Riesz multiwavelet bases

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Abstract

Compactly supported Riesz wavelets are of interest in several applications such as image processing, computer graphics and numerical algorithms. In this paper, we shall investigate compactly supported MRA Riesz multiwavelet bases in $L^2(\mathbb{R})$. An algorithm is presented to derive Riesz multiwavelet bases from refinable function vectors. To illustrate our algorithm and results in this paper, we present several examples of Riesz multiwavelet bases with short support in $L^2(\mathbb{R})$.

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1. Introduction and motivation

Let $\{\psi^1, \ldots, \psi^r\}$ be a finite set of $L^2(\mathbb{R})$ functions. Using integer dilates and shifts, one obtains a wavelet system $\{\psi_{j,k}^\ell := 2^{j/2} \psi^\ell(2^j \cdot -k): j, k \in \mathbb{Z}, \ell = 1, \ldots, r\}$. If the wavelet system forms a Riesz basis for $L^2(\mathbb{R})$, then we say that $\{\psi^1, \ldots, \psi^r\}$ generates a Riesz wavelet basis in $L^2(\mathbb{R})$.

Due to some desirable properties, Riesz wavelets have been found to be of interest in several applications. For example, Riesz wavelets have been used for numerical solutions to operator equations and for characterization of various function spaces, see [6,8,10] and many references therein. Recently, a two-dimensional Riesz wavelet basis derived from the Loop scheme has been employed in mesh compression algorithms in computer graphics with impressive performance [30]; see [21] for the mathematical analysis of the wavelet system derived from the Loop scheme in [30].
Biorthogonal wavelets with rational filters, which are closely related to MRA Riesz wavelets, have been constantly under investigation in electrical engineering (see [1] and references therein). For example, as demonstrated in [1], biorthogonal wavelets with rational filters can provide further improvements in wavelet-based image compression algorithms over traditional compactly supported biorthogonal wavelets with finite impulse filters.

On one hand, due to the complexity of implementing wavelets on a mesh or a complex with an arbitrary topology, the size of the wavelets in various applications plays a critical role and is preferred to be as small as possible (see [8, 10, 20, 21, 29, 30]). For example, it may be easier to adapt wavelet bases with shorter support on the real line to wavelets on the interval [0, 1] (see [9, 18, 23]) and the obtained boundary wavelets may have a simpler structure so that it may be easier to implement such wavelet systems, see [9, 18, 23] and references therein.

On the other hand, it is required in some applications for various purposes that a wavelet should have a high enough smoothness order. For example, for wavelet applications in computer graphics, it is strongly preferred that the wavelets are at least twice differentiable so that the reconstructed subdivision surfaces have continuous curvatures. But it is a well-known fact that in order to have a wavelet system with a higher smoothness order, it is necessary to enlarge the support of the wavelets. Comparing with traditional biorthogonal wavelets with compact support, Riesz wavelets provide a better trade-off in order to achieve both high smoothness order and short support of the wavelets. This is possible partially because the dual Riesz basis of a given Riesz wavelet basis generally has infinite support. In other words, one achieves the short support property of a Riesz wavelet at the cost of the infinite support of its dual system. The infinite support of its dual system is not a serious drawback or not a drawback at all in certain applications for several reasons. First of all, the dual wavelet filters associated with the dual Riesz system with infinite support are generally rational filters, which can be efficiently implemented in applications, as argued in many papers in the literature (see [1] and references therein). Secondly, for some applications such as wavelets in computer graphics [30] and wavelet-based numerical algorithms for solving operator equations [8], the dual system of a Riesz wavelet basis is not explicitly used and a fast wavelet reconstruction algorithm is still available since the Riesz wavelet itself has short support.

MRA Riesz wavelets derived from scalar refinable functions have been investigated in a few papers in the literature, for example, see [6, 16, 20, 21, 29, 31]. In this paper, we are interested in MRA Riesz multiwavelet bases in $L_2(\mathbb{R})$. As mentioned before, a high smoothness order of a scalar refinable function and the shortness of its support are two mutually contradicting requirements. By increasing the number of generating functions, it is known that a refinable function vector in general could be much smoother than the corresponding scalar refinable function when their supports are more or less the same [11, 13, 18, 23, 28]. Since short support of a Riesz wavelet is critical for various applications, it is natural to consider Riesz multiwavelet bases derived from refinable function vectors. However, to our best knowledge, only Riesz wavelet bases derived from scalar refinable functions have been discussed in the literature [6, 16, 20, 21, 29, 31], except biorthogonal multiwavelets with compact support (for example, see [13] for many references on biorthogonal multiwavelets), there are very few papers in the literature discussing Riesz multiwavelet bases with compact support. Since multiwavelets have some interesting and desirable properties as discussed above, it is the purpose of this paper to further extend the results on Riesz wavelet bases derived from scalar refinable functions in [5, 6, 16, 20, 21, 29, 31] to Riesz multiwavelet bases derived from refinable function vectors. In this paper, we will present some sufficient conditions and an algorithm for constructing Riesz multiwavelet bases in $L_2(\mathbb{R})$ from refinable function vectors. Our step-by-step algorithm may allow interested readers to construct Riesz multiwavelet bases with short support for various applications.

This paper is also motivated by the interesting example of a Riesz multiwavelet system constructed in Jia and Liu [24] and the Riesz wavelet bases derived from B-splines in Han and Shen [20]. Our approach in this paper on Riesz multiwavelet bases is closely related and similar to that in [20]. However, as we shall see later, this paper is not a straightforward generalization of the results in [20] and some difficulties arising from refinable function vectors and matrix refinement equations have to be overcome in order to construct Riesz multiwavelet bases from refinable function vectors. As is often the case for studying multiwavelets and refinable function vectors, some technical results and proofs are needed in our investigation of Riesz multiwavelet bases. Though some results and proofs in this paper seem technical or complicated at the first glance, one may find the ideas in these proofs and results useful in other problems related to refinable function vectors and multiwavelets.

Recently, multivariate vector subdivision schemes have been studied in [4, 22] and are of interest in generating subdivision curves and surfaces in computer-aided geometric design. Some interesting examples of smooth two-dimensional symmetric refinable function vectors are constructed in [4, 22]. Some of the refinable function vectors...
constructed in [4,22] have very short support (for example, $[-1, 1]^2$ or $[-2, 2]^2$) and good smoothness orders (for example, $C^1$ or $C^2$). It is quite appealing to build Riesz multiwavelet bases from such refinable function vectors, since they may have some interesting applications in computer graphics and numerical algorithms. This is also another motivation for us to investigate Riesz multiwavelet bases. To avoid further complicated notations and technical results, we restrict ourselves to Riesz multiwavelet bases in dimension one in this paper. Most techniques and ideas developed in this paper will be useful for studying multivariate Riesz multiwavelet bases later.

In general, a multiwavelet is derived from a refinable function vector via a multiresolution analysis. We say that $\phi = (\phi^1, \ldots, \phi^r)^T$ is a **refinable function vector** if

$$
\phi = 2 \sum_{k \in \mathbb{Z}} a(k) \phi(2 \cdot -k),
$$

(1.1)

where $a : \mathbb{Z} \mapsto \mathbb{C}^{r \times r}$ is a sequence of $r \times r$ matrices on $\mathbb{Z}$, called the (**matrix** mask) with multiplicity $r$ for $\phi$. In the frequency domain, the matrix refinement equation in (1.1) can be rewritten as

$$
\hat{\phi}(2 \xi) = \hat{a}(\xi) \hat{\phi}(\xi), \quad \xi \in \mathbb{R},
$$

(1.2)

where $\hat{a}$ is the **Fourier series** of the mask $a$ given by

$$
\hat{a}(\xi) := \sum_{k \in \mathbb{Z}} a(k) e^{-i k \xi}, \quad \xi \in \mathbb{R},
$$

(1.3)

and the Fourier transform $\hat{f}$ of $f \in L_2(\mathbb{R})$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(t) e^{-i t \xi} \, dt$ and can be extended to tempered distributions. For simplicity, we also call $\hat{a}$ the mask for $\phi$.

A multiwavelet $\psi$ is generally derived from the refinable function vector $\phi$ via

$$
\hat{\psi}(2 \xi) := \hat{b}(\xi) \hat{\phi}(\xi), \quad \xi \in \mathbb{R}
$$

(1.4)

for some $r \times r$ matrix $\hat{b}$ of $2\pi$-periodic measurable functions. We say that $\psi = (\psi^1, \ldots, \psi^r)^T$ generates a **Riesz multiwavelet basis** in $L_2(\mathbb{R})$ if the linear span of $\{\psi^{\ell}_{j,k} := 2^{j/2} \psi^\ell(2^j \cdot -k) : j, k \in \mathbb{Z}, \ell = 1, \ldots, r\}$ is dense in $L_2(\mathbb{R})$, and there exist two positive constants $A$ and $B$ such that

$$
A \sum_{\ell=1}^r \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |c^{\ell}_{j,k}|^2 \leq \left\| \sum_{\ell=1}^r \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c^{\ell}_{j,k} \psi^{\ell}_{j,k} \right\|_{L_2(\mathbb{R})}^2 \leq B \sum_{\ell=1}^r \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |c^{\ell}_{j,k}|^2
$$

(1.5)

for all finitely supported sequences $\{c^{\ell}_{j,k}\}_{j,k \in \mathbb{Z}, \ell=1,\ldots,r}$. It is of interest to investigate under what conditions the multiwavelet $\psi$ in (1.4) will generate a Riesz multiwavelet basis in $L_2(\mathbb{R})$. In this paper, we shall present some sufficient conditions on $a$ and $b$ so that $\psi$ generates a Riesz multiwavelet basis in $L_2(\mathbb{R})$.

For a smooth function $f$, in this paper $f^{(j)}$ denotes the $j$th derivative of the function $f$. We say that $\psi = (\psi^1, \ldots, \psi^r)^T$ has $m$ **vanishing moments** if $\hat{\psi}^{(j)}(0) = 0$ for all $\ell = 1, \ldots, r$ and $j = 0, \ldots, m - 1$. The order of vanishing moments is one of the most important properties of a wavelet system.

The structure of this paper is as follows. In Section 2, we shall present an algorithm for constructing compactly supported Riesz multiwavelet bases in $L_2(\mathbb{R})$ with any preassigned order of vanishing moments from refinable function vectors. In order to construct a Riesz multiwavelet basis in $L_2(\mathbb{R})$, we need to calculate a quantity $\mu_2(\hat{a})$ in our algorithm. In Section 3, we shall discuss how to estimate and calculate the quantity $\mu_2(\hat{a})$. In Section 4, we shall present several examples of Riesz multiwavelet bases with high vanishing moments and short support in $L_2(\mathbb{R})$ to illustrate the algorithm in Section 2. Finally, in Section 5, we shall present a proof for the algorithm in Section 2. Some auxiliary and technical results that we need in our proof of the algorithm will be given in Section 5.

### 2. An algorithm for constructing Riesz multiwavelet bases

In this section, we shall present a step-by-step algorithm for constructing Riesz multiwavelet bases on the real line. The proof of the algorithm will be given in Section 5.

The following lemma is needed in our algorithm for constructing Riesz multiwavelet bases with high vanishing moments.
Lemma 2.1. Let \( a : \mathbb{Z} \to \mathbb{C}^{r \times r} \) be a sequence of \( r \times r \) matrices on \( \mathbb{Z} \) such that
\[
1 \text{ is a simple eigenvalue of } \hat{a}(0) \text{ and } \det(2^j I_r - \hat{a}(0)) \neq 0 \text{ for all } j \in \mathbb{N}.
\] (2.1)

Then one can uniquely define a sequence of \( r \times 1 \) vectors \( m^a(j), j \in \mathbb{N} \cup \{0\} \) via the following recursive formula:
\[
\hat{a}(0)m^a(0) = m^a(0) \quad \text{with the first nonzero component of } m^a(0) \text{ being } 1,
\]
\[
m^a(j) := [2^j I_r - \hat{a}(0)]^{-1} \sum_{\ell=0}^{j-1} \frac{j!}{\ell!(j-\ell)!} \hat{a}^{(j-\ell)}(0)m^a(\ell), \quad j \in \mathbb{N}.
\] (2.2)

If \( \phi \) is an \( r \times 1 \) vector of compactly supported tempered distributions on \( \mathbb{R} \) such that \( \hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi) \), then there exists a constant \( c \) such that \( \hat{\phi}(j)(0) = cm^a(j) \) for all \( j \in \mathbb{N} \cup \{0\} \).

Proof. By the Leibniz differentiation formula, it follows from \( \hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi) \) that
\[
2^j \hat{\phi}(j)(0) = \hat{a}(0)\hat{\phi}(j)(0) + \sum_{\ell=0}^{j-1} \frac{j!}{\ell!(j-\ell)!} \hat{a}^{(j-\ell)}(0)\hat{\phi}^{(\ell)}(0).
\]

Now the claim follows directly from the above identity. \(\square\)

In order to state the algorithm for constructing Riesz multiwavelet bases, we need to recall a quantity \( v_2(a) \) defined in [15]. The convolution of two sequences is defined to be
\[
[u * v](j) := \sum_{k \in \mathbb{Z}} u(k)v(j-k), \quad u \in (\ell_0(\mathbb{Z}))^{r \times m}, \quad v \in (\ell_0(\mathbb{Z}))^{m \times n},
\]
where \((\ell_0(\mathbb{Z}))^{r \times m}\) denotes the linear space of all finitely supported sequences of \( r \times m \) matrices on \( \mathbb{Z} \). Clearly, \( u * \hat{u} = \hat{u}u \). Let \( a \) be a matrix mask with multiplicity \( r \). We say that \( a \) satisfies the sum rules of order \( m \) [13,15,25] if there exists a sequence \( y \in (\ell_0(\mathbb{Z}))^{1 \times r} \) such that \( \hat{y}(0) \neq 0 \),
\[
[y_a(\cdot)]^{(j)}(0) = \hat{y}^{(j)}(0) \quad \text{and} \quad [y_a(\cdot)]^{(j)}(\pi) = 0, \quad \forall j = 0, \ldots, m - 1.
\] (2.3)

For \( y \in (\ell_0(\mathbb{Z}))^{1 \times r} \) and a positive integer \( m \), as in [15], we define the space \( V_{m,y} \) by
\[
V_{m,y} := \{ v \in (\ell_0(\mathbb{Z}))^{r \times 1} : [y_a(\cdot)]^{(j)}(0) = 0, \quad \forall j = 0, \ldots, m - 1 \}.
\] (2.4)

By convention, \( V_{0,y} := (\ell_0(\mathbb{Z}))^{r \times 1} \). Note that the above equations in (2.3) and (2.4) depend only on the values \( \hat{y}^{(j)}(0) \), \( j = 0, \ldots, m - 1 \).

For a mask \( a \) with multiplicity \( r \), a sequence \( y \in (\ell_0(\mathbb{Z}))^{1 \times r} \) and a nonnegative integer \( m \), we define
\[
\rho_{m}(a, y) := \sup \left\{ \limsup_{n \to \infty} \|a_n * v\|_{(\ell_2(\mathbb{Z}))^{r \times 1}}^{1/n} : v \in V_{m,y} \right\}.
\] (2.5)

where \( a_n(\xi) := a(2^{n-1}\xi) \cdots a(2\xi) a(\xi) \) and \( \|u\|_{\ell^2(\mathbb{Z})}^2 := \sum_{k \in \mathbb{Z}} |u(k)|^2 \) for \( u \in \ell^2(\mathbb{Z}) \). Define
\[
\rho(a) := \inf \left\{ \rho_{m}(a, y) : (2.3) \text{ holds for some } m \in \mathbb{N} \cup \{0\} \text{ and some } y \in (\ell_0(\mathbb{Z}))^{1 \times r} \text{ with } \hat{y}(0) \neq 0 \right\}.
\] (2.6)

As in [15, p. 61], we define the following quantity:
\[
v_2(a) := -1/2 - \log_2 \rho(a).
\] (2.7)

The above quantity \( v_2(a) \) plays a very important role in characterizing the convergence of a vector cascade algorithm in Sobolev spaces (see [2,3,15,17,28,32] and many references therein for detail). In the above definition of \( \rho(a) \), it seems that the sequences \( y \) (more precisely, the vectors \( \hat{y}^{(j)}(0), j = 0, \ldots, m - 1 \)) are not uniquely determined. We shall see in Proposition 5.2 that up to a scalar multiplicative constant, the vectors \( \hat{y}^{(j)}(0), j \in \mathbb{N} \cup \{0\} \) are quite often uniquely determined.

For a finitely supported matrix mask \( a \), the quantity \( v_2(a) \) can be numerically computed by finding the spectral radius of certain finite matrix using an interesting algorithm in [26]. However, for an infinitely supported mask \( a \), the
algorithm in [26] cannot be directly used to calculate $v_2(a)$. In the following, we introduce a quantity $\mu_2(a)$, which is closely related to the quantity $v_2(a)$.

$$\mu_2(a) := \sup \left\{ v_2(\tilde{a}) : \hat{a}(\xi) = q_1(\xi)d(\xi), \hat{a}(\xi) = q_2(\xi)d(\xi) \text{ satisfying } q_1(0) = q_2(0) \neq 0, \right.$$ \( |q_1(\xi)| \leq |q_2(\xi)| \forall \xi \in \mathbb{R} \right\}, \tag{2.8}$$

where $q_2$ and all components of $d$ are $2\pi$-periodic trigonometric polynomials. We shall see from Theorem 3.1 that we always have $\mu_2(a) \leq v_2(a)$. Moreover, if $a$ is finitely supported, then $\mu_2(a) = v_2(a)$. By the algorithm in [26] and Theorem 3.2 in this paper, we can estimate the quantity $\mu_2(a)$.

Riesz multiwavelet bases in $L_2(\mathbb{R})$ can be obtained via the following algorithm, whose proof will be given in Section 5.

**Algorithm 2.2.** Let $a : \mathbb{Z} \mapsto \mathbb{C}^{r \times r}$ be a finitely supported matrix mask on $\mathbb{Z}$ with multiplicity $r$ such that $v_2(a) > 0$. Let $m$ be a positive integer.

1. (Since $v_2(a) > 0$, (2.1) must hold [15, Theorem 4.3] and therefore Lemma 2.1 applies.)

2. (Choose a finitely supported sequence $b : \mathbb{Z} \mapsto \mathbb{C}^{r \times r}$ such that

$$\sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!} \hat{b}^{(j-\ell)}(0)m^a(j) = 0, \quad j = 0, \ldots, m-1, \tag{2.9}$$

and

$$\det \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi + \pi) \\ \hat{b}(\xi) & \hat{b}(\xi + \pi) \end{bmatrix} \neq 0 \quad \forall \xi \in \mathbb{R}. \tag{2.10}$$

3. Estimate $\mu_2(\tilde{a})$, where the matrix mask $\tilde{a}$ is defined to be

$$\tilde{a}(\xi) := [I_r 0] \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi + \pi) \\ \hat{b}(\xi) & \hat{b}(\xi + \pi) \end{bmatrix}^{-1} \begin{bmatrix} I_r \\ 0 \end{bmatrix}. \tag{2.11}$$

If $\mu_2(\tilde{a}) > 0$, then $\psi$ generates a Riesz multiwavelet basis in $L_2(\mathbb{R})$ and $\psi$ has $m$ vanishing moments, where the wavelet function vector $\psi$ is defined to be $\hat{\psi}(2\xi) := b(\xi)\phi(\xi)$ and $\phi$ denotes the compactly supported refinable function vector associated with mask $a$ satisfying $\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ with $\hat{\phi}(0) \neq 0$.

**Remark.** For the scalar case $r = 1$, using extensive techniques from Fourier analysis, we are able to show that $\mu_2(\tilde{a}) = v_2(\tilde{a})$ and in fact, Algorithm 2.2 presents a necessary and sufficient condition for $\psi$ to generate a Riesz wavelet basis in $L_2(\mathbb{R})$ with $m$ vanishing moments. It is quite possible that this is also true for the general case of multiwavelets and is currently under investigation.

**3. Estimate the quantity $\mu_2(\tilde{a})$ and $v_2(\tilde{a})$**

In this section, we shall discuss how to estimate the quantities $\mu_2(\tilde{a})$ and $v_2(\tilde{a})$ in Algorithm 2.2.

**Theorem 3.1.** Let $d$ be an $r \times r$ matrix of $2\pi$-periodic functions in $C^\infty(\mathbb{R})$. Let $q_1$ and $q_2$ be $2\pi$-periodic functions in $C^\infty(\mathbb{R})$. Define two masks $a_1$ and $a_2$ by $\hat{a}_1(\xi) := q_1(\xi)d(\xi)$ and $\hat{a}_2(\xi) := q_2(\xi)d(\xi)$.

(i) If $q_1(0) = q_2(0) \neq 0$ and $q_1(\xi)/q_2(\xi)$ is bounded in a neighborhood of $\xi = \pi$, and if $a_2$ satisfies the sum rules of order $m$, then $a_1$ must satisfy the sum rules of order at least $m$.

(ii) If $|q_1(\xi)| \leq |q_2(\xi)|$ for all $\xi \in \mathbb{R}$ and $q_1(0) = q_2(0) \neq 0$, then $v_2(a_2) \leq v_2(a_1)$.

**Proof.** Suppose that $a_2$ satisfies the sum rules of order $m$ with a sequence $\gamma_2 \in (\ell_0(\mathbb{Z}))^{1 \times r}$, that is,
\[ \hat{\gamma}_2(2\xi)\hat{a}_2(\xi) = \hat{\gamma}_2(\xi) + o(|\xi|^{-1}) \quad \text{and} \quad \hat{\gamma}_2(2\xi)\hat{a}_2(\xi + \pi) = o(|\xi|^{-1}), \quad \xi \to 0. \] (3.1)

Since \( q_1(0) = q_2(0) \neq 0 \), by [15, Lemma 2.2] or [19, Lemma 3.2], there exists a \( 2\pi \)-periodic trigonometric polynomial \( q \) such that

\[ q(0) = 1 \quad \text{and} \quad q(2\xi)q_1(\xi) = q(\xi)q_2(\xi) + o(|\xi|^{-1}), \quad \xi \to 0. \] (3.2)

Denote \( \hat{\gamma}_1(\xi) := q(\xi)\hat{\gamma}_2(\xi) \). Now we show that \( a_1 \) must satisfy the sum rules of order at least \( m \) with the sequence \( y_1 \).

By (3.1) and (3.2), we deduce that when \( \xi \to 0 \),

\[
\hat{\gamma}_1(2\xi)\hat{a}_1(\xi) = q(2\xi)q_1(\xi)\hat{\gamma}_2(2\xi)\hat{d}(\xi) = q(\xi)q_2(\xi)\hat{\gamma}_2(2\xi)\hat{d}(\xi) + o(|\xi|^{-1}) = q(\xi)[\hat{\gamma}_2(2\xi)\hat{a}_2(\xi) + o(|\xi|^{-1})] = q(\xi)(\hat{\gamma}_2(\xi) + o(|\xi|^{-1})) = \hat{\gamma}_1(\xi) + o(|\xi|^{-1}).
\]

That is, \( [\hat{\gamma}_1(2\cdot)\hat{a}_1(\cdot)]^{(j)}(0) = \hat{\gamma}_1^{(j)}(0) \) for all \( j = 0, \ldots, m - 1 \).

On the other hand, let \( q_3(\xi) := q(2\xi)q_1(\xi)/q_2(\xi) \). By \( q_1(0) = q_2(0) \neq 0 \), we have \( q_3(0) = q(0) = 1 \) and \( q_3(\xi) \) is bounded in a neighborhood of \( \xi = \pi \) since \( q_1(\xi)/q_2(\xi) \) is bounded in a neighborhood of \( \xi = \pi \). Therefore, by (3.1), when \( \xi \to 0 \), we have

\[
\hat{\gamma}_1(2\xi)\hat{a}_1(\xi + \pi) = q(2\xi)q_1(\xi + \pi)\hat{\gamma}_2(2\xi)\hat{d}(\xi + \pi) = q_3(\xi + \pi)\hat{\gamma}_2(2\xi)\hat{a}_2(\xi + \pi) = o(|\xi|^{-1}).
\]

That is, \( [\hat{\gamma}_1(2\cdot)\hat{a}_1(\cdot)]^{(j)}(0) = 0 \) for all \( j = 0, \ldots, m - 1 \). Hence, if \( a_2 \) satisfies the sum rules of order \( m \) with a sequence \( y_2 \), then \( a_1 \) must satisfy the sum rules of order \( m \) with the sequence \( y_1 \) such that \( \hat{\gamma}_1(\xi) = q(\xi)\hat{\gamma}_2(\xi) \). So, (i) holds.

By the assumption in (ii), \( |q_1(\xi)/q_2(\xi)| \leq 1 \) for all \( \xi \in \mathbb{R} \) and therefore, \( q_1(\xi)/q_2(\xi) \) is bounded in a neighborhood of \( \xi = \pi \). Suppose that \( a_2 \) satisfies the sum rules of order \( m \) with a sequence \( y_2 \in (\ell_0(\mathbb{Z}))^{1 \times r} \). By what has been proved, \( a_1 \) must also satisfy the sum rules of order \( m \) with the vector \( y_1 \), where \( \hat{\gamma}_1(\xi) = q(\xi)\hat{\gamma}_2(\xi) \) and \( q \) is a \( 2\pi \)-periodic trigonometric polynomial with \( q(0) = 1 \). Since \( \hat{\gamma}_1(\xi) = q(\xi)\hat{\gamma}_2(\xi) \) for a \( 2\pi \)-periodic trigonometric polynomial \( q \) with \( q(0) = 1 \), by [15, Lemma 3.3], it is easy to see that \( V_{m,y_1} = V_{m,y_2} \). Denote

\[
\overline{a}_1(n) := \overline{a}_1(2^{n-1}\xi) \cdots \overline{a}_1(2\xi)\overline{a}_1(\xi) \quad \text{and} \quad \overline{a}_2(n) := \overline{a}_2(2^{n-1}\xi) \cdots \overline{a}_2(2\xi)\overline{a}_2(\xi).
\]

Since \( \overline{a}_1(\xi) = q_1(\xi)d(\xi) \) and \( \overline{a}_2(\xi) = q_2(\xi)d(\xi) \), we have

\[
\overline{a}_1(n)(\xi) = q_1(2^{n-1}\xi) \cdots q_1(2\xi)q_1(\xi)d_n(\xi)
\]

and

\[
\overline{a}_2(n)(\xi) = q_2(2^{n-1}\xi) \cdots q_2(2\xi)q_2(\xi)d_n(\xi),
\]

where \( d_n(\xi) := d(2^{n-1}\xi) \cdots d(2\xi)d(\xi) \).

For any \( v \in V_{m,y_2} \), by the assumption in (ii), we have

\[
\|a_1 \ast v\|_{(\ell_2(\mathbb{Z}))^{r \times 1}}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{v}(\xi)^T\overline{a}_1(n)(\xi)^T\overline{a}_1(n)(\xi)\overline{v}(\xi)\,d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |q_1(2^{n-1}\xi) \cdots q_1(2\xi)q_1(\xi)|^2 \overline{v}(\xi)^T\overline{a}_1(n)(\xi)^Td_n(\xi)\overline{v}(\xi)\,d\xi
\]

\[
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |q_2(2^{n-1}\xi) \cdots q_2(2\xi)q_2(\xi)|^2 \overline{v}(\xi)^T\overline{a}_2(n)(\xi)^Td_n(\xi)\overline{v}(\xi)\,d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{v}(\xi)^T\overline{a}_2(n)(\xi)^T\overline{a}_2(n)(\xi)\overline{v}(\xi)\,d\xi
\]

\[
= \|a_2 \ast v\|_{(\ell_2(\mathbb{Z}))^{r \times 1}}^2.
\]
Consequently,
\[
\limsup_{n \to \infty} \| a_{1,n} \ast v \|_{1/n}^{1/n} \leq \limsup_{n \to \infty} \| a_{2,n} \ast v \|_{1/n}^{1/n} \quad \forall v \in \mathcal{V}_{m, y_2}.
\]

Since \( \mathcal{V}_{m, y_1} = \mathcal{V}_{m, y_2} \), by (2.5) and the above inequality, we must have \( \rho_m(a_1, y_1) \leq \rho_m(a_2, y_2) \). Hence, we conclude that \( \rho(a_1) \leq \rho(a_2) \) and therefore,
\[
v_2(a_2) := -1/2 - \log_2 \rho(a_2) \leq -1/2 - \log_2 \rho(a_1) =: v_2(a_1).
\]

This completes the proof. \( \square \)

Now we state the result for estimating the quantities \( \mu_2(\hat{a}) \) and \( v_2(\hat{a}) \) in Algorithm 2.2.

**Theorem 3.2.** Write \( \hat{\hat{a}}(\xi) := q(\xi)^{-1} \hat{\hat{a}}(\xi) \) such that \( \hat{\hat{a}} \in (\ell_0(\mathbb{Z}))^{r \times r} \) is a finitely supported sequence of \( r \times r \) matrices on \( \mathbb{Z} \) and \( q \) is a \( 2\pi \)-periodic trigonometric polynomial satisfying
\[
q(0) = 1 \quad \text{and} \quad 0 < q_{\min} \leq q(\xi) \leq q_{\max} < \infty \quad \forall \xi \in \mathbb{R}
\]
for some positive numbers \( q_{\min} \) and \( q_{\max} \). (In case that \( q(\xi) \) is not real-valued but \( q(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \), we can consider \( \hat{\hat{a}}(\xi) = \frac{1}{|q(\xi)|^r} \overline{q(\xi)} \hat{a}(\xi) \), that is, replace \( q \) and \( \hat{a} \) by \( |q(\xi)|^2 \) and \( \overline{q(\xi)} \hat{a}(\xi) \), respectively.) For any positive even integer \( n \), define
\[
q_1(\xi) := \frac{2}{q_{\max} + q_{\min}} \sum_{j=0}^{n-1} \left( 1 - \frac{2q(\xi)}{q_{\max} + q_{\min}} \right)^j \left( 1 - \frac{2q(\xi)}{q_{\max} + q_{\min}} \right)^n \left( \frac{(1 - q(\xi))^2}{q_{\max} - q(\xi)} \right)
\]
and
\[
q_2(\xi) := \frac{2}{q_{\max} + q_{\min}} \sum_{j=0}^{n-1} \left( 1 - \frac{2q(\xi)}{q_{\max} + q_{\min}} \right)^j \left( 1 - \frac{2q(\xi)}{q_{\max} + q_{\min}} \right)^n \left( \frac{(1 - q(\xi))^2}{q_{\min} - q(\xi)} \right).
\]

Then both \( q_1 \) and \( q_2 \) converge to \( 1/q \) exponentially fast in the space \( L_{\infty}(\mathbb{R}) \) as \( n \to \infty \). In addition,
\[
q_1(0) = q_2(0) = 1 \quad \text{and} \quad q_1(\xi) \leq \frac{1}{q(\xi)} \leq q_2(\xi) \quad \forall \xi \in \mathbb{R}.
\]

Define \( \hat{a}_1, \hat{a}_2 \in (\ell_0(\mathbb{Z}))^{r \times r} \) by
\[
\hat{\hat{a}}_1(\xi) := q_1(\xi) \hat{a}(\xi) \quad \text{and} \quad \hat{\hat{a}}_2(\xi) := q_2(\xi) \hat{a}(\xi).
\]

If \( \hat{a} \) satisfies the sum rules of order \( \hat{m} \), then both \( a_1 \) and \( a_2 \) must satisfy the sum rules of order at least \( \hat{m} \). Moreover, we have
\[
v_2(\hat{a}_2) \leq \mu_2(\hat{a}) \leq v_2(\hat{a}_1).
\]

**Proof.** By calculation, we have the following identity:
\[
\frac{1}{x} = \frac{1}{x} - x^n + \sum_{j=0}^{n-1} (1 - x)^j \quad \forall x > 0, \ n \in \mathbb{N}.
\]

Setting \( x = 2q(\xi)/(q_{\max} + q_{\min}) \) and applying the above identity twice, we have
\[
\frac{1}{q(\xi)} = \frac{2}{q_{\max} + q_{\min}} \sum_{j=0}^{n-1} \left( 1 - \frac{2q(\xi)}{q_{\max} + q_{\min}} \right)^j \left( 1 - \frac{2q(\xi)}{q_{\max} + q_{\min}} \right)^n \left( \sum_{\ell=0}^{n-1} (1 - q(\xi))^{\ell} + \frac{(1 - q(\xi))^{n+1}}{q(\xi)} \right).
\]
Taking $n_1 = 2$ and replacing $q(\xi)$ in the denominator of the last term in the above identity by $q_{\min}$ and $q_{\max}$, we see that (3.3) holds since $n$ is an even integer and

$$q_2(\xi) - \frac{1}{q(\xi)} = \left(1 - \frac{2q(\xi)}{q_{\max} + q_{\min}}\right)^n \left(1 - q(\xi)\right)^\nu \left(\frac{1}{q_{\min}} - \frac{1}{q(\xi)}\right) \geq 0 \quad \forall \xi \in \mathbb{R}.$$ 

Since

$$\left|1 - \frac{2q(\xi)}{q_{\max} + q_{\min}}\right| \leq \frac{q_{\max} - q_{\min}}{q_{\max} + q_{\min}} < 1 \quad \forall \xi \in \mathbb{R},$$

both $q_1$ and $q_2$ converge to $1/q$ exponentially fast in the space $L_\infty(\mathbb{R})$ as $n \to \infty$. By Theorem 3.1, we conclude that (3.5) holds.

If $\hat{a}$ satisfies the sum rules of order $\hat{m}$, by $q_1(0) = q_2(0) = 1/q(0) \neq 0$, it follows from Theorem 3.1 that both $\hat{a}_1$ and $\hat{a}_2$ must satisfy the sum rules of order $\hat{m}$. The relation $v_2(\hat{a}_2) \leq v_2(\hat{a}) \leq v_2(\hat{a}_1)$ is a direct consequence of Theorem 3.1. □

4. Some examples of Riesz multiwavelet bases on the real line

By Algorithm 2.2 in Section 2, in this section we shall present several examples of MRA Riesz multiwavelet bases in $L_2(\mathbb{R})$ with short support derived from some refinable function vectors.

Example 4.1. The Hermite cubic spline function vector $\phi = (\phi^1, \phi^2)^T$ is given by

$$\{\phi^1(x) := (1 - 3x^2 - 2x^3)\chi_{[-1,0]} + (1 - 3x^2 + 2x^3)\chi_{[0,1]},$$
$$\phi^2(x) := (x + 2x^2 + x^3)\chi_{[-1,0]} + (x - 2x^2 + x^3)\chi_{[0,1]}\}.$$ 

It is evident that $\phi \in (C^1(\mathbb{R}))^{2 \times 1}$ is a Hermite interpolant of order 1, that is,

$$\phi^1(k) = \delta(k), \quad [\phi^1]'(k) = 0, \quad \phi^2(k) = 0, \quad [\phi^2]'(k) = \delta(k) \quad \forall k \in \mathbb{Z},$$

where $\delta$ is the Dirac sequence such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$. See [13,15] for more discussion on refinable Hermite interpolants. Moreover, $\phi$ is a refinable function vector satisfying $\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$, where the mask $a$ is given by

$$a(-1) = \begin{bmatrix} 1/4 & 3/8 \\ -1/16 & -1/16 \end{bmatrix}, \quad a(0) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1/4 & -3/8 \\ 1/16 & -1/16 \end{bmatrix},$$

with $a(k) = 0$ for all $k \in \mathbb{Z}\setminus\{-1,0,1\}$. By Lemma 2.1, it follows from (2.2) that

$$m^a(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad m^a(1) = \begin{bmatrix} 0 \\ 1/15 \end{bmatrix}, \quad m^a(2) = \begin{bmatrix} 15 \\ 0 \end{bmatrix}, \quad m^a(3) = \begin{bmatrix} 0 \\ 1/315 \end{bmatrix}, \quad m^a(4) = \begin{bmatrix} 1/560 \\ 0 \end{bmatrix}.$$ 

By calculation, $v_2(a) = 2.5$. Consider a sequence $b \in (\ell_0(\mathbb{Z}))^{2 \times 2}$ supported on $\{-1,0,1\}$. To have a nontrivial solution to the system of linear equations in (2.9), we find that $m \leq 4$. Taking $m = 3$ and imposing symmetry on the sequence $b$ we find a parameterized solution for $b$ as follows:

$$b(-1) = \begin{bmatrix} -2 \\ -s \\ -15 \\ -15s - 2 \end{bmatrix}, \quad b(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b(1) = \begin{bmatrix} -2 \\ s \\ 15 \\ 15s - 2 \end{bmatrix},$$

and $b(k) = 0$ for all $k \in \mathbb{Z}\setminus\{-1,0,1\}$. Consequently, we have

$$\det\begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi + \pi) \\ \hat{b}(\xi) & \hat{b}(\xi + \pi) \end{bmatrix} = \frac{1}{8}(39s - 7 + 21s + 111)\cos(2\xi).$$

The above determinant is nonzero for all $\xi \in \mathbb{R}$ if and only if $|39s - 7| > |21s + 111|$, that is, $s \in (-\infty, -1/15) \cup (1, \infty)$. Now by (2.11), we have $\hat{a}(\xi) = q(\xi)^{-1}\hat{a}(\xi)$, where

$$q(\xi) = \frac{1}{60s + 4}(39s - 7 + 21s + 111)\cos(2\xi),$$

and
Using \( n = 4 \) in Theorem 3.2, we find the lower bound of the quantities \( \mu_2(\hat{a}) \) and \( v_2(\hat{a}) \) in Example 4.1 for \( s \in (-16, -1/15) \cup (1, 16) \) (left) and \( s \in (-1/2, -1/5) \) (right).

\[
\hat{a}(-2) = \frac{1}{240s + 16} \begin{bmatrix}
15s + 17 & -14s - 2 \\
-48 & 84s + 16
\end{bmatrix},
\]

\[
\hat{a}(0) = \frac{1}{240s + 16} \begin{bmatrix}
90s - 26 & 0 \\
0 & 312s + 32
\end{bmatrix},
\]

\[
\hat{a}(2) = \frac{1}{240s + 16} \begin{bmatrix}
15s + 17 & 14s + 2 \\
48 & 84s + 16
\end{bmatrix},
\]

with \( \hat{a}(k) = 0 \) for all \( k \in \mathbb{Z} \setminus \{-2, -1, 0, 1, 2\} \). For \( s \in (-\infty, -1/15) \cup (1, \infty) \), we have

\[
q_{\min} := \min_{\xi \in \mathbb{R}} q(\xi) = \frac{|39s - 7| - |21s + 11|}{|60s + 4|} = \min \left\{ 1, \frac{9|s - 1|}{|30s + 2|} \right\}
\]

and

\[
q_{\max} := \max_{\xi \in \mathbb{R}} q(\xi) = \frac{|39s - 7| + |21s + 11|}{|60s + 4|} = \max \left\{ 1, \frac{9|s - 1|}{|30s + 2|} \right\}.
\]

For any \( s \in (-\infty, -1/15) \cup (1, \infty) \), \( \psi \) has at least 3 vanishing moments and \( \psi \) is supported on \([-1, 1]\). When \( s = -7/33 \), \( \psi \) has 4 vanishing moments.

Since \( \hat{a}(0) = \text{diag}(1, \frac{60s + 8}{15s + 1}) \), we see that \( |60s + 8| < |15s + 1| \) if and only if \(-7/15 < s < -1/5\). Taking \( n = 4 \) in Theorem 3.2, we can estimate \( \mu_2(\hat{a}) \) and \( v_2(\hat{a}) \). See Fig. 1 for more detail. We find that when \( s = -125/512 \), \( 1.932383 < \mu_2(\hat{a}) \leq v_2(\hat{a}) < 1.935011 \), which is almost the best one. Therefore, when \( s = -125/512 \), \( \psi \) has 3 vanishing moments and by Algorithm 2.2, \( \psi \) generates a Riesz multiwavelet basis for \( L_2(\mathbb{R}) \) and therefore [8] for the Sobolev space \( W^a(\mathbb{R}) \) with \(-1.932383 \leq \alpha < 2.5\), where \( W^a(\mathbb{R}) := \{ f \in L_2(\mathbb{R}) : \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^a \, d\xi < \infty \} \). See Fig. 2 for the refinable function vector \( \phi \) and the multiwavelet \( \psi \) with \( s = -125/512 \). When \( s = -7/33 \), \( \psi \) has 4 vanishing moments and \( 0.414076 \leq \mu_2(\hat{a}) \leq v_2(\hat{a}) < 0.414265 \). By calculation, the example given in [24] has 2 vanishing moments and \( 0.497088 < \mu_2(\hat{a}) \leq v_2(\hat{a}) < 0.500007 \). The example in [24] has been adapted in [24] to obtain an interesting Riesz multiwavelet basis in \( H^1_0([0, 1]) := \{ f \in L_2([0, 1]) : f' \in L_2 \text{ and } f(0) = f(1) = 0 \} \). We mention that a biorthogonal multiwavelet with compact support is obtained in [9] with \( v_2(\hat{a}) \approx 0.824926 \) and \( \psi \) is supported on \([-2, 2]\) with 2 vanishing moments.

**Example 4.2.** In this example we take the piecewise linear function vector \( \phi \) supported on \([-1, 1]\) given in [23]. As discussed in [23], \( \phi \) is a refinable function vector satisfying \( \hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi) \), where the mask \( a \) is given by

\[
a(-2) = \begin{bmatrix}
0 & -\sqrt{3}/36 \\
0 & 0
\end{bmatrix}, \quad a(-1) = \begin{bmatrix}
-1/12 & 5\sqrt{3}/36 \\
0 & 0
\end{bmatrix},
\]

\[
a(0) = \begin{bmatrix}
1/2 & 5\sqrt{3}/36 \\
0 & 1/3
\end{bmatrix}, \quad a(1) = \begin{bmatrix}
-1/12 & -\sqrt{3}/36 \\
\sqrt{3}/3 & 1/3
\end{bmatrix},
\]

with \( a(k) = 0 \) for all \( k \in \mathbb{Z} \setminus \{-2, -1, 0, 1\} \). By Lemma 2.1, it follows from (2.2) that
Now, by (2.11), we have

\[
\text{By calculation, } v_2(q) = 1.5. \text{ Consider a sequence } b \in (\ell_0(\mathbb{Z}))^{2 \times 2} \text{ such that } b(k) = 0 \text{ for all } k \in \mathbb{Z} \setminus \{-2, -1, 0, 1\}. \text{ In order to have a nontrivial solution to the system of linear equations in (2.9), we find that } m \leq 3. \text{ Taking } m = 3 \text{ and imposing symmetry on the sequence } b \text{ we find a unique solution}
\]

\[
b(-2) = \begin{bmatrix} 0 & \sqrt{3}/36 \\ 0 & \sqrt{6}/36 \end{bmatrix}, \quad b(-1) = \begin{bmatrix} -5/12 & 7\sqrt{3}/36 \\ -5\sqrt{2}/12 & 7\sqrt{6}/36 \end{bmatrix}, \\
b(0) = \begin{bmatrix} -1/2 & 7\sqrt{3}/36 \\ 0 & -7\sqrt{6}/36 \end{bmatrix}, \quad b(1) = \begin{bmatrix} -5/12 & \sqrt{3}/36 \\ 5\sqrt{2}/12 & -\sqrt{6}/36 \end{bmatrix}
\]

with \(b(k) = 0\) for all \(k \in \mathbb{Z} \setminus \{-2, -1, 0, 1\}\). By calculation, we have

\[
\det \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi + \pi) \\ \hat{b}(\xi) & \hat{b}(\xi + \pi) \end{bmatrix} = \frac{\sqrt{6}}{9} \cos(2\xi) - 5 \neq 0 \quad \forall \xi \in \mathbb{R}.
\]

Now, by (2.11), we have

\[
\hat{a}(\xi) = q(\xi)^{-1} \hat{a}(\xi) \quad \text{with } q(\xi) = \frac{1}{4}[5 - \cos(2\xi)],
\]

and

\[
\hat{a}(-2) = \begin{bmatrix} 0 & -\sqrt{3}/8 \\ 0 & -1/32 \end{bmatrix}, \quad \hat{a}(-1) = \begin{bmatrix} -1/8 & \sqrt{3}/4 \\ -\sqrt{3}/48 & -3/32 \end{bmatrix}, \quad \hat{a}(0) = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ -\sqrt{3}/24 & 1/2 \end{bmatrix}, \\
\hat{a}(1) = \begin{bmatrix} -1/8 & -\sqrt{3}/8 \\ 7\sqrt{3}/24 & 1/2 \end{bmatrix}, \quad \hat{a}(2) = \begin{bmatrix} 0 & 0 \\ -\sqrt{3}/24 & -3/32 \end{bmatrix}, \quad \hat{a}(3) = \begin{bmatrix} 0 & 0 \\ -\sqrt{3}/48 & -1/32 \end{bmatrix},
\]

with \(\hat{a}(k) = 0\) for all \(k \in \mathbb{Z} \setminus \{-2, -1, 0, 1, 2, 3\}\). Note that \(1 = q_{\min} \leq q(\xi) \leq q_{\max} = 3/2\). Taking \(n = 4\) in Theorem 3.2, we find that \(1.499952 < \mu_2(a) \leq 0.500002\). The multiwavelet \(\psi\) has 3 vanishing moments and by
Algorithm 2.2, $\psi$ generates a Riesz multiwavelet basis in $L^2(\mathbb{R})$. See Fig. 3 for the refinable function vector $\phi$ and the multiwavelet $\psi$. We note that [23, Example 3.1.1] has 2 vanishing moments and $1.000000 \leq \mu_2(\tilde{a}) \leq \nu_2(\tilde{a}) \leq 1.0000001$.

**Example 4.3.** Let $a$ be a mask given by

\[
a(-2) = \begin{bmatrix} 1/256 & 0 \\ 7/512 & 1/32 \end{bmatrix}, \quad a(-1) = \begin{bmatrix} 1/4 & 15/32 \\ -1/32 & -7/64 \end{bmatrix}, \quad a(0) = \begin{bmatrix} 63/128 & 0 \\ 0 & 3/16 \end{bmatrix},
\]

\[
a(1) = \begin{bmatrix} 1/4 & -15/32 \\ 1/32 & -7/64 \end{bmatrix}, \quad a(2) = \begin{bmatrix} 1/256 & 0 \\ -7/512 & 1/32 \end{bmatrix},
\]

with $a(k) = 0$ for all $k \in \mathbb{Z} \setminus \{-2, -1, 0, 1, 2\}$. By calculation, we have $\nu_2(a) = 4.5$ and the refinable function vector $\phi$ is a vector of $C^3$ spline functions.

By Lemma 2.1, it follows from (2.2) that

\[
m^a(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad m^a(1) = \begin{bmatrix} 0 \\ -1/256 \end{bmatrix}, \quad m^a(2) = \begin{bmatrix} 1/126 \\ 0 \end{bmatrix}, \quad m^a(3) = \begin{bmatrix} 0 \\ -1/315 \end{bmatrix},
\]

\[
m^a(4) = \begin{bmatrix} 13/3780 \\ 0 \end{bmatrix}, \quad m^a(5) = \begin{bmatrix} 0 \\ -31/110880 \end{bmatrix}, \quad m^a(6) = \begin{bmatrix} 1/11880 \\ 0 \end{bmatrix}, \quad m^a(7) = \begin{bmatrix} -13/5141 \\ 0 \end{bmatrix}, \quad m^a(8) = \begin{bmatrix} 0 \\ -5321/454053600 \end{bmatrix}, \quad m^a(9) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Consider a sequence $b \in (\ell_0(\mathbb{Z}))^{2 \times 2}$ supported on $\{-2, -1, 0, 1, 2\}$. To have a nontrivial solution to the system of linear equations in (2.9), we find that $m \leq 8$. Taking $m = 7$ and imposing symmetry on the sequence $b$ we find a parameterized solution for $b$ as follows:

\[
b(-2) = \begin{bmatrix} 180493 \\ 1528678 \end{bmatrix}, \quad b(-1) = \begin{bmatrix} 83200 \\ 764339 \end{bmatrix}, \quad b(0) = \begin{bmatrix} 275 \\ 555064 \end{bmatrix},
\]

\[
b(1) = \begin{bmatrix} 5321 \end{bmatrix}, \quad b(2) = \begin{bmatrix} 180493 \\ 1528678 \end{bmatrix}, \quad b(3) = \begin{bmatrix} 180493 \\ 1528678 \end{bmatrix}, \quad b(4) = \begin{bmatrix} 180493 \\ 1528678 \end{bmatrix}.
\]
Fig. 4. (a) and (b) are the two components $\phi_1$ and $\phi_2$ of the refinable function vector $\phi$ in Example 4.3. (c) and (d) are the two components $\psi_1$ and $\psi_2$ of the multiwavelet $\psi$ with $s = 103/128$ in Example 4.3. The multiwavelet $\psi$ has 7 vanishing moments and $\psi$ generates a Riesz multiwavelet basis in $L^2(\mathbb{R})$.

with $b(0) = I_2$ and $b(k) = 0$ for all $k \in \mathbb{Z}\setminus\{-2, -1, 0, 1, 2\}$. Now by (2.11), we have $\hat{a}(\xi) = q(\xi)^{-1}\hat{\alpha}(\xi)$, where $\hat{\alpha}$ is supported on $[-5, 5]$ and

$$q(\xi) := A(\xi)/A(0), \quad \text{where } A(\xi) := \det \begin{bmatrix} \hat{\alpha}(\xi) & \hat{\alpha}(\xi + \pi) \\ \hat{b}(\xi) & \hat{b}(\xi + \pi) \end{bmatrix}.$$ 

Note that

$$\hat{\alpha}(0) = \begin{bmatrix} 1 \\ 0 \\ \frac{8(3530845 - 2887888)}{317528 + 9731455} \end{bmatrix}.$$ 

We find that the eigenvalue other than 1 of $\hat{\alpha}(0)$ is less than 1 in modulus if and only if $0.599964 \approx 690472/1150855 < s < 23420632/18515305 \approx 1.264934$.

Taking $n = 2$ in Theorem 3.2, we can estimate $\mu_2(\hat{\alpha})$ and $\nu_2(\hat{\alpha})$. We find that when $s = 103/128$, $4.3183708 < \mu_2(\hat{\alpha}) \leq \nu_2(\hat{\alpha}) < 4.428120$, which is almost the best one. Therefore, when $s = 103/128$, $\psi$ has 7 vanishing moments and $\psi$ generates a Riesz multiwavelet basis for the Sobolev space $W^\alpha(\mathbb{R})$ with $-4.3183708 < \alpha < 4.5$. See Fig. 4 for the refinable function vector $\phi$ and the multiwavelet $\psi$ with $s = 103/128$.

5. Proof of Algorithm 2.2 and some auxiliary results

In this section, generalizing the results in [20] and using some techniques in [15], we shall establish some results on Riesz multiwavelet bases in $L^2(\mathbb{R})$. At the end of this section, we shall give a proof to Algorithm 2.2 stated in Section 2.

As in [20], we say that a function $f : \mathbb{R} \rightarrow \mathbb{C}$ has polynomial decay if $(1 + |\cdot|)^j f \in L_\infty(\mathbb{R})$ for all $j \in \mathbb{N}$. Similarly, we say that a function vector $f$ has polynomial decay if each component of $f$ has polynomial decay.

For a sequence $a$ of $r \times s$ matrices on $\mathbb{Z}$, we say that $a$ has polynomial decay if $\sup_{k \in \mathbb{Z}}(1 + |k|)^j \|a(k)\| < \infty$ for all $j \in \mathbb{N}$. Clearly, the sequence $a$ has polynomial decay if and only if $\hat{a} \in (C^\infty(\mathbb{R}))^{r \times s}$.

Combining [15, Theorem 3.6] and [20, Lemma 3.1], we have the following result.
Lemma 5.1. Let \( y \in (\ell_0(\mathbb{Z}))^{1 \times r} \) such that \( \hat{y}(0) \neq 0 \). Let \( f \in (L_2(\mathbb{R}))^{r \times 1} \) be a function vector with polynomial decay and \( m \) be a positive integer. Then \( \{\hat{y}(\cdot) \hat{f}(\cdot)\}^{1 \times r}_{k=0} \) is a function vector with polynomial decay and \( v_j \) for some function vectors \( h_j \in (L_2(\mathbb{R}))^{r \times 1} \) with polynomial decay and \( v_j \in \mathcal{V}_{m,y} \), where \( \mathcal{V}_{m,y} \) is defined in (2.4) and \( v_j \ast h_j \) is a sequence of matrices on \( \mathbb{Z} \).

The following result is a special case of [13, Theorem 3.1], which plays a critical role in the CBC (coset by coset) algorithm for constructing compactly supported biorthogonal multiwavelets with arbitrarily high vanishing moments in [13]. The following result will be needed later. For the purpose of completeness, we present a proof here.

Proposition 5.2. Let \( a, \hat{a}, \hat{b} : \mathbb{Z} \rightarrow \mathbb{C}^{r \times r} \) be two sequences of \( r \times r \) matrices on \( \mathbb{Z} \) such that \( \hat{a} \) is a dual mask of \( a \), that is,

\[
\hat{a}(\xi)\hat{a}(\xi)^T + \hat{a}(\xi + \pi)\hat{a}(\xi + \pi)^T = I_r. \tag{5.1}
\]

Suppose that (2.1) holds. If \( \hat{a} \) satisfies the sum rules of order \( \hat{m} \) with a sequence \( \hat{y} \in (\ell_0(\mathbb{Z}))^{1 \times r} \), then there exists a nonzero constant \( c \) such that \( \hat{y}(j)(0) = cm^a(j)^T \) for all \( j = 0, \ldots, \hat{m} - 1 \), where the vectors \( m^a(j), j \in \mathbb{N} \cup \{0\} \) are uniquely defined in (2.2) of Lemma 2.1.

Proof. By assumption, \( \hat{a} \) satisfies the sum rules of order \( \hat{m} \) with the sequence \( \hat{y} \), which is equivalent to

\[
\hat{y}(2\xi)\hat{a}(\xi) = \hat{y}(\xi) + o(\xi^{\hat{m}-1}), \quad \hat{y}(2\xi)\hat{a}(\xi + \pi) = o(\xi^{\hat{m}-1}), \quad \xi \rightarrow 0. \tag{5.2}
\]

It follows from (5.1) and (5.2) that

\[
\hat{y}(2\xi) = \hat{y}(2\xi)\hat{a}(\xi)\hat{a}(\xi)^T + \hat{y}(2\xi)\hat{a}(\xi + \pi)\hat{a}(\xi + \pi)^T = \hat{y}(\xi)\hat{a}(\xi)^T + o(\xi^{\hat{m}-1}), \quad \xi \rightarrow 0.
\]

That is,

\[
\hat{a}(\xi)\overline{\hat{a}(\xi)^T} = \overline{\hat{y}(2\xi)^T} + o(\xi^{\hat{m}-1}), \quad \xi \rightarrow 0.
\]

By the uniqueness of the vectors \( m^a(j), j \in \mathbb{N} \cup \{0\} \), there exists a nonzero constant \( c \) such that \( \overline{\hat{y}(j)(0)^T} = \hat{c}m^a(j) \) for all \( j = 0, \ldots, \hat{m} - 1 \). This completes the proof. \( \square \)

Proposition 5.3. Let \( a, b, \hat{a}, \hat{b} : \mathbb{Z} \rightarrow \mathbb{C}^{r \times r} \) be sequences of \( r \times r \) matrices on \( \mathbb{Z} \) such that

\[
\begin{bmatrix}
\hat{a}(\xi) & \hat{a}(\xi + \pi) \\
\hat{b}(\xi) & \hat{b}(\xi + \pi)
\end{bmatrix}
= \begin{bmatrix}
I_r & 0 \\
0 & I_r
\end{bmatrix}, \tag{5.3}
\]

If (2.1) holds, then \( \hat{a} \) satisfies the sum rules of order \( \hat{m} \) if and only if \( b \) satisfies

\[
\sum_{\ell=0}^{j} \frac{j!}{(j - \ell)!} \hat{b}^{(j-\ell)}(0) m^a(\ell) = 0 \quad \forall j = 0, \ldots, \hat{m} - 1, \tag{5.4}
\]

where the vectors \( m^a(j), j \in \mathbb{N} \cup \{0\} \) are uniquely given in (2.2). Similarly, if (2.1) holds with \( \hat{a}(0) \) being replaced by \( \hat{b}(0) \), then \( \hat{b} \) satisfies the sum rules of order \( m \) if and only if \( \hat{b} \) satisfies

\[
\sum_{\ell=0}^{j} \frac{j!}{(j - \ell)!} \hat{b}^{(j-\ell)}(0) m^a(\ell) = 0 \quad \forall j = 0, \ldots, m - 1, \tag{5.5}
\]

where the vectors \( m^a(j), j \in \mathbb{N} \cup \{0\} \) are uniquely defined in (2.2) with \( \hat{a} \) being replaced by \( \hat{b} \).

Proof. Suppose that \( \hat{a} \) satisfies the sum rules of order \( \hat{m} \) with a sequence \( \hat{y} \in (\ell_0(\mathbb{Z}))^{1 \times r} \). Then \( \hat{y}(0) \neq 0 \) and

\[
\hat{y}(2\xi)\hat{a}(\xi) = \hat{y}(\xi) + o(\xi^{\hat{m}-1}), \quad \hat{y}(2\xi)\hat{a}(\xi + \pi) = o(\xi^{\hat{m}-1}), \quad \xi \rightarrow 0.
\]

By (5.3), we have

\[
0 = \hat{y}(2\xi)\hat{a}(\xi)\overline{\hat{a}(\xi)^T} + \hat{y}(2\xi)\hat{a}(\xi + \pi)\overline{\hat{a}(\xi + \pi)^T} = \hat{y}(\xi)\overline{\hat{a}(\xi)^T} + o(\xi^{\hat{m}-1}), \quad \xi \rightarrow 0.
\]
That is, \( \hat{b}(\xi) \tilde{y}(\xi)^T = o(|\xi|^{\hat{m}-1}) \), as \( \xi \to 0 \), or equivalently,
\[
0 = \left[ \hat{b}(\cdot) \tilde{y}(\cdot) \right]^{(j)} (0) = \sum_{\ell=0}^{j-1} \frac{j!}{(j-\ell)!} \hat{b}^{(j-\ell)}(0) \tilde{y}^{(\ell)}(0)^T, \quad j = 0, \ldots, \hat{m} - 1. \tag{5.6}
\]

Since (2.1) holds and \( \hat{a} \) is a dual mask of \( a \), by Proposition 5.2, there exists a nonzero constant \( c \) such that \( \tilde{y}^{(\ell)}(0) = c m^{a(\ell)}(\ell)^T \) for all \( \ell = 0, \ldots, \hat{m} - 1 \). Therefore, (5.4) holds.

Conversely, suppose that (5.4) holds. Since (2.1) holds, we have some \( \tilde{y} \in (\ell_0(\mathbb{Z}))[1, \infty) \times \mathbb{R} \) such that \( \tilde{y}^{(j)}(0) = m^{a(j)}(j)^T \) for all \( j = 0, \ldots, \hat{m} - 1 \). By the definition of the vectors \( m^{a(j)} \), this is equivalent to \( \tilde{y}(2\xi) = \hat{y}(\xi) \hat{a}(\xi)^T + o(|\xi|^{\hat{m}-1}) \), as \( \xi \to 0 \). Now we see that (5.4) is equivalent to \( \hat{y}(\xi) \hat{b}(\xi)^T = o(|\xi|^{\hat{m}-1}) \), as \( \xi \to 0 \). From (5.3), we have
\[
\tilde{a}(\xi)^T \hat{a}(\xi) + \tilde{b}(\xi)^T \hat{b}(\xi) = I_r \quad \text{and} \quad \tilde{a}(\xi)^T \hat{a}(\xi + \pi) + \tilde{b}(\xi)^T \hat{b}(\xi + \pi) = 0.
\]

Therefore, by \( \hat{y}(\xi) \hat{b}(\xi)^T = o(|\xi|^{\hat{m}-1}) \) and \( \hat{y}(\xi) \hat{a}(\xi)^T = \hat{y}(2\xi) + o(|\xi|^{\hat{m}-1}) \), as \( \xi \to 0 \), we have
\[
\hat{y}(\xi) = \hat{y}(\xi) \hat{a}(\xi)^T \hat{a}(\xi) + \hat{y}(\xi) \hat{b}(\xi)^T \hat{b}(\xi) = \hat{y}(2\xi) \hat{a}(\xi) + o(|\xi|^{\hat{m}-1})
\]
and
\[
0 = \hat{y}(\xi) \tilde{a}(\xi)^T \hat{a}(\xi + \pi) + \hat{y}(\xi) \tilde{b}(\xi)^T \hat{b}(\xi + \pi) = \hat{y}(2\xi) \hat{a}(\xi + \pi) + o(|\xi|^{\hat{m}-1}).
\]

Since \( \hat{y}(0) = m^{a(0)}(0)^T \neq 0 \), \( \hat{a} \) must satisfy the sum rules of order \( \hat{m} \) with the sequence \( \tilde{y} \). \( \square \)

We need the following result which is critical in our proof of the main result in this section and may be also useful for investigating other problems on biorthogonal multiwavelets.

**Lemma 5.4.** Let \( \hat{a}, \hat{\tilde{a}} \) be two \( r \times r \) matrices of 2\( \pi \)-periodic functions in \( C^\infty(\mathbb{R}) \) such that \( \hat{a} \) is a dual mask of \( a \) (that is, (5.1) holds) and (2.1) holds for both \( \hat{a}(0) \) and \( \hat{\tilde{a}}(0) \). Suppose that \( a \) and \( \hat{a} \) satisfy the sum rules of orders \( m \) and \( \hat{m} \) with some sequences \( y, \tilde{y} \in (\ell_0(\mathbb{Z}))[1, \infty) \times \mathbb{R} \), respectively. Then there exist function vectors \( f, \tilde{f} \in (L_2(\mathbb{R}))^{r \times 1} \cap (C^{m+\hat{m}-2}(\mathbb{R}))^{r \times 1} \) with polynomial decay (in fact, one is compactly supported and the other has exponential decay) such that

(i) \( \left[ \tilde{f}(\cdot)^T \hat{f}(\cdot) \right]^{(j)}(0) = \delta(j) \) for all \( j = 0, \ldots, m + \hat{m} - 2 \) and
\[
\left[ \hat{a}(\cdot) \hat{\tilde{a}}(\cdot) \right]^{(j)}(0) = \left[ \tilde{f}(2\cdot)^T \right]^{(j)}(0) \quad \forall j = 0, \ldots, m - 1,
\]
\[
\left[ \hat{a}(\cdot) \hat{\tilde{a}}(\cdot) \right]^{(\ell)}(0) = \left[ \tilde{f}(2\cdot)^T \right]^{(\ell)}(0) \quad \forall \ell = 0, \ldots, \hat{m} - 1. \tag{5.7}
\]

(ii) The shifts of \( f \) and \( \tilde{f} \) are biorthogonal: \( \langle f, \tilde{f}(\cdot - k) \rangle = \delta(k)I_r \) for all \( k \in \mathbb{Z} \).

(iii) \( \left[ \hat{y}(\cdot) \hat{\tilde{y}}(\cdot) \right]^{(j)}(2\pi k) \) are \( \hat{y} \)-functions \( \hat{\tilde{y}} \) are \( \hat{\tilde{y}} \)-functions such that all \( k \in \mathbb{Z} \) and \( j = 0, \ldots, m + \hat{m} - 2 \).

**Proof.** Without loss of generality, we assume that \( m \geq \hat{m} \). Since \( a \) satisfies the sum rules of order \( m \) and (2.1) holds, by [19, Theorem 2.3] or [15, Proposition 2.4], there exists an invertible \( r \times r \) matrix \( U(\xi) \) of 2\( \pi \)-periodic trigonometric polynomials such that

1. \( U(\xi)^{-1} \) is an \( r \times r \) matrix of 2\( \pi \)-periodic trigonometric polynomials.
2. \( \left[ \hat{y}(\cdot) U(\cdot)^{-1} \right]^{(j)}(0) = * \) for all \( j = 0, \ldots, m - 1 \), where \( * \) denotes some number.
3. \( U(2\hat{\xi}) \hat{a}(\xi) U(\xi)^{-1} \) takes the form
\[
\begin{bmatrix}
\hat{a}_{1,1}(\xi) & \hat{a}_{1,2}(\xi) \\
\hat{a}_{2,1}(\xi) & \hat{a}_{2,2}(\xi)
\end{bmatrix}
\]

such that
\[
\hat{a}_{1,1}(0) = 1, \quad \hat{a}_{1,1}(\pi) = 0, \quad \hat{a}_{1,2}(0) = 0, \quad \hat{a}_{1,2}(\pi) = 0, \quad \hat{a}_{2,1}(0) = \hat{a}_{2,1}(\pi) = 0, \quad \forall j = 0, \ldots, m - 1. \tag{5.9}
\]

(5.9)
where \( \hat{a}_{1,1}, \hat{a}_{1,2}, \hat{a}_{2,1} \) and \( \hat{a}_{2,2} \) are some \( 1 \times 1, 1 \times (r - 1), (r - 1) \times 1 \) and \( (r - 1) \times (r - 1) \) matrices of \( 2\pi \)-periodic functions in \( C^\infty(\mathbb{R}) \).

Replacing the functions \( \hat{a}, \hat{\alpha}, \hat{\eta} \) and \( \hat{\psi} \) by \( U(2\xi)\hat{a}(\xi)U(\xi)^{-1}, U(2\xi)^T \hat{\alpha}(\xi)[U(\xi)^T]^{-1}, \hat{\eta}(\xi)U(\xi)^{-1} \) and \( \hat{\psi}(\xi)[U(\xi)^T]^{-1} \), respectively, by [15, Proposition 2.4] or [19, Lemma 2.1], we see that all the conditions in Lemma 5.4 still hold. So, without loss of generality, by [15, Proposition 2.4], we can assume that \( \hat{a} \) takes the form in (5.8) and (5.9) with \( \hat{\eta}(\xi) = [d(\xi), 0, \ldots, 0] \) for some \( 2\pi \)-periodic trigonometric polynomial \( d(\xi) \) with \( d(0) = 1 \).

In the following, we show that \( \hat{\eta}(j)(0) = [1/d(\xi)](0) \) for all \( j = 0, \ldots, m - 1 \), that is, we can take \( \hat{\psi}(\xi) = [1/d(\xi), 0, \ldots, 0] \). Since \( \hat{a} \) satisfies the sum rules of order \( m \) with the sequence \( \hat{\eta} \) and (2.1) holds with \( \hat{\alpha}(0) \) being replaced by \( \hat{\psi}(0) \), by Proposition 5.2, there must exist a nonzero constant \( c \) such that \( \hat{\psi}(j)(0) = cm^a(j)^T \) for all \( j = 0, \ldots, m - 1 \), where the vectors \( m^a(j), j \in \mathbb{N} \cup \{0\} \), are uniquely defined in (2.2). Since \( \hat{a} \) takes the form in (5.8) and (5.9), we know that

\[
\hat{a}(j)(0) = \begin{bmatrix}
\hat{a}_{1,1}(j)(0) & 0 \\
0 & \hat{a}_{2,2}(j)(0)
\end{bmatrix}, \quad j = 0, \ldots, m - 1. \tag{5.10}
\]

By induction and \( \hat{m} \leq m \), it follows from (2.2) and (5.10) that all the components except the first component of the vectors \( m^a(j), j = 0, \ldots, m - 1 \), must be zero, that is, \( m^a(j) = [*, 0, \ldots, 0]^T \) for all \( j = 0, \ldots, m - 1 \), where \( * \) denotes some number. Therefore, we can assume that \( \hat{\psi}(\xi) = [\hat{\eta}(\xi), 0, \ldots, 0] \) for some \( \hat{\eta} \in L_0(\mathbb{Z}) \). By the definition of \( m^a(j) \) and \( \hat{\psi}(j)(0) = cm^a(j)^T \) for all \( j = 0, \ldots, m - 1 \), we see that \( \hat{\psi}(2\xi) = \hat{\psi}(\xi)\hat{a}(\xi)^T + o(\|\xi\|^m-1) \). Now it follows from (5.10) that

\[
\hat{\eta}(2\xi) = \hat{\eta}(\xi)[\hat{\alpha}_{1,1}(\xi) + o(\|\xi\|^m-1)]. \tag{5.11}
\]

On the other hand, since \( a \) satisfies the sum rules of order \( m \) with the sequence \( y \), by the definition of sum rules in (2.3), it follows from the special form of \( \hat{a} \) in (5.8) and (5.9) with \( \hat{\psi}(\xi) = [d(\xi), 0, \ldots, 0] \) that

\[
d(\xi) = d(2\xi)\hat{a}_{1,1}(\xi) + o(\|\xi\|^m-1), \quad \xi \to 0. \tag{5.12}
\]

Consequently, by \( m \geq \hat{m} \) and \( d(0) = 1 \), it follows from the above relation that

\[
d(2\xi)^{-1} = d(\xi)^{-1}\hat{a}_{1,1}(\xi) + o(\|\xi\|^m-1), \quad \xi \to 0. \tag{5.13}
\]

Since \( \hat{a}_{1,1}(0) = 1 \), up to a scalar multiplicative constant, the solution \( \{\hat{\eta}(j)(0): j = 0, \ldots, m - 1\} \) to the system of linear equations in (5.11) is unique. Now it follows from (5.11) and (5.13) that \( \hat{\eta}(j)(0) = [1/d(\xi)](j)(0) \) for all \( j = 0, \ldots, m - 1 \). Consequently, we can take \( \hat{\eta}(\xi) = 1/d(\xi) \). Therefore, we have \( \hat{\psi}(\xi) = [d(\xi), 0, \ldots, 0] \) and \( \hat{\psi}(\xi) = [1/d(\xi), 0, \ldots, 0] \) with \( d(0) = 1 \).

In the following, we shall construct the desired function vectors \( f \) and \( \tilde{f} \). It is well known [10] that there is a compactly supported orthogonal refinable function \( \eta_0 \in C^{m+m-2}(\mathbb{R}) \) with dilation factor \( r + 1 \) such that \( \eta_0(0) = 1 \) and \( \eta_0((r+1)\xi) = \tilde{u}(\xi)\eta_0(\xi) \) for some finitely supported mask \( u \in L_0(\mathbb{Z}) \), where \( r \) is the multiplicity of the matrix mask \( a \). Since \( \eta_0 \in C^{m+m-2}(\mathbb{R}) \) is an orthogonal refinable function with dilation factor \( r + 1 \), there exist compactly supported wavelet functions \( \eta_\ell \in C^{m+m-2}(\mathbb{R}) \), \( \ell = 1, \ldots, r \), such that \( \{\eta_{\ell}(\cdot - k): k \in \mathbb{Z}, \ell = 1, \ldots, r\} \) is an orthonormal system and

1. \( \tilde{\eta}_0(j)(2\pi k) = 0 \) for all \( k \in \mathbb{Z}\setminus\{0\} \) and \( j = 0, \ldots, m + \hat{m} - 2 \).
2. \( \tilde{\eta}_0(j)(0) = d(j) \) and \( \tilde{\eta}_\ell(j)(0) = 0 \) for all \( j = 0, \ldots, m + \hat{m} - 2 \) and \( \ell = 1, \ldots, r \).

By [15, Lemma 3.4] and \( d(0) = \tilde{\eta}_0(0) = 1 \), there exists a \( 2\pi \)-periodic trigonometric polynomial \( c(\xi) \) such that \( \|1 - c\|_{L^\infty(\mathbb{R})} < 1/2 \) and

\[
c(j)(0) = \left[\frac{1}{d(\cdot)\tilde{\eta}_0(\cdot)}\right](j)(0) \quad \forall j = 1, \ldots, m + \hat{m} - 2. \]

That is,
Define two function vectors \( f := (f_1, \ldots, f_r)^T \) and \( \tilde{f} := (\tilde{f}_1, \ldots, \tilde{f}_r)^T \) by

\[
\tilde{f}_1(\xi) := c(\xi)\hat{\eta}_0(\xi), \quad f_\ell := \eta_{\ell - 1}, \quad \ell = 2, \ldots, r
\]

and

\[
\hat{f}_1(\xi) := \frac{c(\xi)}{c(\xi)} \eta_0(\xi), \quad \tilde{f}_\ell := \eta_{\ell - 1}, \quad \ell = 2, \ldots, r.
\]

Since \(|1 - c(\xi)| < 1/2\) for all \( \xi \in \mathbb{R} \), the \( 2\pi \)-periodic trigonometric polynomial \( c \) must satisfy \( 1/2 \leq |c(\xi)| \leq 3/2 \) for all \( \xi \in \mathbb{R} \). Therefore, it is easy to see that \( f \) is compactly supported and \( \tilde{f} \) has polynomial decay (in fact, \( \tilde{f} \) has exponential decay).

Now we show that (i) must be true. By the definition of \( f \) and \( \tilde{f} \), it follows from (1) and (2) that

\[
\frac{\hat{f}(\xi)^T}{\hat{f}(\xi)} \tilde{f}(\xi) = \hat{\eta}_0(\xi) \eta_0(\xi) + \sum_{\ell=1}^{r-1} \hat{\eta}_\ell(\xi) \eta_\ell(\xi) = 1 + o(|\xi|^{m+m-2}), \quad \xi \to 0.
\]

That is, \( [\hat{f}(\xi)^T \tilde{f}(\xi)](j)(0) = \delta(j) \) for all \( j = 0, \ldots, m + \tilde{m} - 2 \). Since \( \hat{\eta}_\ell(\xi) = o(|\xi|^{m+m-2}) \) as \( \xi \to 0 \) for all \( \ell = 1, \ldots, r \), when \( \xi \to 0 \), we have

\[
\hat{a}(\xi) \tilde{f}(\xi) = \hat{f}(2\xi) = [\hat{a}_{1,1}(\xi) \hat{f}_1(\xi) - \hat{f}_1(2\xi), \hat{a}_{2,1}(\xi) \hat{f}_1(\xi)]^T + o(|\xi|^{m+m-2}).
\]

and similarly,

\[
\hat{a}(\xi) \tilde{f}(\xi) = \hat{f}(2\xi) = [\hat{a}_{1,1}(\xi) \hat{f}_1(\xi) - \hat{f}_1(2\xi), \hat{a}_{2,1}(\xi) \hat{f}_1(\xi)]^T + o(|\xi|^{m+m-2}).
\]

Since \( \hat{f}_1(\xi) = c(\xi)\hat{\eta}_0(\xi) \), by (5.12) and (5.14), we have

\[
\hat{a}_{1,1}(\xi) \hat{f}_1(\xi) - \hat{f}_1(2\xi) = \hat{a}_{1,1}(\xi)c(\xi)\hat{\eta}_0(\xi) - c(2\xi)\hat{\eta}_0(2\xi) = [\hat{a}_{1,1}(\xi)d(2\xi)c(\xi)d(\xi)\hat{\eta}_0(\xi) - d(\xi)c(2\xi)d(2\xi)\hat{\eta}_0(2\xi)]/[d(\xi)d(2\xi)] = o(|\xi|^{m+m-2}), \quad \xi \to 0.
\]

By (5.9), \( \hat{a}_{2,1}(\xi) = o(|\xi|^{m-1}) \), as \( \xi \to 0 \). Therefore, \( [\hat{a}(\cdot) \tilde{f}(\cdot)](j)(0) = [\hat{f}(2\cdot)](j)(0) \) for all \( j = 0, \ldots, m - 1 \). So, the first part of (5.7) holds.

In order to show the second part of (5.7), let us first show that \( \hat{a} \) must take the following form:

\[
\hat{a}(\xi) = \begin{bmatrix} \hat{a}_{1,1}(\xi) & \hat{a}_{1,2}(\xi) \\ \hat{a}_{2,1}(\xi) & \hat{a}_{2,2}(\xi) \end{bmatrix}
\]

such that

\[
\hat{a}_{1,1}(0) = 1, \quad \hat{a}_{1,1}(\pi) = 0, \quad \hat{a}_{2,1}(0) = 0, \quad \hat{a}_{1,2}(0) = 0
\]

\[
\hat{a}_{1,2}(\pi) = 0, \quad \forall j = 0, \ldots, \tilde{m} - 1.
\]

Since \( \hat{a} \) satisfies the sum rules of order \( \tilde{m} \) with the sequence \( \tilde{y} \) such that \( \tilde{y}(\xi) = [1/\tilde{d}(\xi), 0, \ldots, 0] \) and \( d(0) = 1 \), by the definition of sum rules in (2.3), it is easy to see that

\[
\hat{a}_{1,1}(0) = 1, \quad \hat{a}_{2,1}(\pi) = 0 \quad \text{and} \quad \hat{a}_{1,2}(0) = \hat{a}_{1,2}(\pi) = 0 \quad \forall j = 0, \ldots, \tilde{m} - 1.
\]

On the other hand, by (5.8) and (5.18), it follows from (5.1) that

\[
0 = \hat{a}_{2,1}(\xi)\hat{a}_{1,1}(\xi) + \hat{a}_{2,2}(\xi)\hat{a}_{1,2}(\xi)^T + \hat{a}_{2,1}(\xi + \pi)\hat{a}_{1,1}(\xi + \pi) + \hat{a}_{2,2}(\xi + \pi)\hat{a}_{1,2}(\xi + \pi)^T.
\]

\[
1 = \hat{a}_{1,1}(\xi)\hat{a}_{1,1}(\xi) + \hat{a}_{1,2}(\xi)\hat{a}_{1,2}(\xi)^T + \hat{a}_{1,1}(\xi + \pi)\hat{a}_{1,1}(\xi + \pi) + \hat{a}_{1,2}(\xi + \pi)\hat{a}_{1,2}(\xi + \pi)^T.
\]

By (5.9) and (5.20), it follows from the above identities that

\[
\hat{a}_{2,1}(\xi) = o(|\xi|^{m-1}) \quad \text{and} \quad \hat{a}_{1,1}(\xi)\hat{a}_{1,1}(\xi) = 1 + o(|\xi|^{m+m-2}), \quad \xi \to 0.
\]
By \( m \geq \tilde{m} \), \( \hat{a} \) must take the form in (5.18) and (5.19).

Now by a similar argument, it follows from (5.17) that the second part of (5.7) holds. So, (i) holds. Since \( \{ \eta_{\ell}(-k) : k \in \mathbb{Z}, \ell = 0, \ldots, r \} \) is an orthonormal system, by a simple calculation, it is easy to check that \( \langle f, \hat{f}(-k) \rangle = \delta(k)I_r \), for all \( k \in \mathbb{Z} \). Therefore, (ii) holds.

Since \( \hat{\eta}^{(j)}(2\pi k) = 0 \) for all \( k \in \mathbb{Z} \setminus \{0\} \) and \( j = 0, \ldots, m + \tilde{m} - 2 \), by the Leibniz differentiation formula, we deduce that

\[
\left[ \hat{y}(\cdot) \hat{f}(\cdot) \right]^{(j)}(2\pi k) = \left[ (d(\cdot)c(\cdot))^{(j)} \right](2\pi k) = 0 \quad \forall k \in \mathbb{Z} \setminus \{0\} \) and \( j = 0, \ldots, m + \tilde{m} - 2 \)

and

\[
\left[ \hat{\hat{y}}(\cdot) \hat{\hat{f}}(\cdot) \right]^{(j)}(2\pi k) = \left[ \hat{\hat{\hat{\eta}}}(\cdot) \right]^{(j)}(2\pi k) = 0 \quad \forall k \in \mathbb{Z} \setminus \{0\} \) and \( j = 0, \ldots, m + \tilde{m} - 2 \).

To prove (iii), it suffices to show that

\[
\hat{y}(\xi) \hat{f}(\xi) = 1 + o(\|\xi\|^{m+\tilde{m}-2}) \quad \text{and} \quad \hat{\hat{y}}(\xi) \hat{\hat{f}}(\xi) = 1 + o(\|\xi\|^{m+\tilde{m}-2}), \quad \xi \to 0.
\]

(5.22)

By \( \hat{y}(\xi) = [d(\xi), 0, \ldots, 0] \) and \( \hat{\hat{y}}(\xi) = \{1/d(\xi), 0, \ldots, 0\} \), it follows from (5.14) that

\[
\hat{y}(\xi) \hat{f}(\xi) = d(\xi)c(\xi)\hat{\eta}(\xi) = 1 + o(\|\xi\|^{m+\tilde{m}-2}), \quad \xi \to 0
\]

and as \( \xi \to 0 \),

\[
\hat{\hat{y}}(\xi) \hat{\hat{f}}(\xi) = \frac{1}{d(\xi)c(\xi)} \frac{1}{\hat{\eta}(\xi)} \hat{\eta}(\xi) = \hat{\eta}(\xi) \hat{\eta}(\xi) = 1 + o(\|\xi\|^{m+\tilde{m}-2}).
\]

By (2), we have \( \hat{\eta}(\xi) \hat{\eta}(\xi) = 1 + o(\|\xi\|^{m+\tilde{m}-2}) \). Therefore, (5.22) holds and (iii) is verified. \( \square \)

For \( f, g \in (L_2(\mathbb{R}))^{r \times r} \), the bracket product [27] is defined to be

\[
[f, g](\xi) := \sum_{k \in \mathbb{Z}} f(\xi + 2\pi k) g(\xi + 2\pi k)^T, \quad \xi \in \mathbb{R}.
\]

Let \( \psi = (\psi^1, \ldots, \psi^r)^T \in (L_2(\mathbb{R}))^{r \times 1} \) be a function vector. We say that \( \psi \) generates a Bessel multiwavelet sequence in \( L_2(\mathbb{R}) \) if there exists a positive constant \( C \) such that

\[
\sum_{\ell=1}^{r} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \|f, \psi^\ell_{j,k}\|^2 \leq C \|f\|^2 \quad \forall f \in L_2(\mathbb{R}),
\]

where \( \psi^\ell_{j,k} := 2^{j/2} \psi^\ell(2^j \cdot - k), j, k \in \mathbb{Z} \) and \( \ell = 1, \ldots, r \).

The main result in this section is as follows.

**Theorem 5.5.** Let \( a, b : \mathbb{Z} \to \mathbb{C}^{r \times r} \) be two sequences of \( r \times r \) matrices on \( \mathbb{Z} \) such that \( \hat{a}, \hat{b} \in (C^\infty(\mathbb{R}))^{r \times r} \) and

\[
\det \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi + \pi) \\ \hat{b}(\xi) & \hat{b}(\xi + \pi) \end{bmatrix} \neq 0 \quad \forall \xi \in \mathbb{R}.
\]

(5.23)

Define two sequences \( \hat{a}, \hat{b} : \mathbb{Z} \to \mathbb{C}^{r \times r} \) by

\[
\begin{bmatrix} \hat{a}(\xi) \\ \hat{b}(\xi) \end{bmatrix} = \left( \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi + \pi) \\ \hat{b}(\xi) & \hat{b}(\xi + \pi) \end{bmatrix} \right)^{-1} \begin{bmatrix} I_r \\ 0 \end{bmatrix}.
\]

(5.24)

Suppose that (2.1) holds for both \( \hat{a}(0) \) and \( \tilde{a}(0) \). Then there exist two \( r \times 1 \) vectors \( \phi, \tilde{\phi} \) of tempered distributions such that

\[
\phi(2\xi) = \hat{a}(\xi) \phi(\xi), \quad \tilde{\phi}(2\xi) = \hat{a}(\xi) \tilde{\phi}(\xi), \quad \text{and} \quad \phi(0)^T \phi(0) = 1.
\]

(5.25)

Define two wavelet function vectors \( \psi, \tilde{\psi} \) by
\[ \hat{\psi}(2\xi) := \hat{b}(\xi)\hat{\phi}(\xi) \quad \text{and} \quad \hat{\psi}(2\xi) := \hat{b}(\xi)\hat{\phi}(\xi). \] (5.26)

If \( \nu_2(a) > 0 \) and \( \nu_2(\tilde{a}) > 0 \), then \( \phi, \psi, \tilde{\phi}, \tilde{\psi} \in (L_2(\mathbb{R}))^{r \times 1} \) and
\[
\begin{align*}
\{ \phi, \tilde{\phi}(\cdot - k) \} &= \{ \psi, \tilde{\psi}(\cdot - k) \} = \delta(k)L, \\
\{ \phi, \psi(\cdot - k) \} &= \{ \tilde{\phi}, \tilde{\psi}(\cdot - k) \} = 0 \quad \forall k \in \mathbb{Z}.
\end{align*}
\] (5.27)

Moreover, if
\[
\begin{align*}
(\text{i}) & \quad [\phi, \tilde{\phi}] \in (L_{\infty}(\mathbb{R}))^{r \times r} \quad \text{and} \quad [\tilde{\phi}, \tilde{\psi}] \in (L_{\infty}(\mathbb{R}))^{r \times r}, \\
(\text{ii}) & \quad \text{each of } \psi \text{ and } \tilde{\psi} \text{ generates a Bessel multiwavelet sequence in } L_2(\mathbb{R}),
\end{align*}
\]

then \( \psi \) generates a Riesz multiwavelet basis in \( L_2(\mathbb{R}) \).

**Proof.** Since \( \nu_2(a) > 0 \) and \( \nu_2(\tilde{a}) > 0 \), by the definition of \( \nu_2(a) \) and \( \nu_2(\tilde{a}) \), for any \( \rho, \tilde{\rho} \) such that \( 2^{-\nu_2(a)} < \rho < 1 \) and \( 2^{-\nu_2(\tilde{a})} < \tilde{\rho} < 1 \), there exist \( y, \tilde{y} \in (\ell_0(\mathbb{Z}))^{1 \times r} \) such that \( \hat{y}(0) \neq 0 \), \( \hat{\tilde{y}}(0) \neq 0 \) and
\[
\begin{align*}
(1) & \quad a \text{ satisfies the sum rules of order } m \text{ with the sequence } y, \\
(2) & \quad \tilde{a} \text{ satisfies the sum rules of order } m \text{ with the sequence } \tilde{y}, \\
(3) & \quad \rho_m(a, y) < \rho 2^{-1/2} \quad \text{and} \quad \rho_{\tilde{m}}(\tilde{a}, \tilde{y}) < \tilde{\rho} 2^{-1/2}.
\end{align*}
\]

Since \( \nu_2(a) > 0 \) and \( \nu_2(\tilde{a}) > 0 \), by [15, Proposition 4.1], we must have \( m \geq 1 \) and \( \tilde{m} \geq 1 \). Now by Lemma 5.4, there exist two function vectors \( f \) and \( \tilde{f} \) with polynomial decay such that (i)–(iii) of Lemma 5.4 hold.

Now we consider the cascade sequences \( f_n := Q^n_a f \) and \( \tilde{f}_n := Q^n_{\tilde{a}} \tilde{f} \), where the cascade operators \( Q_a, Q_{\tilde{a}}: (L_2(\mathbb{R}))^{r \times 1} \rightarrow (L_2(\mathbb{R}))^{r \times 1} \) are defined to be
\[
\begin{align*}
\hat{Q}_a f(\xi) & := \hat{\tilde{a}}(\xi/2)\hat{f}(\xi/2) \quad \text{and} \quad \hat{Q}_{\tilde{a}} \tilde{f}(\xi) := \hat{\tilde{a}}(\xi/2)\hat{\tilde{f}}(\xi/2).
\end{align*}
\]

Denote \( g := Q_a f - f \) and \( \tilde{g} := Q_{\tilde{a}} \tilde{f} - \tilde{f} \). Since all \( f, \tilde{f}, a \) and \( \tilde{a} \) have polynomial decay, the function vectors \( g \) and \( \tilde{g} \) must also have polynomial decay. Now we show that
\[
\begin{align*}
\begin{bmatrix} \hat{y}(\cdot) \hat{\tilde{g}}(\cdot) \end{bmatrix}^{(j)}(2\pi k) &= 0 \quad \forall k \in \mathbb{Z} \quad \text{and} \quad j = 0, \ldots, m - 1 \quad \text{(5.28)}
\end{align*}
\]
and
\[
\begin{align*}
\begin{bmatrix} \hat{\tilde{y}}(\cdot) \hat{\tilde{g}}(\cdot) \end{bmatrix}^{(j)}(2\pi k) &= 0 \quad \forall k \in \mathbb{Z} \quad \text{and} \quad j = 0, \ldots, \tilde{m} - 1.
\end{align*}
\] (5.29)

Note that \( \hat{g}(\xi) = \hat{\tilde{a}}(\xi/2)\hat{\tilde{f}}(\xi/2) - \hat{f}(\xi) \). Therefore, for any \( k \in \mathbb{Z} \), by (iii) of Lemma 5.4, we have
\[
\begin{align*}
\begin{bmatrix} \hat{y}(\cdot) \hat{\tilde{g}}(\cdot) \end{bmatrix}^{(j)}(2\pi k) &= -\begin{bmatrix} \hat{y}(\cdot) \hat{\tilde{f}}(\cdot) \end{bmatrix}^{(j)}(2\pi k) + \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!} \begin{bmatrix} \hat{y}(\cdot) \hat{\tilde{a}}(\cdot/2) \end{bmatrix}^{(j-\ell)}(2\pi k) \begin{bmatrix} \hat{\tilde{f}}(\cdot/2) \end{bmatrix}^{(\ell)}(2\pi k) \\
&= -\begin{bmatrix} \hat{y}(\cdot) \hat{\tilde{f}}(\cdot) \end{bmatrix}^{(j)}(2\pi k) + 2^{-j} \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!} \begin{bmatrix} \hat{y}(2\cdot) \hat{\tilde{a}}(\cdot) \end{bmatrix}^{(j-\ell)}(\pi k) \begin{bmatrix} \hat{\tilde{f}}(\cdot) \end{bmatrix}^{(\ell)}(\pi k).
\end{align*}
\]

Since \( a \) satisfies the sum rules of order \( m \) with the sequence \( y \), we have
\[
\begin{align*}
\begin{bmatrix} \hat{y}(2\cdot) \hat{\tilde{a}}(\cdot) \end{bmatrix}^{(j-\ell)}(\pi k) &= \begin{cases} \hat{y}^{(j-\ell)}(\pi k), & \text{if } k \text{ is even}, \\
0, & \text{if } k \text{ is odd}.
\end{cases}
\end{align*}
\] (5.30)

Consequently, when \( k \) is odd, it follows from (5.30) and (iii) of Lemma 5.4 that
\[
\begin{align*}
\begin{bmatrix} \hat{y}(\cdot) \hat{\tilde{g}}(\cdot) \end{bmatrix}^{(j)}(2\pi k) &= -\delta(j)\delta(k) + 2^{-j} \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!} \begin{bmatrix} \hat{y}(2\cdot) \hat{\tilde{a}}(\cdot) \end{bmatrix}^{(j-\ell)}(\pi k) \begin{bmatrix} \hat{\tilde{f}}(\cdot) \end{bmatrix}^{(\ell)}(\pi k) \\
&= -\delta(j)\delta(k) = 0 \quad \forall j = 0, \ldots, m - 1.
\end{align*}
\]
and when $k$ is even, by (iii) of Lemma 5.4 and (5.30), we have

$$\left[\hat{y}(\cdot)\hat{g}(\cdot)\right]^{(j)}(2\pi k) = -\delta(j)\delta(k) + 2^{-j} \sum_{\ell=0}^{j-1} \frac{j!}{\ell!(j-\ell)!} \hat{y}^{(j-\ell)}(\pi k) \hat{f}^{(\ell)}(\pi k)$$

$$= -\delta(j)\delta(k) + 2^{-j} \left[\hat{y}(\cdot)\hat{f}(\cdot)\right]^{(j)}(\pi k)$$

$$= -\delta(j)\delta(k) + 2^{-j} \delta(j)\delta(k)$$

$$= 0.$$

Therefore, (5.28) holds. By a similar argument, (5.29) can be verified.

Since (5.28) and (5.29) hold, by Lemma 5.1, there exist function vectors $h_j, \tilde{h}_\ell \in (L_2(\mathbb{R}))^{r \times 1}$ with polynomial decay such that

$$g = \sum_{j=1}^{N_g} v_j * h_j \quad \text{and} \quad \tilde{g} = \sum_{\ell=1}^{N_{\tilde{g}}} \tilde{v}_\ell * \tilde{h}_\ell,$$

where $v_j \in \mathcal{V}_{m,y}$ for all $j = 1, \ldots, N_g$ and $\tilde{v}_\ell \in \mathcal{V}_{\tilde{m},\tilde{y}}$ for all $\ell = 1, \ldots, N_{\tilde{g}}$.

By induction, we have

$$f_{n+1} - f_n = Q_n^m g = 2^n \sum_{k \in \mathbb{Z}} a_n(k)g(2^n \cdot -k) = 2^n \sum_{k \in \mathbb{Z}} \sum_{j=1}^{N_g} a_n(k)(v_j * h_j)(2^n \cdot -k)$$

$$= 2^n \sum_{j=1}^{N_g} \sum_{k \in \mathbb{Z}} (a_n * v_j)(k)h_j(2^n \cdot -k)$$

and similarly,

$$\tilde{f}_{n+1} - \tilde{f}_n = Q_n^m \tilde{g} = 2^n \sum_{\ell=1}^{N_{\tilde{g}}} (\tilde{a}_n * \tilde{v}_\ell)(k)\tilde{h}_\ell(2^n \cdot -k),$$

where $\tilde{a}_n(\xi) := \tilde{a}(2^{n-1}\xi) \cdots \tilde{a}(\xi)$ and $\tilde{a}_{\ell}(\xi) := \tilde{a}(2^{n-1}\xi) \cdots \tilde{a}(\xi)\tilde{a}(\xi)$. Since all $h_j$ and $\tilde{h}_\ell$ are function vectors in $(L_2(\mathbb{R}))^{r \times 1}$ with polynomial decay, we have $[\tilde{h}_j, \tilde{h}_\ell] \in (L_\infty(\mathbb{R}))^{r \times r}$ for all $j = 1, \ldots, N_g$ and $[\tilde{h}_j, \tilde{h}_\ell] \in (L_\infty(\mathbb{R}))^{r \times r}$ for all $\ell = 1, \ldots, N_{\tilde{g}}$. Therefore, there exists a positive constant $C_1$, depending only on the function vectors $h_j$ and $\tilde{h}_\ell$, such that for all $n \in \mathbb{N}$,

$$\|f_{n+1} - f_n\|_{(L_2(\mathbb{R}))^{r \times 1}} \leq C_1 2^{n/2} \sum_{j=1}^{N_g} \|a_n * v_j\|_{(\ell_2(\mathbb{Z}))^{r \times 1}},$$

$$\|	ilde{f}_{n+1} - \tilde{f}_n\|_{(L_2(\mathbb{R}))^{r \times 1}} \leq C_1 2^{n/2} \sum_{\ell=1}^{N_{\tilde{g}}} \|\tilde{a}_n * \tilde{v}_\ell\|_{(\ell_2(\mathbb{Z}))^{r \times 1}}.$$  (5.31)

On the other hand, since $\rho_m(a, y) < \rho 2^{-1/2}$ and $\rho_{\tilde{m}}(\tilde{a}, \tilde{y}) < \tilde{\rho} 2^{-1/2}$, by $v_j \in \mathcal{V}_{m,y}$ and $\tilde{v}_\ell \in \mathcal{V}_{\tilde{m},\tilde{y}}$, we must have

$$\limsup_{n \to \infty} \|a_n * v_j\|_{(\ell_2(\mathbb{Z}))^{r \times 1}}^{1/n} < \rho 2^{-1/2} \quad \forall j = 1, \ldots, N_g,$$

$$\limsup_{n \to \infty} \|\tilde{a}_n * \tilde{v}_\ell\|_{(\ell_2(\mathbb{Z}))^{r \times 1}}^{1/n} < \tilde{\rho} 2^{-1/2} \quad \forall \ell = 1, \ldots, N_{\tilde{g}}.$$  

Therefore, there exists a positive constant $C_2$ such that for all $j = 1, \ldots, N_g$ and $\ell = 1, \ldots, N_{\tilde{g}}$,

$$\|a_n * v_j\|_{(\ell_2(\mathbb{Z}))^{r \times 1}} \leq C_2 \rho^n 2^{-n/2} \quad \text{and} \quad \|\tilde{a}_n * \tilde{v}_\ell\|_{(\ell_2(\mathbb{Z}))^{r \times 1}} \leq C_2 \tilde{\rho}^n 2^{-n/2} \quad \forall n \in \mathbb{N}. $$

Now by (5.31), we conclude that for all $n \in \mathbb{N}$,
\[ \|f_{n+1} - f_n\|_{L^2(\mathbb{R})^r \times 1} \leq C_1 C_2 N_{\hat{\rho}} \rho^n \quad \text{and} \quad \|\tilde{f}_{n+1} - \tilde{f}_n\|_{L^2(\mathbb{R})^r \times 1} \leq C_1 C_2 N_{\tilde{\rho}} \tilde{\rho}^n. \]  

(5.32)

Since \(0 < \rho, \tilde{\rho} < 1\), the above inequalities in (5.32) imply that both \(\{f_n\}_{n \in \mathbb{N}}\) and \(\{\tilde{f}_n\}_{n \in \mathbb{N}}\) are Cauchy sequences in \((L^2(\mathbb{R}))^r \times 1\).

Note that \(\tilde{f}_n(\xi) = \prod_{j=1}^n \hat{a}(2^{-j} \xi) \hat{f}(2^{-n} \xi)\) and \(\tilde{f}_n(\xi) = \prod_{j=1}^n \hat{\tilde{a}}(2^{-j} \xi) \hat{\tilde{f}}(2^{-n} \xi)\). Since (2.1) holds for both \(\hat{a}(0)\) and \(\hat{\tilde{a}}(0)\), by \(\overline{\hat{f}(0)}^T \hat{\tilde{f}}(0) = \overline{\hat{f}(0)}^T \hat{\tilde{f}}(0) = 1\), we see that \(\lim_{n \to \infty} \|f_n - \tilde{f}_n\|_{L^2(\mathbb{R})^r \times 1} = 0\) and \(\lim_{n \to \infty} \|f_n - \tilde{f}_n\|_{L^2(\mathbb{R})^r \times 1} = 0\) for some nonzero constant \(\tilde{c}\).

By the definition of \(\hat{\tilde{a}}^T\) and \(\hat{\tilde{b}}^T\), we see that (5.3) holds. Note that by Lemma 5.4, \((f, \tilde{f}(\cdot - k)) \equiv \delta(k) I_r\) for all \(k \in \mathbb{Z}\). By induction, it follows from (5.1) that \((f_n, \tilde{f}_n(\cdot - k)) \equiv \delta(k) I_r\) for all \(k \in \mathbb{Z}\) and \(n \in \mathbb{N}\). Consequently, by \(\lim_{n \to \infty} \|f_n - \tilde{f}_n\|_{L^2(\mathbb{R})^r \times 1} = 0\) and \(\lim_{n \to \infty} \|f_n - \tilde{f}_n\|_{L^2(\mathbb{R})^r \times 1} = 0\) for some nonzero constant \(c\), we must have \((\hat{a}, \hat{\tilde{a}}(\cdot - k)) \equiv \delta(k) I_r\) for all \(k \in \mathbb{Z}\). Now by (5.3), (5.27) holds.

Using the standard argument of multiresolution analysis [7,10], one can show that \(\psi\) and \(\tilde{\psi}\) form a pair of biorthogonal multiwavelet bases in \(L^2(\mathbb{R})\). Therefore, in particular, \(\psi\) generates a Riesz multiwavelet basis in \(L^2(\mathbb{R})\). \(\square\)

In order to prove Algorithm 2.2, we need the following result which is related to [14, Theorem 2.2].

\textbf{Proposition 5.6.} Let \(a : \mathbb{Z} \to \mathbb{C}^{r \times r}\) be a finitely supported matrix mask such that \(v_2(a) > 0\). Let \(\phi\) be a compactly supported refinable function vector such that \(\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)\) with \(\hat{\phi}(0) \neq 0\). Then for any \(0 < \alpha < v_2(a)\), there exists a positive constant \(C\) such that

\[ \sum_{k \in \mathbb{Z}} \left(1 + |\xi + 2\pi k|^2\right)^{\alpha/2} \phi(\xi + 2\pi k)^T \phi(\xi + 2\pi k) \leq C \quad \forall \xi \in \mathbb{R}. \]  

(5.33)

\textbf{Proof.} Since \(v_2(a) > 0\), by [15, Theorem 4.3], we have \(\phi \in (L^2(\mathbb{R}))^r \times 1\). By the definition of \(v_2(a)\) and \(\alpha < v_2(a)\), for any \(0 < \epsilon < v_2(a) - \alpha\), there exist \(y \in (\ell_0(\mathbb{Z}))^{1 \times r}\) with \(\hat{y}(0) \neq 0\) and a nonnegative integer \(m\) such that \(a\) satisfies the sum rules of order \(m\) with the sequence \(y\) and \(\rho_m(a, y) < 2^{e-1/2-v_2(a)}\).

Now we prove (5.33) by modifying the proof of [14, Theorem 2.2]. Since \([\hat{\phi}, \hat{\phi}](\xi) > 0\) for all \(\xi \in \mathbb{R}\) and it is a matrix of 2\(\pi\)-periodic trigonometric polynomials, we can find a matrix \(V(\xi)\) of 2\(\pi\)-periodic trigonometric polynomials such that \(V(\xi) \overline{V(\xi)}^T = [\hat{\phi}, \hat{\phi}](\xi)\). Define

\[ \hat{\nu}(\xi) := (1 - e^{-i\xi})^m V(\xi) \quad \text{and} \quad \hat{\nu}(\xi) := \hat{\nu}(\xi) \overline{\hat{\nu}(\xi)}^T = |1 - e^{-i\xi}|^{2m} [\hat{\phi}, \hat{\phi}](\xi). \]  

(5.34)

Since \(\phi\) is compactly supported, \(\hat{\nu}\) and \(\hat{\nu}\) are \(r \times r\) matrices of 2\(\pi\)-periodic trigonometric polynomials. Let \(\hat{a}_n(\xi) := \hat{\phi}(2^n \xi) \cdots \hat{\phi}(2 \xi) \hat{\phi}(\xi)\). As in [14, (2.8)], by \(\hat{\phi}(2^n \xi) = \hat{\phi}(2^n \xi)\), we have

\[ [\hat{\phi}(2^n \xi), \hat{\phi}(2^n \xi)](\xi) = \hat{a}_n(\xi)[\hat{\phi}(\xi), \hat{\phi}(\xi)]\hat{a}_n(\xi)^T. \]

Let \(\Omega := (-\pi, -\pi/2] \cup [\pi/2, \pi]\) and define a 2\(\pi\)-periodic function \(g\) by \(g(\xi) = \chi_\Omega(\xi)\) for \(\xi \in (-\pi, \pi]\), where \(\chi_\Omega\) denotes the characteristic function of the set \(\Omega\). Then we have \(0 \leq g(\xi) \leq |1 - e^{-i\xi}|^{2m}\) for all \(\xi \in \mathbb{R}\) and

\[ \overline{\hat{\phi}(2^n \xi)}^T \hat{\phi}(2^n \xi) \chi_\Omega(\xi) \leq \overline{\hat{\phi}(2^n \xi)}^T \hat{\phi}(2^n \xi) g(\xi) \leq \text{trace}(\hat{\phi}(2^n \xi), \hat{\phi}(2^n \xi)](\xi)) g(\xi) = \text{trace}(\hat{a}_n(\xi)[\hat{\phi}(\xi), \hat{\phi}(\xi)]\hat{a}_n(\xi)^T g(\xi)) \leq \text{trace}(\hat{a}_n(\xi) \hat{\nu}(\xi) \hat{a}_n(\xi)^T). \]

Consequently,

\[ \overline{\hat{\phi}(\xi)}^T \hat{\phi}(\xi) \chi_{2^n \Omega}(\xi) \leq \text{trace}(\hat{a}_n(2^{-n} \xi) \hat{\nu}(2^{-n} \xi) \hat{a}_n(2^{-n} \xi)^T) \quad \forall \xi \in \mathbb{R}. \]  

(5.35)

By induction on \(n\), we observe that

\[ \sum_{k=0}^{2^n - 1} \hat{a}_n(2^{-n} (\xi + 2\pi k)) \hat{\nu}(2^{-n} (\xi + 2\pi k)) \hat{a}_n(2^{-n} (\xi + 2\pi k)) = [T_a^n \hat{\nu}(\xi)]. \]  

(5.36)

where the transition operator \(T_a\) is defined to be

\[ [T_a f](\xi) := \hat{a}(\xi/2) \hat{f}(\xi/2) \overline{\hat{a}(\xi/2)}^T + \hat{a}(\xi/2 + \pi) \hat{f}(\xi/2 + \pi) \overline{\hat{a}(\xi/2 + \pi)}^T. \]  

(5.37)
Since $2^n \Omega = (-2^n \pi, -2^{n-1} \pi] \cup [2^{n-1} \pi, 2^n \pi]$, by (5.35) and (5.36), we conclude that for $\xi \in (-\pi, \pi]$, 
\[
\sum_{2^{n-1} \pi \leq |\xi + 2k\pi| < 2^n \pi} \phi(\xi + 2k\pi)^T \hat{\phi}(\xi + 2k\pi) \leq \sum_{k=-2^n}^{2^n-1} \phi(\xi + 2k\pi)^T \hat{\phi}(\xi + 2k\pi) \chi_{2^n \Omega}(\xi + 2k\pi)
\]
\[
\leq \text{trace} \left( \sum_{k=-2^n}^{2^n-1} \hat{a}_n(2^{-n}(\xi + 2k\pi)) \hat{u}(2^{-n}(\xi + 2k\pi)) \hat{a}_n(2^{-n}(\xi + 2k\pi))^T \right) = 2 \text{trace}(\left[T^u_n \hat{u}\right](\xi)).
\]
Since both $\hat{u}$ and $\hat{a}$ are matrices of $2\pi$-periodic trigonometric polynomials, the linear space containing all $T^u_n \hat{u}$, $n \in \mathbb{N} \cup \{0\}$, is a finite-dimensional space (see [17, Lemma 2.3]). Consequently, any norm on this linear space is equivalent. Therefore, there exists a positive constant $C_1$, independent of $n$, such that
\[
\text{trace}(\left[T^u_n \hat{u}\right](\xi)) \leq \text{trace}\left(\| T^u_n \hat{u} \|_{L^\infty(\mathbb{R})} \right) \leq C_1 \text{trace}\left( \int_{-\pi}^{\pi} |T^u_n \hat{u}(\xi)| \, d\xi \right) = C_1 \text{trace}\left( \int_{-\pi}^{\pi} \left[T^u_n \hat{u}\right](\xi) \, d\xi \right).
\]
By (5.36) and $\hat{u}(\xi) = \tilde{v}(\xi) \overline{\tilde{v}(\xi)}^T$, we have
\[
\text{trace}\left( \int_{-\pi}^{\pi} \left[T^u_n \hat{u}\right](\xi) \, d\xi \right) = 2^n \text{trace}\left( \int_{-\pi}^{\pi} \hat{a}_n(\xi) \tilde{v}(\xi) \overline{\tilde{v}(\xi)}^T \hat{a}_n(\xi)^T \, d\xi \right) = 2^n \text{trace}\left( \int_{-\pi}^{\pi} \tilde{v}(\xi) \overline{\tilde{v}(\xi)}^T \, d\xi \right) = 2^n 2\pi \| a_n \|_{\ell^2(\mathbb{Z})}^2.
\]
By the definition of $v$, each column of $v$ belongs to the space $V_m, y$ since $\tilde{v}^{(j)}(0) = 0$ for all $j = 0, \ldots, m - 1$. Now it follows from $\rho_n(a, y) < 2^{\epsilon - 1/2 - \nu_2(a)}$ that there exists a positive constant $C_2$ such that
\[
\| a_n \|_{\ell^2(\mathbb{Z})}^2 \leq C_2 2^{-n/2} 2^{\nu(a - \nu_2(a))} \quad \forall n \in \mathbb{N}.
\]
Combining all the above inequalities together, we conclude that for $\xi \in (-\pi, \pi]$ and $n \in \mathbb{N}$,
\[
\sum_{2^{n-1} \pi \leq |\xi + 2k\pi| < 2^n \pi} \phi(\xi + 2k\pi)^T \hat{\phi}(\xi + 2k\pi) \leq 4 \pi 2^n C_1 \| a_n \|_{\ell^2(\mathbb{Z})}^2 \leq 4 \pi C_1 C_2^2 2^{2n(\nu(a) - \nu_2(a))}.
\]
(5.38)
Since $\epsilon < \nu_2(a) - \alpha$, we have $\epsilon - \nu_2(a) + \alpha < 0$ and by (5.38), we conclude that for all $\xi \in (-\pi, \pi]$ and $n \in \mathbb{N}$,
\[
\sum_{2^{n-1} \pi \leq |\xi + 2k\pi| < 2^n \pi} (1 + |\xi + 2k\pi|^2)^{\alpha} \phi(\xi + 2k\pi)^T \hat{\phi}(\xi + 2k\pi) \leq C_3 \rho^n,
\]
where $C_3 := 8 \pi 1^{2n} C_1 C_2^2 < \infty$ and $\rho := 2^{\nu(a) - \nu_2(a) + \alpha} < 1$. Hence, for $\xi \in (-\pi, \pi]$, we have
\[
\sum_{|\xi + 2k\pi| \geq n} (1 + |\xi + 2k\pi|^2)^{\alpha} \phi(\xi + 2k\pi)^T \hat{\phi}(\xi + 2k\pi) \leq \sum_{n=1}^{\infty} C_3 \rho^n = C_3 \rho / (1 - \rho) < \infty.
\]
Since $\phi$ is compactly supported, the function vector $\hat{\phi}$ is continuous. Now (5.33) follows directly from the above inequality.

We finish this paper by presenting a proof to Algorithm 2.2.

**Proof of Algorithm 2.2.** Let $\hat{b}, \hat{\phi}, \hat{\psi}$ be defined as in Theorem 5.5. Since $\nu_2(a) > 0$, by [15, Theorem 4.3], (2.1) holds and $a$ must satisfy the sum rules of order at least 1. Note that $\nu_2(\hat{a}) \geq \mu_2(\hat{a}) > 0$. By Theorem 5.5, we have $\phi, \psi, \hat{\phi}, \hat{\psi} \in (L_2(\mathbb{R}))^{r \times 1}$ and (5.25) holds. By $\nu_2(\hat{a}) > 0$ and [15, Proposition 3.1], (2.1) holds with $\hat{u}(0)$ being replaced by $\hat{a}(0)$.
In the following, we show that the conditions in (i) and (ii) of Theorem 5.5 are satisfied. Since \( \mu_2(\hat{a}) > 0 \), by the definition of \( \mu_2(\hat{a}) \) in (2.8), there is a finitely supported mask \( \hat{a} \) such that \( v_2(\hat{a}) > 0 \) and \( \hat{a}(\xi) = q_2(\xi)d(\xi), \hat{\alpha}(\xi) = q_1(\xi)d(\xi), q_1(0) = q_2(0) \neq 0 \) and \( q_1(\xi) \leq q_2(\xi) \) for all \( \xi \in \mathbb{R} \), where \( q_2 \) and all components of \( \hat{a} \) are 2\( \pi \)-periodic trigonometric polynomials. Let \( \phi \) be the compactly supported refinable function vector satisfying \( \hat{\phi}(2\xi) = \hat{\alpha}(\xi)\hat{\phi}(\xi) \) with \( \hat{\phi}(0) = \hat{\phi}(0) \), by \( \hat{a}(0) = \hat{\alpha}(0) \). Since \( v_2(\alpha) > 0 \) and \( v_2(\hat{a}) > 0 \), by Proposition 5.6, there exist \( \alpha > 0 \) and \( C > 0 \) such that (5.33) holds for both \( \phi \) and \( \hat{\phi} \). By the relation between \( \hat{a} \) and \( \hat{\alpha} \), it is easy to see that \( |\hat{\phi}(\xi)| \leq |\hat{\phi}(\xi)| \) for all \( \xi \in \mathbb{R} \), where \( |\cdot| \) is the \( \ell_2 \)-norm on \( \mathbb{C}^n \). So, (5.33) holds with \( \phi \) being replaced by \( \hat{\phi} \). Now it follows directly from (5.33) that \( \|\hat{\phi},\hat{\phi}\|_{(L^2_2(\mathbb{R}))^{r \times r}} \leq C \) and \( \|\hat{\phi},\hat{\phi}\|_{(L^2_2(\mathbb{R}))^{r \times r}} \leq C \). So, condition (i) in Theorem 5.5 holds.

Since (5.33) holds for both \( \phi \) and \( \hat{\phi} \), by the definition of \( \psi \) and \( \hat{\psi} \) and \( \hat{\beta}, \hat{\phi} \in (L^2_2(\mathbb{R}))^{r \times r} \), we see that (5.33) holds with \( \phi \) being replaced by \( \psi \) and \( \hat{\psi} \) (probably with a different constant \( C \)). Now by the Hölder’s inequality, for any \( 0 < \varepsilon < 2\alpha/\alpha + 1 \), we have

\[
\sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2\pi k)|^{2-\varepsilon} \leq \left( \sum_{k \in \mathbb{Z}} (1 + |\xi + 2\pi k|^{2} \alpha^{rac{1-\varepsilon}{\varepsilon}}) \right)^{\varepsilon/2} \leq C_1 \quad \forall \xi \in \mathbb{R},
\]

where

\[
C_1 := C^{1-\varepsilon/2} \left\| \sum_{k \in \mathbb{Z}} (1 + |\cdot + 2\pi k|^{2})^{-(2\varepsilon-1)\alpha} \right\|_{L^\infty(\mathbb{R})}^{\varepsilon/2} < \infty,
\]

since \( (2\varepsilon-1)\alpha > 1 \). So, \( \sum_{k \in \mathbb{Z}} |\hat{\psi}(\cdot + 2\pi k)|^{2-\varepsilon} \in L^\infty(\mathbb{R}) \).

By assumption in (2.9), we have \( \hat{\psi}(0) = 0 \). Since (2.1) holds with \( \hat{\psi}(0) \) being replaced by \( \hat{\alpha}(0) \), and since \( \alpha \) satisfies the sum rules of order at least 1, by Proposition 5.3, we must have \( \hat{\psi}(0) = \hat{\beta}(0)\hat{\phi}(0) = 0 \).

Since (5.33) holds with \( \phi \) being replaced by \( \psi \) and \( \hat{\psi} \) is differentiable with \( \hat{\psi}(0) = 0 \), it is not difficulty to verify that \( \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j\cdot)|^2 \in L^\infty(\mathbb{R}) \). By the same proof, we also have \( \sum_{k \in \mathbb{Z}} |\hat{\psi}(\cdot + 2\pi k)|^{2-\varepsilon} \in L^\infty(\mathbb{R}) \) and \( \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j\cdot)|^2 \in L^\infty(\mathbb{R}) \). So, both \( \psi \) and \( \hat{\psi} \) satisfy the conditions in [12, Proposition 2.6] and consequently, both \( \psi \) and \( \hat{\psi} \) generate Bessel multiwavelet sequences in \( L_2(\mathbb{R}) \). Therefore, condition (ii) of Theorem 5.5 is verified.

Hence, all the conditions in Theorem 5.5 are satisfied, which completes the proof. \( \square \)

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