i(x) = 3$. Then we divide each cell x of G which is neither the general cell nor a terminal cell, into two subcells $x^{(1)}$ and $x^{(2)}$, called the *first subcell of x* and the *second subcell of x*, respectively.



(a) (b)

$m(x_g) = (2, 3, 3, 3)$ $m(x_3) = (1, 3, 3, 3)$

$m(x_1) = (1, 3, 3, 2)$ $m(x_4) = (1, 3, 2, 3)$

$m(x_2) = (1, 3, 2, 3)$ $m(x_5) = (1, 1, 3, 3)$

(c)

Fig. 3. Reduction of a graph structure: (a) a graph structure G; (b) the reduced digraph structure G' of G; (c) d-vectors of the cells of G.

The set of cells of G' consists of all such subcells, the general cell of G, and the terminal cells of G. The general cell of G' is the general cell of G. The predecessors of cells of G' are determined by the d-vectors as follows. The predecessors of the first subcell $x^{(1)}$ [a terminal cell x_t] are all $x(i)^{(1)}$'s with $m_i(x) = 1$ [$x_t(i)^{(1)}$'s with $m_i(x_t) = 1$], where if $x(i)$ [$x_t(i)$] is the general cell x_g, $x(i)^{(1)}$ [$x_t(i)^{(1)}$] means x_g [x_g]. The predecessors of the second subcell $x^{(2)}$ [the general cell x_g] are all $x(i)^{(2)}$'s with $m_i(x) = 2$ [$x_g(i)^{(2)}$ with $m_i(x_g) = 2$], where if $x(i)$ [$x_g(i)$] is a terminal cell x_t, $x(i)^{(2)}$ [$x_g(i)^{(2)}$] means x_t [x_t].

Example 2. In Fig. 3, we give the reduction of a graph structure. There we show maximal descending paths as bold arcs.

For any descending path $[x_0, \dots, x_l]$ in G, there is a path $[x_0^{(1)}, \dots, x_l^{(1)}]$ in G' since $d_G(x_{j-1}) < d_G(x_j)$ for $j = 1, \dots, l$, and for any maximal descending path $[x_0, \dots, x_l]$, there is a path $[x_0^{(1)}, \dots, x_{l-1}^{(1)}, x_l, x_{l-1}^{(2)}, \dots, x_0^{(2)}]$ in G' since a path $[x_j, \dots, x_l]$ in G is also a maximal descending path of x_j for $j = 1, \ldots, l$. Hence, corresponding to the only maximal descending paths of x_g, there are cyclic paths in G', which have length $2r_G$.

Lemma 3.2. *Given a graph structure $G = (X, U, x_g, d)$, let $G' = (Y, V, x_g, d)$ be the reduced digraph structure of G defined above. Then G' is a quasi-circuit structure of length $2r_G$.*

Proof. By definition, each cell of G' has at least one predecessor. Let $Y_0 = \{x_g\}$ and for $k = 1, \ldots, 2r_G - 1$, let $Y_k = Y_k^{(1)} \cup Y_k^{t} \cup Y_k^{(2)}$, where

$$Y_k^{(1)} = \{x^{(1)} \mid x \text{ is a non-terminal cell of } G,\ x \neq x_g, \text{ and } d_G(x) = k\},$$

$$Y_k^{t} = \{x_t \mid x_t \text{ is a terminal cell of } G \text{ and } d_G(x_t) = k\},$$

$$Y_k^{(2)} = \{x^{(2)} \mid x \text{ is a non-terminal cell of } G,\ x \neq x_g, \text{ and } d_G(x) + 2l_G(x) = k\}.$$

For each cell $x (x \neq x_g)$ of G, $d_G(x) \leq d_G(x) + 2l_G(x) \leq 2r_G - 1$. Hence $Y = \bigcup_{k=0}^{2r_G-1} Y_k$. For $k = 0, \ldots, 2r_G - 1$, Y_k is nonempty since for each cell x_j on a maximal descending path $[x_0, \ldots, x_{r_G}]$ of x_g, $d_G(x_j) = j$ and $d_G(x_j) + 2l_G(x_j) = 2r_G - j$. It is clear that $Y_k \cap Y_l = \emptyset$ if $k \neq l$. Therefore, Y is partitioned into disjoint nonempty subsets $Y_0, \ldots, Y_{2r_G-1}$ with $Y_0 = \{x_g\}$.

We show that all predecessors of cells in Y_k are contained in Y_{k-1} for $k = 1, \ldots, 2r_G - 1$ and all predecessors of x_g in Y_0 are contained in Y_{2r_G-1}.

Let $0 \leq k \leq 2r_G - 1$. Let $y \in Y_k$.

Assume that $y \in Y_k^{(1)}$. Then there exists a non-terminal cell x of G such that $x^{(1)} = y$ and $d_G(x) = k$. There are two cases to consider.

(1) $d_G(x) \geq 2$: Any $x(i)$ with $m_i(x) = 1$ is not the general cell since $d_G(x(i)) = d_G(x) - 1 \geq 1$. Hence the predecessors of $y (= x^{(1)})$ are all $x(i)^{(1)}$'s with $m_i(x) = 1$ by definition. Since $m_i(x) = 1$ and $d_G(x) = k$, $d_G(x(i)) = d_G(x) - 1 = k - 1$. Thus all predecessors of y are contained in $Y_{k-1}^{(1)}$.

(2) $d_G(x) = 1$: In this case, $y \in Y_1^{(1)}$. The cell $x(i)$ with $m_i(x) = 1$ is the general cell since $d_G(x(i)) = d_G(x) - 1 = 0$. Hence the predecessor of y $(= x^{(1)})$ is only the general cell x_g by definition. Thus the predecessor of y is contained in Y_0.

A similar argument shows that if $y \in Y_k^{t}$, all predecessors of y are contained in $Y_{k-1}^{(1)}$ when $k \geq 2$ and in Y_0 when $k = 1$.

Assume that $y \in Y_k^{(2)}$. Then there exists a non-terminal cell x of G such that $x^{(2)} = y$ and $d_G(x) + 2l_G(x) = k$. There are two cases to consider.

(1) $l_G(x) \geq 2$: Every $x(i)$ with $m_i(x) = 2$ is non-terminal since $l_G(x(i)) = l_G(x) - 1 \geq 1$. Hence the predecessors of $y (= x^{(2)})$ are all $x(i)^{(2)}$'s with $m_i(x) = 2$ by definition. Since $m_i(x) = 2$ and $d_G(x) + 2l_G(x) = k$, $d_G(x(i)) + 2l_G(x(i)) = (d_G(x) + 1) + 2(l_G(x) - 1) = d_G(x) + 2l_G(x) - 1 = k - 1$. Thus all predecessors of y are contained in $Y_{k-1}^{(2)}$.

(2) $l_G(x) = 1$: Every $x(i)$ with $m_i(x) = 2$ is terminal since $l_G(x(i)) = l_G(x) - 1 = 0$. Hence the predecessors of $y (= x^{(2)})$ are all $x(i)$'s with $m_i(x) = 2$ by definition. Since

$l_G(x(i))=0$, $m_i(x)=2$, and $d_G(x)+2l_G(x)=k$, $d_G(x(i))=d_G(x(i))+2l_G(x(i))=d_G(x)-1=k-1$. Thus all predecessors of y are contained in $Y^{\natural}_{k-1}$.

A similar argument shows that if $y \in Y_0$, then all predecessors of y are contained in $Y^{(2)}_{2r_G-1}$ when $r_G \geq 2$ and in $Y^{\natural}_{2r_G-1}$ when $r_G = 1$.

Thus G' is a quasi-circuit structure of length $2r_G$.

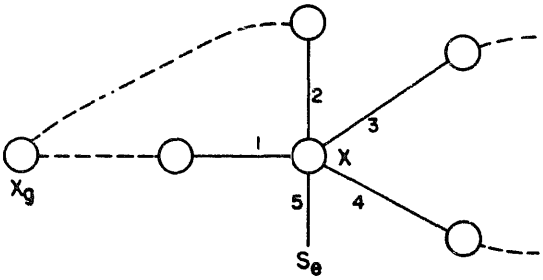
Now we construct an automaton M_R which realizes the above reduction. The realization of the quasi-circuit structure D_{2r_G} from a given graph structure G by M_R means that every cell x (with M_R) of G computes and stores its own d-vector $m(x)$. We assume that M_R has $d+1$ registers $s, r_1, \ldots, r_d$. The first register s holds a signal in $S_R=\{G_0, G_1, G_2, H_0, H_1, I, J, Q_0\}$ and each of the remaining d registers $r_1, \ldots, r_d$ holds an element of $\{0,1,2,3\}$. Thus each state of M_R has the form $(s, r_1, \ldots, r_d)$. The transition table of M_R is given by Table 1. In Table 1, by $s(x)$ and $r_i(x)$, we denote the contents of the register s and the register r_i of a cell x, respectively, and by R_0, we denote the set of integers i with $r_i(x)=0$.

Now we explain the behavior of the network defined by a given graph structure G and M_R. (See Fig. 4.) At time 0, $s(x)=G_0$ or $s(x)=Q_0$ according as x is the general cell or not, and $r_i(x)=0$ for $i=1,\ldots,d$. The behavior of the network is partitioned into three parts.

(1) *The general cell sends the signal H_0 to all cells along descending paths.* Each cell x with $s(x)=Q_0$ sets $s(x)=H_0$ if there is an adjacent cell y such that $s(y)=H_0$ or G_0.

Table 1

Before the step		After the step	
$s(x)$	the condition for the transition	$s(x)$	$r_i'(x)$ (the content of r_i after the step)
G_0		G_1	$r_i'(x)=\begin{cases}3 & \text{if } x(i) \text{ does not exist}\\ 0 & \text{else}\end{cases}$
Q_0	$\exists i \in R_0\ s(x(i))=H_0, G_0$	H_0	$r_i'(x)=\begin{cases}1 & \text{if } s(x(i))=H_0, G_0\\ 3 & \text{if } x(i) \text{ does not exist}\\ 0 & \text{else}\end{cases}$
	otherwise	Q_0	unchanged
H_0	$\exists i \in R_0\ s(x(i))=Q_0$	H_1	$r_i'(x)=\begin{cases}3 & \text{if } s(x(i))=H_0\\ r_i(x) & \text{else}\end{cases}$
	otherwise	I	$r_i'(x)=\begin{cases}3 & \text{if } s(x(i))=H_0\\ r_i(x) & \text{else}\end{cases}$
H_i (G_1)	$\forall i \in R_0\ s(x(i))=J, I$	J (G_2)	$r_i'(x)=\begin{cases}2 & \text{if } i \in R_0\\ r_i(x) & \text{else}\end{cases}$
	otherwise	H_1 (G_1)	$r_i'(x)=\begin{cases}3 & \text{if } s(x(i))=J, I\\ r_i(x) & \text{else}\end{cases}$
G_2		G_2	unchanged
I		I	unchanged
J		J	unchanged



time	$s(x(1))$	$s(x(2))$	$s(x(3))$	$s(x(4))$	$s(x)$ $(r_1(x), \cdots, r_5(x))$
$d_G(x)-1$	H_0	Q_0	Q_0	Q_0	Q_0 $(0,0,0,0,0)$
$d_G(x)$	H_1	H_0	Q_0	Q_0	H_0 $(1,0,0,0,0)$
$d_G(x)+1$	H_1	I	H_0	H_0	H_1 $(1,3,0,0,3)$
$d_G(x)+2\,l_G(x(3))+1$	H_1	I	J	H_1	H_1 $(1,3,0,0,3)$
$d_G(x)+2(l_G(x(3))+1)$	H_1	I	J	H_1	H_1 $(1,3,3,0,3)$
$d_G(x)+2\,l_G(x)$	H_1	I	J	J	H_1 $(1,3,3,0,3)$
$d_G(x)+2\,l_G(x)+1$	H_1	I	J	J	J $(1,3,3,2,3)$

Fig. 4. Behavior of M_R on cells of a graph structure, where $d_G(x(1))<d_G(x)=d_G(x(2))<d_G(x(3))=d_G(x(4))$, $x(2)$ is terminal, and $x(4)$ is on the maximal descending path of x but $x(3)$ is not (i.e. $l_G(x(3))<l_G(x(4))=l_G(x)-1$).

At the same time for each $i=1,\ldots,d$ the cell x sets $r_i(x)=1$ if $s(x(i))=H_0$ or G_0 (i.e. $d_G(x(i))<d_G(x)$), $r_i(x)=3$ if $x(i)$ does not exist, and $r_i(x)=0$, otherwise (i.e. $d_G(x(i))\geq d_G(x)$). This move of x occurs at time $d_G(x)$.

(2) *Each cell recognizes whether it is terminal or not.* Each cell x with $s(x)=H_0$ sets $s(x)=H_1$ if there exists i with $s(x(i))=Q_0$ (i.e. x is not terminal) and sets $s(x)=I$ if there is no i with $s(x(i))=Q_0$ (i.e. x is terminal). At the same time, the cell x changes the value of $r_i(x)$ from 0 to 3 for every i with $s(x(i))=H_0$ (i.e. $d_G(x(i))=d_G(x)$). This move of x occurs at time $d_G(x)+1$.

(3) *Each terminal cell sends the reflexive signal J toword the general cell along the descending paths (in reverse direction).* Each non-terminal cell x with $s(x)=H_1$ changes the value of $r_i(x)$ as soon as the cell x detects that $s(x(i))$ has changed from H_1 into J or I. If there exists $j (j\neq i)$ with $r_j(x)=0$ and $s(x(j))=H_1$ at that time, $r_i(x)$ changes to 3 (from 0) and $s(x)$ is unchanged, and otherwise $r_i(x)$ changes to 2 (from 0) and $s(x)$ changes to J (from H_1). In the latter case, $x(i)$ is on a maximal descending path of x, but in the former case, $x(i)$ is not on any maximal descending path of x. The time when $r_i(x)$ changes to 2 (for the latter case) is $d_G(x)+2l_G(x)+1$.

Hence each cell x of G has $m(x)$ as $(r_1(x), \ldots, r_d(x))$ at time $d_G(x)+2l_G(x)+1$. Precisely the following fact is obtained.

Fact 3.3. Given a graph structure G and the above automaton M_R, for each cell x of G the time when $r_i(x)$ changes to 1 (from 0) is $d_G(x)$ and for each non-terminal cell x of G the time when $r_i(x)$ changes to 2 (from 0) is $d_G(x)+2l_G(x)+1$.

3.3. *The (3r+1)-solution*

In the previous section, we presented an automaton which reduces any graph structure to a quasi-circuit structure. The synchronization of the quasi-circuit structure or a part of the quasi-circuit structure that completely covers the original graph structure gives the synchronization of the graph structure. Moreover by Lemma 3.1, the synchronization of any quasi-circuit structure is reduced to that of a circuit structure. Thus, in this section, we consider the synchronization of circuit structures and finally give solutions for the class $\mathcal{G}$ of all graph structures.

A solution for the class of circuit structures was given by Kobayashi [8, 9]. Its synchronization time for C_n is $2n-1$ time units, which is minimum. The authors have obtained independently a similar solution M_c called the *circuit solution.*

Using the circuit solution M_c, we can easily give a solution M_{4r} for $\mathcal{G}$ whose synchronization time for a graph structure G is $4r_G$ time units. The automata M_{4r} on a given graph structure G simulate the automata M_R (as given in the previous section) to reduce G into a quasi-circuit structure D_{2r_G}, and simulate the modified automata M_c' of M_c (cf. Section 3.1) which are assumed to be placed on the reduced quasi-circuit structure D_{2r_G} of G. Thus, for each cell x which is neither the general cell nor a terminal cell, M_{4r} on x simulates M_R on x and simulates M_c' on the first and the second subcells ($x^{(1)}$ and $x^{(2)}$ respectively) of x. For the general cell x_g [a terminal cell x_t], M_{4r} on x_g [x_t] simulates M_R on x_g [x_t] and simulates M_c' on x_g [x_t]. Fact 3.3. and Lemma 3.2 imply that by M_R, the predecessors of $x^{(1)}$ [x_t] are determined at time $d_{D_{2r_G}}(x^{(1)})$ [$d_{D_{2r_G}}(x_t)$] and those of $x^{(2)}$ [x_g] are determined at time $d_{D_{2r_G}}(x^{(2)})+1$ [$2r_G+1$]. Thus the general cell of G can start to simulate M_c' one time unit later after the simulation of M_R starts (at time 0). We make M_{4r} so that each cell fires when M_c' simulated by M_{4r} on it moves to the firing state F (of M_c'). Since the automata M' synchronize D_{2r_G} at time $2(2r_G)-1=4r_G-1$, we obtain the solution M_{4r} for $\mathcal{G}$ whose synchronization time for G is $1+(4r_G-1)=4r_G$ time units.

We can improve the solution M_{4r} by considering the following two facts.

(1) The synchronization of a graph structure G is achieved by synchronizing the first subcells, the general cell, and terminal cells of the reduced quasi-circuit structure D_{2r_G} of G.

(2) A terminal cell farthest from x_g divides each cyclic path passing through it in D_{2r_G} into two halves.

Hence we have the following modified problem. Let (C_{2n}, x_n) be a *circuit structure C_{2n} with a designated cell x_n*, in which the cell x_n can move to a particular state at time

n $(=d_{C_{2n}}(x_n))$. In other words, when the cell x_n receives the first signal from the general cell, it can know that it divides C_{2n} into two halves $[x_0, \ldots, x_{n-1}]$ and $[x_n, \ldots, x_{2n-1}]$. The problem is to construct an automaton with one input terminal which synchronizes all cells on the semicircuit $[x_0, \ldots, x_n]$ for the class of (C_{2n}, x_n). This automaton is called the *semicircuit solution.*

We first explain the circuit solution $M_c = (S_c, s_e, Q, P_{00}, F, \lambda_c)$, and then explain how to modify M_c to obtain a semicircuit solution M_h. Since it is laborious to describe the details of the behavior of M_c, we explain only the basic idea for the case when n is an integer of the form 2^m, and present the transition table of M_c in the Appendix.

The propagation of 'signals' of M_c in C_n is depicted in Fig. 5 by using a diagram.

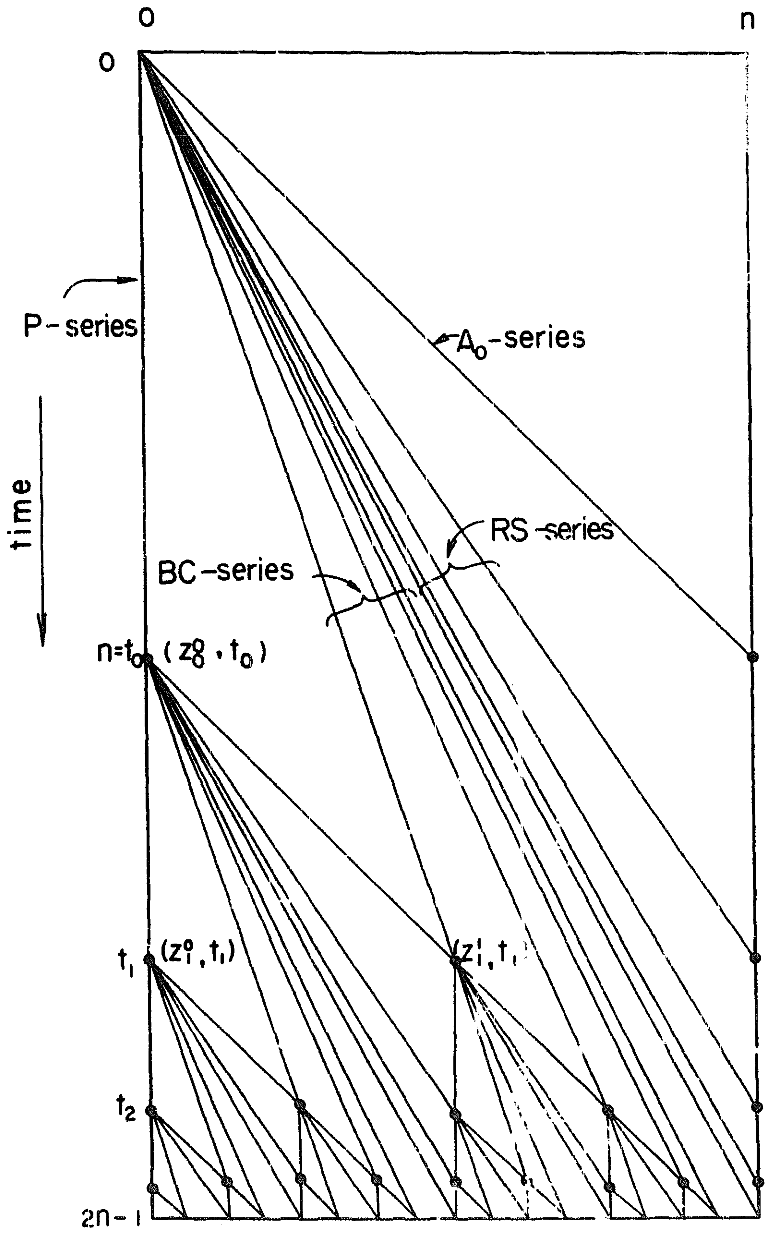


Fig. 5. Propagation of 'signals' of M_c in C_n.

The horizontal axis represents cells of C_n and the vertical axis represents time. The (z, t) entry represents the state of a cell x_z with $d_{C_n}(x_z) = z$ at time t. Note that two points $(0, t)$ and (n, t) are the same point. Let z_i^k and t_i be $k(n/2^i)$ and $2n - n/2^i (i = 0, \ldots, m-1, k = 0, \ldots, 2^{m-1} - 1)$ respectively.

There is one special signal P_{00} called the general signal. P_{00} is the general state of M_c. Hence P_{00} is at (0, 0). Moreover P_{00}'s are generated at (z_i^k, t_i) in the following way. Each P_{00} generates the following series (sequence of signals in the diagram):
- P-series with velocity $v = 0$ (cells/time unit),
- BC-series with $v = \frac{1}{3}, \frac{3}{7}, \ldots, (2^i - 1)/(2^{i+1} - 1), \ldots,$
- A_0-series with $v = 1$, and
- RS-series with $v = \frac{2}{3}, \frac{4}{7}, \ldots, 2^i/(2^{i+1} - 1), \ldots.$

When
(1) an A_0-series meets a P-series, or
(2) an RS-series meets a P-series, or
(3) a BC-series meets an A_0-series,

a new general signal P_{00} is generated at the point. These new P_{00}'s generate new P, BC, A_0, RS-series and these series in turn generate new P_{00}'s and so on. In Fig. 5, the A_0-series and the RS-series from (0, 0) meet the P-series from (0, 0) at (z_0^0, t_0), $(z_1^0, t_1), \ldots, (z_j^0, t_j), \ldots.$ The BC-series from (0, 0) meet the A_0-series from $(0, n) = (z_0^0, t_0)$ at $(z_1^1, t_1), (z_2^3, t_2), \ldots, (z_j^{2^j-1}, t_j), \ldots.$ Then, at time $t_{m-1} = 2n - 2$, every cell x_z with $z = 2k$ $(k = 0, \ldots, 2^{m-1})$ is in the state P_{00}, and hence all cells of C_n can fire at time $2n - 1$.

More precisely, M_c has two general signals P_{00} and P_{11}. P_{11} is used for the case, where n is not of the form 2^m. The behavior of M_c for C_{13} is given in Fig. 6. In Fig. 6 the two signals K and T are used to produce BC-series and RS-series as trigger signals respectively. For more details, see [8].

Now we give the semicircuit solution $M_h = (S_h, s_e, Q, P, F, \lambda_h)$. The state set S_h includes that of M_c. We explain only the basic idea for the case, where n is 2^m, and present the transition table of M_h in the Appendix.

The propagation of 'signals' of M_h in (C_{2n}, x_n) is depicted in Fig. 7, in which the propagation of 'signals' of M_c in C_n is also shown for the reference. We construct M_h so that the signals generated at (z, t) in (C_{2n}, x_n) are identical to those at $(z, t-n)$ in C_n for $0 \leq z \leq n-1$ and $t \geq 2n + z$, that is, P_{00}'s are generated at $(z_j^0, t_j + n)$ and $(z_j^{2^j-1}, t_j + n)$ (for $j = 0, \ldots, m-1$) in (C_{2n}, x_n).

M_h has two special signals P and P'_{00} in addition to P_{00}. P is the general state of M_h. Hence P is at (0, 0). P generates a P'-series with velocity $v = 0$, an A-series with $v = 1$, and VW-series with $v = \frac{1}{5}, \frac{3}{11}, \ldots, (2^i - 1)/(3 \cdot 2^i - 1), \ldots.$ P'_{00} is generated at (n, n), that is P'_{00} is the state of the designated cell x_n at time $n(= d_{C_{2n}}(x_n))$. P'_{00} generates a P''_{00}-series with $v = 0$, an A_0-series with $v = 1$, and RS-series with $v = \frac{2}{3}, \frac{4}{7}, \ldots, 2^i/(2^{i+1} - 1), \ldots.$

General signals P_{00}'s are generated by the following rules in addition to the rules of M_c. When (4) an A_0-series meets a P'-series, or (5) a VW-series meets an A_0-series, a new general signal P_{00} is generated at the point.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	P_{00}												
1	P_1	A_{00}											
2	P_0	B^2_2	A_{01}										
3	P_1	B_0	L^1	A_{00}									
4	P_0		C^2_1	T	A_{01}								
5	P_1	K	C_2	S^1_0		A_{00}							
6	P_0		C_0	B^2_2	S_1	T	A_{01}						
7	P_1	K		B_{13}	S^1_2	T		A_{00}					
8	P_0		K	B_2	B_2	R_2		T	A_{01}				
9	P_1	K		B_0	B_3	L^1	R_1	T		A_{00}			
10	P_0		K		C_1	B^2_2	R_2	T		T	A_{01}		
11	P_1	K		K	C_2	B_0	L^1	S_0		T		A_{00}	
12	P_0		K		C_0		C^2_1	T	S_1	T		T	A_{01}
13	P_{11}	K		K		B_1	C_2	S^1_0	S_2	T		T	
14	P_0	A_{10}	K		K	B_2	C_3	B^2_2	S_1	R_2		T	
15	P_1	B^2_1	A_{11}	K		B_0		C_{23}	S^1_2		R_1	T	
16	P_0	B_2	L^1	A_{10}	K		C_1	C_3	B_2	S_1	R_2	T	
17	P_1	B_0	L^2	T	A_{11}	K	C_2	K	C_{23}	S^1_2		S_0	
18	P_0		C_1	S^1_0		A_{10}	C_0		C_0	B^2_2	S_1	T	S_1
19	P_1	K	C_2	B^2_1	S_1	T	P_{20}	B_1		B_{13}	S^1_2	T	S_2
20	P_{00}		C_0	B_2	S^1_2	T	P_2	P_{00}		B_2	B^2_2	R_2	
21	P_1	A_{00}		B_{13}	L^2	R_2	P_2	P_1	A_{00}	B_3	B_3	L^1	R_1
22	P_0	B^2_2	A_{01}	B_2	B_2	L^1	P_{21}	P_0	B^2_2	A_{01}	B_2	B^2_2	R_2
23	P_{11}	B_0	L^1	P_{11}	B_3	L^2	P_2	P_{11}	B_0	L^1	P_{11}	B_3	L^1
24	P_0	P_{22}	C^2_1	P_0	P_{22}	B_2	P_{22}	P_0	P_{22}	C^2_1	P_0	P_{22}	B^2_2
25	F	F	F	F	F	F	F	F	F	F	F	F	F

Fig. 6. Behavior of M_c for C_{13}.

In Fig. 7, the A_0-series from (n, n) meets the P'-series from $(0, 0)$ at $(0, 2n)$. Hence P_{00} is generated at $(0, 2n) = (z^0_0, t_0 + n)$ by the rule (4). The RS-series from (n, n) meet the P-series from $(0, 2n)$ at $(z^0_1, t_1 + n)$, $(z^0_2, t_2 + n), \ldots, (z^0_j, t_j + n), \ldots$. The VW-series from $(0, 0)$ meet the A_0-series from $(0, 2n)$ at $(z^1_1, t_1 + n)$, $(z^3_2, t_2 + n), \ldots, (z^{2j-1}_j, t_j + n), \ldots$.

From the above consideration, we can conclude that M_h on (C_{2n}, x_n) generates the same signals at (z, t) as M_c on C_n does at $(z, t-n)$ for $0 \leqslant z \leqslant n-1$, $t \geqslant 2n + z$. Thus

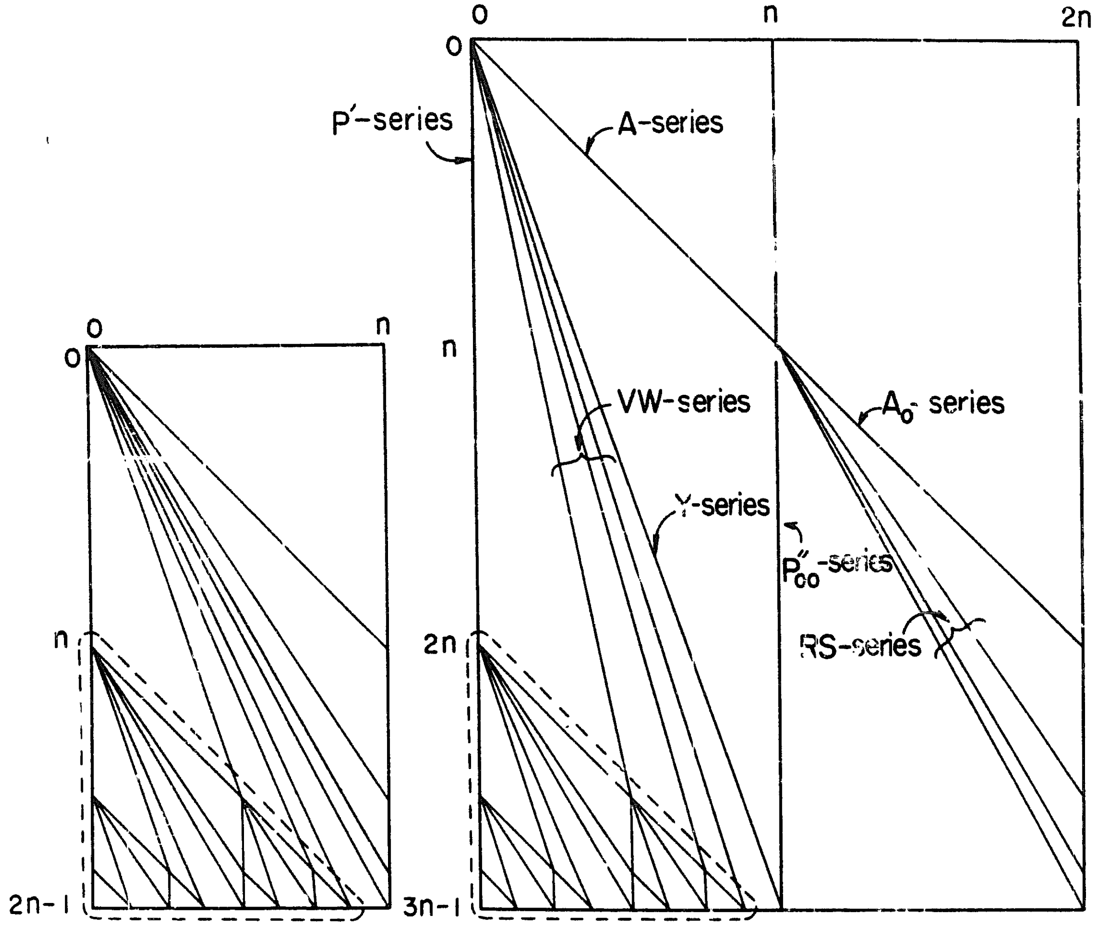


Fig. 7. Propagation of 'signals' of M_h in (C_{2n}, x_n).

each of cells $x_0, \ldots, x_{n-1}$ fires at time $3n-1$. The device of the firing of the designated cell x_n is special. We make the designated cell x_n fire at time $3n-1$ by the Y-series with $v = \frac{1}{3}$ generated by P at $(0, 0)$. Hence all cells on the semicircuit $[x_0, \ldots, x_n]$ of C_{2n} fire at time $3n-1$ simultaneously. Fig. 8 gives the semicircuit solution for (C_{12}, x_6).

Now we construct a solution for $\mathscr{G}$ called the $(3r+1)$-solution M_{3r+1} by using the semicircuit solution M_h. In the same way as M_{4r} does, each M_{3r+1} on a cell x of a given graph structure G simulates (1) one M_R on x, and (2) one M'_h on x if x is either x_g or a terminal cell and two M'_h on $x^{(1)}$ and $x^{(2)}$ otherwise. (M'_h is the modified automaton of M_h.) The general cell starts to simulate M'_h at time 1. Here we must determine the designated cells of the reduced quasi-circuit structure D_{2r_G} of G. It is desirable to designate only all terminal cells x with $d_G(x) = r_G$ as the designated cells of D_{2r_G}. But it costs too much time. Hence we define that all terminal cells of G are the designated cells of D_{2r_G}. This designating is made when each terminal cell knows that it is terminal in the process of the reduction of G to D_{2r_G} by M_R. Then for every first subcell $x^{(1)}$, there is no designated cell on all paths from x_g to $x^{(1)}$ in D_{2r_G}, and for

	0	1	2	3	4	5	6	7	8	9	10	11
0	P											
1	P'_0	A										
2	P'_1	V^1_1	A									
3	P'_2	V^2_2		A								
4	P'_3	V^3_3			A							
5	P'_0	V_4	Y^1			A						
6	P'_1	V_0	Y^2				P'_{00}					
7	P'_2		W^3_1				P''_{00}	A_{00}				
8	P'_3		W_2	V^1_1			P''_{00}	L^2	A_{01}			
9	P'_0		W_3	V^2_2			P''_{00}		L^1	A_{00}		
10	P'_1	K'	W_4	V^3_3			P''_{00}		L^2	T	A_{01}	
11	P'_2		W_0	V_4	Y^1		P''_{00}			S^1_0		A_{00}
12	P_{00}			V_{15}	Y^2		P''_{00}			L^2	S_1	T
13	P_1	A_{00}		V_2	V^3_3		P''_{00}				S^1_2	T
14	P_0	B^2_2	A_{01}	V_3	V_4	Y^1	P''_{00}				L^2	R_2
15	P_{11}	B_0	L^1	P_{11}	V_5	Y^2	P''_{00}					L^1
16	P_0	P_{22}	C^2_1	P_0	P_{22}	V^3_3	P''_{00}					L^2
17	F	F	F	F	F	F	F					

Fig. 8. Behavior of M_h for (C_{12}, x_6).

every cyclic path in D_{2r_G}, there is a unique designated cell y and $d_{D_{2r_G}}(y) = r_G$. There are designated cells that are not on cyclic paths. But the behavior of these cells cannot affect the behavior of the first subcells, the general cell, and the designated cells on cyclic paths. Thus the modified semicircuit solutions M'_h on all first subcells, the general cell, and all designated cells on cyclic paths move to the firing state F (of M'_h) at time $1+(3r_G-1)=3r_G$ by an argument similar to that used for proving Lemma 3.1. But for a designated cell y not on a cyclic path in D_{2r_G}, M'_h on it moves to F at time $3d_{D_{2r_G}}(y) < 3r_G$ since a designated cell fires when the Y-series with $v=\frac{1}{3}$ from the general cell arrives to it. Hence we make M_{3r+1} fire as follows. For a non-terminal cell x [the general cell x_g] of G, x [x_g] fires at time $t+1$ if M'_h on the first subcell $x^{(1)}$ of x [on x_g] moves to F at time t. For a terminal cell x_t, x_t fires at time $t+1$ if M'_h on the first subcell of an adjacent cell y of x_t with $d_G(y) < d_G(x_t)$ moves to F at time t. It requires one more time unit.

Theorem 3.4. *The automaton M_{3r+1} is a solution for $\mathcal{G}$ and its synchronization time for G in $\mathcal{G}$ is $3r_G+1$ time units.*

4. The $(3r)$-solution

In this section, we improve the $(3r+1)$-solution M_{3r+1}. The improved solution is called the $(3r)$-*solution* M_{3r}. Its synchronization time for a graph structure G in $\mathscr{G}_s$ and $\mathscr{G}$-$\mathscr{G}_s$ are respectively $3r_G$ and $3r_G+1$ time units, where $\mathscr{G}_s$ is the subclass of $\mathscr{G}$ defined below.

In a graph structure G, a cell x with $d_G(x)=r_G$ is called a *radial* cell. A cell x for which there is no adjacent cell y with $d_G(y) \geq d_G(x)$, is called a *solitary* cell. Note that each solitary cell is terminal but a terminal cell is not necessarily solitary. Let $\mathscr{G}_s$ be the class of all graph structures G such that all radial cells of G are solitary and the number of cells of G is more than or equal to 2.

The fundamental behavior of M_{3r} is identical to that of M_{3r+1}. The $(3r)$-solution M_{3r} reduces a given graph structure to a quasi-circuit structure simulating a slightly distinct automaton M_S from M_R which was simulated by M_{3r+1}, and simulates the modified semicircuit solutions M'_h. Moreover, it is so devised that when all radial cells are solitary in a given graph structure, the general cell can know this fact and sends the signal informing it to all other cells by M_S.

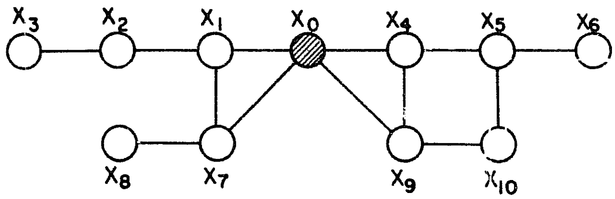
For every terminal cell x_t of a given graph structure G, we assume that two M'_h are placed on x_t and one of them behaves as if it is on a designated cell of D_{2r_G} (hence, it plays the same role as M'_h in M_{3r+1} on x_t) and the other does as if it is not on a designated cell of D_{2r_G}. By $M'_h(x_t)$ and $M'_h(x'_t)$, we denote the former M'_h and the latter M'_h respectively. For a non-terminal cell x of G [the general cell x_g], by $M'_h(x^{(1)})$ and $M'_h(x^{(2)})$ [$M'_h(x_g)$] we also denote M'_h on the first subcell of x and M'_h on the second subcell of x [M'_h on x_g] respectively. Then the following facts are obtained.

Fact 4.1. (1) For a radial cell x_r, $M'_h(x'_r)$ does not move to F at any time and $M'_h(x_r)$ moves to F at time $3r_G$.

(2) For a non-radial terminal cell x_t, $M'_h(x'_t)$ moves to F at time $3r_G$ and $M'_h(x_t)$ moves to F at time $3d_G(x_t) < 3r_G$.

(3) For a non-terminal cell x [x_g], $M'_h(x^{(1)})$ [$M'_h(x_g)$] moves to F at time $3r_G$ (cf. Section 3.3).

Next, we explain the automaton M_S which reduces a graph structure to a quasi-circuit structure and makes all cells to recognize whether all radial cells are solitary or not. M_S has $d+1$ registers $s, r_1, \ldots, r_d$. The register s holds a signal in $S_S = S_R \cup \{G_s, I_s, I_0, J_s, J_0\}$ where S_R is given by M_R (simulated by M_{3r+1}), and each of $r_1, \ldots, r_d$ holds an element of $\{0, 1, 2, 3\}$. The behavior of M_S is similar to that of M_R, that is, the registers $r_1, \ldots r_d$ play the same role as those of M_R do and the new signals G_s, I_s (I_0), and $J_s(J_0)$ to be held in the register s essentially play the same role as G_2, I, and J do in M_R respectively. Hence we explain only how M_S makes all cells of G recognize whether all radial cells are solitary or not. (See Fig. 9 and 10 where $r_G=3$.) The transition table of M_S is presented in Table 2, where $s(x)$ and $r_i(x)$ mean the same ones as in Table 1 and by $R_k (k=0, 1)$ we denote the set of integers i such that $r_i(x)=k$ and $x(i)$ exists.

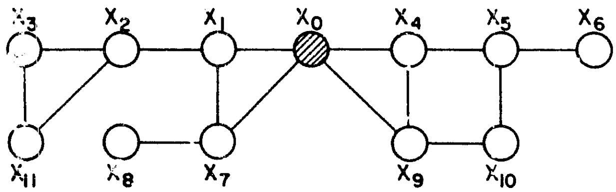


	x_0	x_1 (x_4)	x_2 (x_5)	x_3 (x_6)
0	G_0			
1	G_1	H_0		
2	G_1	H_1	H_0	
3	G_1	H_1	H_1	I_0
4	G_1	H_1	J_0	I_0
5	G_1	J_0	J_0	I_0
6	G_S	J_0	J_0	I_0
7	G_S	J_S	J_0	I_0
8	G_S	J_S	J_S	I_0

	x_7	x_8
0		
1	H_0	
2	H_1	I_0
3	J_0	I_0
4	J_0	I_0
5	J_0	I_0
6	J_0	I_0
7	J_S	I_0
8	J_S	I_S

	x_9	x_{10}
0		
1	H_0	
2	H_1	H_0
3	H_1	I
4	J	I
5	J	I
6	J	I
7	J_S	I
8	J_S	I_S

Fig. 9. Behavior of M_S for a graph structure G, where x_0 is the general cell and all radial cells x_3 and x_6 are solitary.



	x_0	x_1	x_2	x_3 (x_{11})
0	G_0			
1	G_1	H_0		
2	G_1	H_1	H_0	
3	G_1	H_1	H_1	H_0
4	G_1	H_1	H_1	I
5	G_1	H_1	J	I
6	G_1	J	J	I
7	G_2	J	J	I
	⋮	⋮	⋮	⋮

	x_4	x_5	x_6
0			
1	H_0		
2	H_1	H_0	
3	H_1	H_1	I_0
4	H_1	J_0	I_0
5	J_0	J_0	I_0
6	J_0	J_0	I_0
7	J_0	J_0	I_0
	⋮	⋮	⋮

	x_7	x_8
0		
1	H_0	
2	H_1	I_0
3	J_0	I_0
4	J_0	I_0
5	J_0	I_0
6	J_0	I_0
7	J_0	I_0
	⋮	⋮

	x_9	x_{10}
0		
1	H_0	
2	H_1	H_0
3	H_1	I
4	J	I
5	J	I
6	J	I
7	J	I
	⋮	⋮

Fig. 10. Behavior of M_S for a graph structure G, where x_0 is the general cell and x_3, x_6, and x_{11} are radial but neither x_3 nor x_{11} is solitary.

Table 2

	Before the step		After the step
$s(x)$	the condition for the transition	$s(x)$	$r_i'(x)$ (the content of r_i after the step)
G_0		G_1	$r_i'(x)=\begin{cases}3 & \text{if } x(i) \text{ does not exist}\\ 0 & \text{else}\end{cases}$
Q_0	$\forall i\in R_0\ s(x(i))=H_0, G_0$	I_0	$r_i'(x)=\begin{cases}1 & \text{if } s(x(i))=H_0, G_0\\ 3 & \text{else}\end{cases}$
	$\exists i\in R_0\ s(x(i))=H_0, G_0$ and $\exists j\in R_0\ s(x(i))=Q_0$	H_0	$r_i'(x)=\begin{cases}1 & \text{if } s(x(i))=H_0, G_0\\ 3 & \text{if } x(i) \text{ does not exist}\\ 0 & \text{else}\end{cases}$
	otherwise	Q_0	unchanged
H_0	$\forall i\in R_0\ s(x(i))=H_0$	I	$r_i'(x)=\begin{cases}3 & \text{if } i\in R_0\\ r_i(x) & \text{else}\end{cases}$
	otherwise	H_1	$r_i'(x)=\begin{cases}3 & \text{if } s(x(i))=H_0\\ r_i(x) & \text{else}\end{cases}$
H_1 (G_1)	$\forall i\in R_0\ s(x(i))=J_0, I_0$	J_0 (G_s)	$r_i'(x)=\begin{cases}2 & \text{if } i\in R_0\\ r_i(x) & \text{else}\end{cases}$
	$\forall i\in R_0\ s(x(i))=J, I$	J (G_2)	$r_i'(x)=\begin{cases}2 & \text{if } i\in R_0\\ r_i(x) & \text{else}\end{cases}$
	otherwise	H_1 (G_1)	$r_i'(x)=\begin{cases}3 & \text{if } s(x(i))=J_0, I_0, J, I\\ r_i(x) & \text{else}\end{cases}$
I_0 (I)	$\exists i\in R_1\ s(x(i))=J_s, G_s$	I_s (I_s)	unchanged
	otherwise	I_0 (I)	unchanged
J_0 (J)	$\exists i\in R_1\ s(x(i))=J_s, G_s$	J_s (J_s)	unchanged
	otherwise	J_0 (J)	unchanged
G_2		G_2	unchanged
G_s		G_s	unchanged
I_s		I_s	unchanged
J_s		J_s	unchanged

(1) The general cell sends the signal H_0 to all cells in the same way as it does by M_R. Each cell x recognizes at time $d_G(x)-1$ whether it is solitary or not. Note that x is solitary if and only if for every adjacent cell y, $s(y)$ is H_0 or G_0 and $s(x)$ is Q_0.

(2) A non-solitary cell x recognizes at time $d_G(x)$ whether it is terminal or not in the same way as it does by M_R.

(3) Each terminal cell x_t sends the reflexive signal toward the general cell in the same way as it does by M_R. Here if x_t is solitary, the signal J_0 is sent at time $d_G(x_t)$, and if x_t is non-solitary, the signal J is sent at time $d_G(x_t)+1$.

(4) If all radial cells are solitary, the general cell x_g recognizes this fact at time $2r_G - 1$, that is, for all $x_g(i)$ with $r_i(x_g) = 0$, $s(x_g(i)) = J_0$, I_0 at that time. At the next time, x_g sets $s(x_g) = G_s$ from G_1 and then sends a signal J_s informing this fact to all cells. Each cell x receives this signal at time $2r_G - 1 + d_G(x)$. Note that if there is a non-solitary radial cell, there is a cell $x_g(i)$ with $r_i(x_g) = 0$ and $s(x_g(i)) = H_1$ at time $2r_G - 1$.

Thus the following fact is obtained.

Fact 4.2. If all radial cells are solitary, then every cell x recognizes this fact at time $2r_G + d_G(x) - 1$.

From Fact 4.1 and 4.2, we define the firing of M_{3r} by the following rules.

(1) For a terminal cell x_t,

(i) x_t fires when $M'_h(x_t)$ moves to F if x_t recognizes that all radial cells are solitary before $M'_h(x_t)$ moving to F,

(ii) x_t fires when $M'_h(x'_t)$ moves to F if x_t recognizes that all radial cells are solitary after $M'_h(x_t)$ moving to F, and

(iii) otherwise, x_t fires at the next time when $M'_h(x_t(i)^{(1)})$ with $d_G(x_t(i)) < d_G(x_t)(r_i(x_t) = 1)$ moves to F.

(2) For a non-terminal cell x [the general cell x_g],

(i) x [x_g] fires when $M'_h(x^{(1)})$ [$M'_h(x_g)$] moves to F if x [x_g] recognizes that all radial cells are solitary, and

(ii) otherwise, x [x_g] fires at the next time when $M'_h(x^{(1)})$ [$M'_h(x_g)$] moves to F.

First, we consider the case where all radial cells are solitary. For a radial cell x_r, $M'_h(x_r)$ has not yet moved to F at time $2r_G + d_G(x_r) - 1 = 3r_G - 1$ when x_r recognizes that all radial cells are solitary. Hence x_r fires at time $3r_G$ by the rule (i) of (1) and Fact 4.1 (1). For a non-radial terminal cell x_t, $M'_h(x_t)$ moves to F at time $3d_G(x_t)$ and then recognizes that all radial cells are solitary at time $2r_G + d_G(x_t) - 1 > 3d_G(x_t)$. Hence x_t fires at time $3r_G$ by the rule (ii) of (1) and Fact 4.1 (2). All other cells fire at time $3r_G$ by the rule (i) of (2), Fact 4.2, and Fact 4.1 (3). From the above discussion, if all radial cells are solitary, then all cells fire at time $3r_G$.

In the case where all radial cells are not solitary, it is clear by the rules (iii) of (1) and (ii) of (2) that all cells fire at time $3r_G + 1$.

Theorem 4.3. *The automaton M_{3r} is a solution for $\mathscr{G}$, and its synchronization time is $3r_G$ time units for G in $\mathscr{G}_s$ and is $3r_G + 1$ time units for G in $\mathscr{G} - \mathscr{G}_s$.*

Finally, we point out that M_{3r} gives the minimum synchronization time for some subclass of $\mathscr{G}$.

For a graph structure G, let $t_{min}(G)$ be the minimum value of the synchronization time $t(G, M)$ over all solutions for $\mathscr{G}$. Kobayashi [6, 7] gave an algorithm to calculate

$t_{\min}(G)$ for each graph structure G. Intuitively $t_{\min}(G)$ is about $\max\{d_G(x)+d_G(x, y) \mid x$ and y are cells of $G\}$. For any cells x and y, $d_G(x)+d_G(x, y)$ means the time required for x to leave the quiescent state and then for y to receive a signal from x.

Let $\mathscr{G}_{m1}$ be the class of all graph structures G in $\mathscr{G}_s$ having two radial cells x and y such that $d_G(x, y)=2r_G$. Let $\mathscr{G}_{m2}$ be the class of all graph structures G in $\mathscr{G}-\mathscr{G}_s$ having three radial cells x, y, and z such that x and y are adjacent and $d_G(x, z)=d_G(y, z)=2r_G$.

By Kobayashi's algorithm, we obtain that for G in $\mathscr{G}_{m1}$, $t_{\min}(G)=3r_G$, and that for G in $\mathscr{G}_{m2}$, $t_{\min}(F)=3r_G+1$. On the other hand, Theorem 4.3 shows that for G in $\mathscr{G}_{m1}(\subseteq \mathscr{G}_s)$, $t(G, M_{3r})=3r_G$, and that for G in $\mathscr{G}_{m2}$ $(\subseteq \mathscr{G}-\mathscr{G}_s)$, $t(G, M_{3r})=3r_G+1$. Then we obtain the following result.

Theorem 4.4. *Let* $\mathscr{G}_m=\mathscr{G}_{m1}\cup\mathscr{G}_{m2}\cup\{G\in\mathscr{G} \mid$ *the number of cells of* G *is equal to* $1\}$. *For any graph structure* G in $\mathscr{G}_m$, $t(G, M_{3r})=t_{\min}(G)$.

5. Summary

We have given new solutions of the firing squad synchronization problem for the class of graph structures. The synchronization times of our solutions are proportional to the radius of a graph structure. Considering that the synchronization times of the solutions previously known are proportional to the number of nodes of a graph structure except special cases, our results are remarkable improvement for the problem. First, we have pointed out that the synchronization of a quasi-circuit structure is reduced to that of circuit structures. Using this fact, we have given two preliminary solutions whose synchronization times for a graph structure G are respectively $4r_G$ and $3r_G+1$ time units. Finally, we have given our final solutions whose synchronization time for a graph structure G is $3r_G$ or $3r_G+1$ time units depending upon a property of radial cells of G. Moreover, we have shown that this solution gives the minimum synchronization time for an infinite number of graph structures.

Appendix. The state transition tables of the circuit solution M_c and the semicircuit solution M_h.

Tables give the state of a cell at time $t+1$ corresponding to its own and its predecessor's state at time t. In the tables, the symbol $*$ means any state other than being specified. Table 3 gives the state transition table of M_c. Table 4 gives the state transition table of M_h other than the ones given in Table 3.

State of cell at time t	
State of predecessor at time t	New state of cell at time t + 1

A_{00}	
P_1	B_2^2
*	T

A_{01}	
B_2^2	L^1
*	Q

A_{10}	
P_0	B_1^2
*	T

A_{11}	
B_1^2	L^1
*	Q

P_2	
S_0	P_{20}
S_0^1	P_{20}
S_2	P_{20}
S_2^1	P_{20}
R_2	P_{21}
L^2	P_{22}
*	P_2

P_{20}	
L^2	P_{22}
B_1^2	P_{22}
*	P_2

P_{21}	
*	P_2

P_{22}	
*	F

P_{00}	
B_2^2	F
P_{22}	F
P_{00}	F
*	P_1

P_{11}	
*	P_0

L^1	
B_0	C_1^2
B_3	B_2^2
*	L^2

L^2	
B_0	C_1
B_3	B_2
B_{13}	B_2
C_{23}	B_2
*	Q

P_0	
R_2	P_{11}
P_{21}	P_{11}
C_1^2	F
P_{22}	F
B_2^2	F
*	P_1

P_1	
S_0	P_{00}
S_0^1	P_{00}
S_2	P_{00}
S_2^1	P_{00}
P_{20}	P_{00}
*	P_0

Q	
P_{00}	A_{00}
A_{00}	A_{01}
A_{01}	A_{00}
P_{11}	A_{10}
A_{10}	A_{11}
A_{11}	A_{10}
S_0	S_1
S_0^1	S_1
R_2	R_1
K	K
P_0	K
C_0	B_1
C_3	C_2
B_0	C_1
C_3	C_2
B_0	C_1
B_3	B_2
S_2	S_1
S_2^1	S_1
*	Q

K	
P_{11}	A_{10}
A_{10}	A_{11}
A_{11}	A_{10}
*	Q

B_0	
A_{11}	P_{21}
P_{11}	P_{22}
*	Q

B_1	
P_{20}	P_{00}
*	B_2

B_2	
A_{01}	P_{11}
P_{21}	P_{11}
K	B_0
P_0	B_0
C_3	C_{23}
C_0	B_{13}
P_{00}	F
P_{22}	F
*	B_3

B_3	
A_{11}	P_{21}
P_{11}	P_{22}
K	K
B_0	C_1
B_3	B_2
A_{00}	A_{01}
*	Q

B_1^2	
*	B_2

B_2^2	
P_{00}	F
P_{22}	F
C_1	B_0
C_0	B_{13}
C_3	C_{23}
*	B_3

B_{13}	
P_{20}	P_{00}
*	B_2

C_0	
A_{10}	P_{20}
*	C_1

C_1	
P_{21}	P_{11}
P_{22}	F
*	C_2

C_2	
A_{00}	P_{00}
P_{20}	P_{00}
K	C_0
*	C_3

C_1^2	
P_{22}	F
*	C_2

C_{23}	
A_{00}	P_{00}
P_{20}	P_{00}
K	C_0
*	C_3

C_3	
A_{10}	P_{20}
K	K
C_1	K
P_{00}	A_{00}
A_{00}	A_{01}
C_0	B_1
C_3	C_2
*	Q

R_1	
*	R_2

T	
L^2	S_0^1
C_1^2	S_0^1
R_2	S_0
S_2	R_2
S_2^1	R_2
*	T

R_2	
L^2	L^1
B_2^2	L^1
*	Q

S_1	
L^2	S_2^1
B_2^2	S_2^1
B_1^2	S_2^1
*	S_2

S_2	
S_0^1	S_1
S_2^1	S_1
*	Q

S_2^1	
B_{13}	B_2^2
C_{23}	B_2^2
B_2	L^2

S_0	
*	T

S_0^1	
C_1	B_1^2
C_2	B_2^2

Table 3

Q	
P	A
A	A
P'_{00}	A_{00}
P'_0	K'
K'	K'
V^3_3	Y^1
W^3_1	V^1_1
W_0	V_1
W_5	W_3
V_0	W_1
V_5	V_3

P'_0	
P'_{00}	F
A_{01}	P_{11}
*	P'_1

P'_1	
*	P'_2

P'_2	
A_{00}	P_{00}
*	P'_3

P'_3	
*	P'_0

P_{00}	
L^2	F

P_0	
L^2	F

V_1	
P_{20}	P_{00}
*	V_2

V^1_1	
*	V^2_2

V_2	
*	V_3

V^2_2	
*	V^3_3

V_3	
A_{01}	P_{11}
P_{21}	P_{11}
*	V_4

V^3_3	
F_{00}	F
P_{22}	F
*	V_4

V_4	
K'	V_0
*	V_5

V_5	
A_{11}	P_{21}
P_{11}	P_{22}
A_{00}	A_{01}
K'	K'
*	Q

V_0	
A_{11}	P_{21}
P_{11}	P_{22}
*	Q

V_{15}	
P_{20}	F_{00}
*	V_2

W_1	
P_{21}	P_{11}
*	W_2

W_2	
*	W_3

W_3	
A_{00}	P_{00}
P_{20}	P_{00}
*	W_4

W_4	
K'	W_0
*	W_5

W_5	
A_{10}	P_{20}
A_{01}	A_{00}
P_{00}	A_{00}
K'	K'
*	Q

W_0	
A_{10}	P_{20}
*	Q

W^3_1	
P_{22}	F
*	W_2

W_{35}	
A_{00}	P_{00}
P_{20}	P_{00}
*	W_4

Y^1	
*	Y^2

Y^2	
V_0	W^3_1
V_5	V^3_3
V_{15}	V^3_3
W_{35}	V^3_3

P'_{00}	
P'_0	F
*	P''_{00}

P''_{00}	
V^3_3	F
W^3_1	F
*	F''_{00}

K'	
P_{11}	A_{10}
A_{10}	A_{11}
A_{11}	A_{10}
*	Q

A	
P'_0	V^1_1
*	Q

A_{00}	
P''_{00}	L^2

A_{01}	
L^2	L^1

S^1_1	
Q	L^2

S^1_2	
Q	L^2

Table 4

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