J. Math. Anal. Appl. 389 (2012) 48–53

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the best constants of Hardy inequality in $\mathbb{R}^{n-k}\times(\mathbb{R}_+)^k$ and related improvements ☆

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ARTICLE INFO

Article history:
Received 9 October 2011
Available online 18 November 2011
Submitted by M. Milman

Keywords:
Hardy inequality
Sharp constants

ABSTRACT

We compute the explicit sharp constants of Hardy inequalities in the cone $\mathbb{R}^n_{k_+}:=\mathbb{R}^{n-k}\times(\mathbb{R}_+)^k=\{(x_1,\dots,x_n)\mid x_{n-k+1}>0,\dots,x_n>0\}$ with $1\leqslant k\leqslant n$. Furthermore, the spherical harmonic decomposition is given for a function $u\in C_0^\infty(\mathbb{R}^n_{k_+})$. Using this decomposition and following the idea of Tertikas and Zographopoulos, we obtain the Filippas–Tertikas improvement of the Hardy inequality.

1. Introduction

Let Σ be a domain in $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$, and let $\mathcal{C}_\Sigma\subset\mathbb{R}^n$ be the cone associated with Σ:

$$\mathcal{C}_\Sigma:=\{t\sigma\mid t>0,\ \sigma\in\Sigma\}.$$

The Hardy inequality in $\mathcal{C}_\Sigma$ states that, for all $u\in C_0^\infty(\mathcal{C}_\Sigma)$, there holds (cf. [11,10])

$$\int_{\mathcal{C}_\Sigma}\left|\nabla u(x)\right|^2dx\geqslant\left(\frac{(n-2)^2}{4}+\lambda_1(\Sigma)\right)\int_{\mathcal{C}_\Sigma}\frac{u(x)^2}{|x|^2}dx \tag{1.1}$$

and the constant $(\frac{(n-2)^2}{4}+\lambda_1(\Sigma))$ in (1.1) is sharp, where $\lambda_1(\Sigma)$ is the Dirichlet principal eigenvalue of the spherical Laplacian $-\Delta_{\mathbb{S}^{n-1}}$ on Σ. In some special cases, the exact value of $\lambda_1(\Sigma)$ can be computed. We note the value of $\lambda_1(\Sigma)$ has been full-filled in the case of $n=2$ (cf. [1]). To the best of our knowledge (cf. [2–4,9–11]), when $n\geqslant 3$, $\lambda_1(\Sigma)$ is known only in the case of $\Sigma=\mathbb{S}^{n-1}_+$, the semi-sphere mapped in the upper half space $\mathbb{R}^n_+=\{(x_1,\dots,x_n)\mid x_n>0\}$. In fact, it can be computed via the following sharp Hardy inequality (cf. [6])

$$\int_{\mathbb{R}^n_+}\left|\nabla u(x)\right|^2dx\geqslant\frac{n^2}{4}\int_{\mathbb{R}^n_+}\frac{u(x)^2}{|x|^2}dx. \tag{1.2}$$

☆ The first author is supported by Program for Innovative Research Team in UIBE. The second author is supported by the National Natural Science Foundation of China (No. 11101096).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.11.033

One of the aim of this note is to compute the explicit sharp constants of Hardy inequalities in the cone $\mathbb{R}^n_{k_+} = \{(x_1, \ldots, x_n) \mid x_{n-k+1} > 0, \ldots, x_n > 0\}$, where $1 \leqslant k \leqslant n$. To this end, we have:

Theorem 1.1. *Let $n \geqslant 3$. There holds, for all $u \in C_0^\infty(\mathbb{R}^n_{k_+})$,*

$$\int_{\mathbb{R}^n_{k_+}} |\nabla u|^2 \, dx \geqslant \frac{(n-2+2k)^2}{4} \int_{\mathbb{R}^n_{k_+}} \frac{u^2}{|x|^2} \, dx, \tag{1.3}$$

and the constant $\frac{(n-2+2k)^2}{4}$ in (1.3) is sharp.

We note the proof of Theorem 1.1 above is similar to that of Theorem 1.2 and Corollary 1.3 in [7] and also to that of Theorem 6.1 in [8]. Combing the inequality (1.1) and Theorem 1.1 yields

Corollary 1.2. $\lambda_1(\mathbb{S}^{n-1} \cap \mathbb{R}^n_{k_+}) = k(n+k-2)$ *for all $n \geqslant 3$.*

Next, we consider the spherical harmonic decomposition of a function $u \in C_0^\infty(\mathbb{R}^n_{k_+})$. We show that for a function $u \in C_0^\infty(\mathbb{R}^n_{k_+})$, it has the expansion in spherical harmonics (for details, see Section 3)

$$u(x) = \sum_{l=k}^{\infty} f_l(r)\phi_l(\sigma),$$

where $r = |x|$ and $\phi_l(\sigma)$ ($l \geqslant k$) are the orthonormal eigenfunctions of the spherical Laplacian $-\Delta_{\mathbb{S}^{n-1}}$ with responding eigenvalues $l(n+l-2)$. Using this decomposition and following the idea of Tertikas and Zographopoulos [14], one can easily obtain several improvements of inequality (1.3) when u is supported in a bounded domain $\Omega \subset \mathbb{R}^n_{k_+}$. For example, we have the following Filippas–Tertikas improvement (cf. [5]):

Theorem 1.3. *Let $n \geqslant 3$. There holds, for all $u \in C_0^\infty(B_R \cap \mathbb{R}^n_{k_+})$,*

$$\int_{B_R \cap \mathbb{R}^n_{k_+}} |\nabla u|^2 \geqslant \frac{(n-2+2k)^2}{4} \int_{B_R \cap \mathbb{R}^n_{k_+}} \frac{u^2}{|x|^2} + \frac{1}{4} \sum_{i=1}^{\infty} \int_{B_R \cap \mathbb{R}^n_{k_+}} \frac{u^2}{|x|^2} X_1^2\left(\frac{|x|}{R}\right) \cdot \ldots \cdot X_i^2\left(\frac{|x|}{R}\right),$$

where

$$X_1(s) = (1 - \ln s)^{-1}, \qquad X_i(s) = X_1\big(X_{i-1}(t)\big)$$

for $i \geqslant 2$ and $B_R = \{x \in \mathbb{R}^n \colon |x| < R\}$.

2. Proof of Theorem 1.1

Let $l > 0$. A simple calculation shows, for $x_n > 0$,

$$(x_n)^{-l}\left(-\Delta + \frac{l(l-1)}{x_n^2}\right)\big(x_n^l g(x)\big) = -\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{2l}{x_n}\frac{\partial}{\partial x_n}\right) g(x). \tag{2.1}$$

Notice that $\frac{\partial^2}{\partial x_n^2} + \frac{2l}{x_n}\frac{\partial}{\partial x_n}$ is nothing but the $(2l+1)$-dimensional Laplacian of a radial function if $2l$ is a positive integer. So following the proof of Theorem 1.2 in [7] or Theorem 6.1 in [8], we have:

Lemma 2.1. *There holds, for $l \in \{1/2, 1, 3/2, 2, \ldots, n/2, \ldots\}$ and $u \in C_0^\infty(\mathbb{R}^n_+)$,*

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx + l(l-1) \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} \, dx \geqslant \frac{(n+2l-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{|x|^2} \, dx \tag{2.2}$$

and the constant $\frac{(n+2l-2)^2}{4}$ in (2.2) is sharp.

Proof. Recall the sharp Hardy inequality on $\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}$:

$$\int_{\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}} |\nabla v|^2 \geqslant \frac{(n+2l-2)^2}{4} \int_{\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}} \frac{v^2}{x_1^2 + \cdots + x_{n-1}^2 + |y|^2}, \tag{2.3}$$

where $v \in C_0^\infty(\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1})$. The constant that appear in (2.3) is also sharp if one consider only the functions like $\widetilde{v}(x, |y|) \in C_0^\infty(\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1})$. Set $x_n = |y|$ and $\varphi(x_1, \ldots, x_n) = \widetilde{v}(x, |y|)$, we can deduce, by (2.3) and (2.1),

$$\begin{aligned}
\int_{\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}} |\nabla \widetilde{v}|^2 &= - \int_{\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}} \widetilde{v}\big(x, |y|\big) \left(\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} + \sum_{k=1}^{2l+1} \frac{\partial^2}{\partial y_k^2} \right) \widetilde{v}\big(x, |y|\big) \\
&= - \int_{\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}} \varphi(x) \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{2l}{x_n} \frac{\partial}{\partial x_n} \right) \varphi(x) \\
&= - \int_{\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}} x^{-l} \varphi(x) \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{l(l-1)}{x_n^2} \right) \big(x^l \varphi(x)\big) \\
&= -\big|\mathbb{S}^{2l+1}\big| \int_{\mathbb{R}_+^n} x^l \varphi(x) \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{l(l-1)}{x_n^2} \right) \big(x^l \varphi(x)\big) \\
&\geqslant \frac{(n+2l-2)^2}{4} \int_{\mathbb{R}_x^{n-1} \times \mathbb{R}_y^{2l+1}} \frac{\widetilde{v}^2}{x_1^2 + \cdots + x_{n-1}^2 + |y|^2} \\
&= \frac{(n+2l-2)^2 |\mathbb{S}^{2l+1}|}{4} \int_{\mathbb{R}_+^n} \frac{\varphi^2 x_n^{2l}}{|x|^2},
\end{aligned}$$

where $|\mathbb{S}^{2l+1}|$ is the volume of $\mathbb{S}^{2l+1}$. It remains to set $u = x_n^l \varphi$. □

Remark 2.2. If we let $l(l-1) = 0$ in Lemma 2.1, then $l = 1$ and we obtain the sharp Hardy inequality on the half space $\mathbb{R}_+^n$ (see [6] for a different proof)

$$\int_{\mathbb{R}_+^n} \big|\nabla u(x)\big|^2 \, dx \geqslant \frac{n^2}{4} \int_{\mathbb{R}_+^n} \frac{u(x)^2}{|x|^2} \, dx.$$

Notice that this inequality is one of the objects in Theorem 1.1 and the dimension $2l + 1 = 3$ play an important role. So, in order to prove Theorem 1.1, we can repeat the same argument of Corollary 1.3 in [7] by choosing such dimension 3.

Proof of Theorem 1.1. Notice that

$$- \prod_{i=n-k+1}^{n} x_i^{-1} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \left(\prod_{i=n-k+1}^{n} x_i g(x) \right) = - \sum_{j=1}^{n-k} \frac{\partial^2 g(x)}{\partial x_j^2} - \sum_{j=n-k+1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{2}{x_j} \frac{\partial}{\partial x_j} \right) g(x). \tag{2.4}$$

We consider the sharp Hardy inequality on $\mathbb{R}_x^{n-k} \times \mathbb{R}_y^{3k}$:

$$\int_{\mathbb{R}_x^{n-k} \times \mathbb{R}_y^{3k}} \big(|\nabla_x v|^2 + |\nabla_y v|^2\big) \geqslant \frac{(n+2k-2)^2}{4} \int_{\mathbb{R}_x^{n-k} \times \mathbb{R}_y^{3k}} \frac{v^2}{\sum_{i=1}^{n-k} x_i^2 + \sum_{j=1}^{3k} y_j^2},$$

where $v \in C_0^\infty(\mathbb{R}_x^{n-k} \times \mathbb{R}_y^{3k})$. Set

$$x_{n-k+1} = \sqrt{y_1^2 + y_2^2 + y_3^2}, \qquad x_{n-k+2} = \sqrt{y_4^2 + y_5^2 + y_6^2}, \qquad \ldots, \qquad x_n = \sqrt{y_{3k-2}^2 + y_{3k-1}^2 + y_{3k}^2}$$

and consider all the functions like

$$v(x_1, \ldots, x_{n-k}, y_1, \ldots, y_{3k}) = \widetilde{v}(x_1, \ldots, x_n).$$

The constant $\frac{(n+2k-2)^2}{4}$ is also sharp for such functions (see e.g. [12]). Following the proof of Lemma 2.1, we have, using (2.4),

$$\begin{aligned}\int_{\mathbb{R}_x^{n-1}\times\mathbb{R}_y^{3k}} |\nabla\widetilde{v}|^2 &= -\left|\mathbb{S}^3\right|^k \int_{\mathbb{R}^n_{k_+}} \prod_{i=n-k+1}^n x_i\widetilde{v}(x)\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}\left(\prod_{i=n-k+1}^n x_i\widetilde{v}(x)\right)\\ &\geqslant \frac{(n+2k-2)^2}{4}\int_{\mathbb{R}_x^{n-k}\times\mathbb{R}_y^{3k}} \frac{\widetilde{v}^2}{\sum_{i=1}^{n-k}x_i^2+\sum_{j=1}^{3k}y_j^2}\\ &= \frac{(n+2k-2)^2|\mathbb{S}^3|^k}{4}\int_{\mathbb{R}^n_{k_+}} \frac{\widetilde{v}^2\prod_{i=n-k+1}^n x_i^2}{|x|^2}.\end{aligned}$$

It remains to set $u=\widetilde{v}\prod_{i=n-k+1}^n x_i$ and the desired result follows. □

3. Spherical harmonic decomposition

For a function $u\in C_0^\infty(\mathbb{R}^n_{k_+})$, we denote by $\widetilde{u}$ the odd extension of variables $\{x_{n-k+1},\dots,x_n\}$ of u, i.e. $\widetilde{u}(x)$ satisfies

$$\widetilde{u}(x_1,\dots,x_n)=u(x_1,\dots,x_n),\quad \forall(x_1,\dots,x_n)\in\mathbb{R}^n_{k_+}$$

and

$$\widetilde{u}(x_1,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_n)=-\widetilde{u}(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_n)$$

for all $n-k+1\leqslant j\leqslant n$. Then $\widetilde{u}\in C_0^\infty(\mathbb{R}^n)$ and moreover,

$$\int_{\mathbb{R}^n}|\nabla\widetilde{u}|^2=2^k\int_{\mathbb{R}^n_{k_+}}|\nabla u|^2,\qquad \int_{\mathbb{R}^n}\frac{\widetilde{u}^2}{|x|^2}=2^k\int_{\mathbb{R}^n_{k_+}}\frac{u^2}{|x|^2}. \tag{3.1}$$

Decomposing $\widetilde{u}$ into spherical harmonics we get (see e.g. [14])

$$\widetilde{u}=\sum_{l=0}^\infty \widetilde{u}_l:=\sum_{l=0}^\infty f_l(r)\phi_l(\sigma), \tag{3.2}$$

where $\phi_l(\sigma)$ are the orthonormal eigenfunctions of the Laplace–Beltrami operator with responding eigenvalues

$$c_l=l(n+l-2),\quad l\geqslant 0.$$

The functions $f_l(r)$ belong to $C_0^\infty(\mathbb{R}^n)$, satisfying $f_l(r)=O(r^l)$ and $f_l'(r)=O(r^{l-1})$ as $r\to 0$. Without loss of generality, we assume

$$\int_{\mathbb{S}^{n-1}}\left|\phi_l(\sigma)\right|^2d\sigma=1,\quad \forall l\geqslant 0.$$

By (3.2),

$$f_l(r)=\int_{\mathbb{S}^{n-1}}\widetilde{u}(x)\phi_l(\sigma)\,d\sigma$$

and

$$\int_0^\infty f_l^2(r)r^{n+l-1}\,dr=\int_0^\infty\int_{\mathbb{S}^{n-1}}\widetilde{u}(x)f_l(r)\phi_l(\sigma)r^{n+l-1}\,d\sigma\,dr=\int_{\mathbb{R}^n}\widetilde{u}(x)f_l\big(|x|\big)\phi_l(\sigma)|x|^l\,dx. \tag{3.3}$$

Lemma 3.1. *$f_l=0$ for all $0\leqslant l\leqslant k-1$.*

Before the proof of Lemma 3.1, we need some multi-index notation. We denote by $\mathbb{N}_0$ the set of nonnegative integer. A multi-index is denoted by $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$. For $\alpha\in\mathbb{N}_0^n$ and $x\in\mathbb{R}^n$ a monomial in variables $x_1,\dots,x_n$ of index α is defined by

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}.$$

The number $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is called the total degree of x^α. Notice that $\phi_l(\sigma)$ is nothing but the spherical harmonic of degree l (see e.g. [13], Chapter IV), it has the expansion

$$\phi_l(\sigma) = \frac{1}{|x|^l} \sum_{|\alpha|=l} C_\alpha x^\alpha \tag{3.4}$$

for some constants $C_\alpha \in \mathbb{R}$.

Proof of Lemma 3.1. By (3.3) and (3.4),

$$\int_0^\infty f_l^2(r) r^{n+l-1}\,dr = \int_{\mathbb{R}^n} \widetilde{u}(x) f_l\big(|x|\big)\phi_l(\sigma)|x|^l\,dx = \sum_{|\alpha|=l} C_\alpha \int_{\mathbb{R}^n} \widetilde{u}(x) f_l\big(|x|\big)x^\alpha\,dx.$$

So to finish the proof, it is enough to show

$$\int_{\mathbb{R}^n} \widetilde{u}(x) f_l\big(|x|\big)x^\alpha\,dx = 0$$

for all $|\alpha| = l$ with $0 \leqslant l \leqslant k-1$.

For $|\alpha| = \alpha_1 + \cdots + \alpha_n = l \leqslant k-1$, there must exist j, $n-k+1 \leqslant j \leqslant n$, such that $\alpha_j = 0$ (we note if $\alpha_j > 0$ for all $n-k+1 \leqslant j \leqslant n$, then $\alpha_{n-k+1} + \cdots + \alpha_n \geqslant k$ and this is a contradiction to $|\alpha| \leqslant k-1$). Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} \widetilde{u}(x) f_l\big(|x|\big)x^\alpha\,dx &= \int_{\mathbb{R}^n} \widetilde{u}(x) f_l\big(|x|\big)x_1^{\alpha_1}\cdots x_{j-1}^{\alpha_{j-1}}\cdot x_{j+1}^{\alpha_{j+1}}\cdots x_n^{\alpha_n}\,dx \\ &= \int_{\mathbb{R}^{n-1}} x_1^{\alpha_1}\cdots x_{j-1}^{\alpha_{j-1}}\bigg(\int_{\mathbb{R}} \widetilde{u}(x) f_l\big(|x|\big)\,dx_j\bigg)x_{j+1}^{\alpha_{j+1}}\cdots x_n^{\alpha_n}\,dx_1\cdots dx_{j-1}\,dx_{j+1}\cdots dx_n.\end{aligned}$$

Since $\widetilde{u}(x)$ is an odd function of variable x_j, so does $\widetilde{u}(x) f_l(|x|)$. Therefore,

$$\int_{\mathbb{R}} \widetilde{u}(x) f_l\big(|x|\big)\,dx_j = 0$$

and hence

$$\int_{\mathbb{R}^n} \widetilde{u}(x) f_l\big(|x|\big)x^\alpha\,dx = 0.$$

The proof of Lemma 3.1 is now completed. □

Remark 3.2. By Lemma 3.1, the function $\widetilde{u}$, the odd extension of variables $\{x_{n-k+1}, \ldots, x_n\}$ of u, has the expansion in spherical harmonics

$$u(x) = \sum_{l=k}^\infty f_l(r)\phi_l(\sigma),$$

so does the function u itself in $\mathbb{R}^n_{k_+}$.

Proof of Theorem 1.3. If we extend u as zero in $\mathbb{R}^n_{k_+} \setminus B_R$, we may consider $u \in C_0^\infty(\mathbb{R}^n_{k_+})$. By (3.1), it is enough to show that

$$\int_{B_R} |\nabla\widetilde{u}|^2 \geqslant \frac{(n-2+2k)^2}{4}\int_{B_R} \frac{\widetilde{u}^2}{|x|^2} + \frac{1}{4}\sum_{i=1}^\infty \int_{B_R} \frac{\widetilde{u}^2}{|x|^2} X_1^2\bigg(\frac{|x|}{R}\bigg)\cdot\cdots\cdot X_i^2\bigg(\frac{|x|}{R}\bigg)$$

holds for all $\widetilde{u} \in C_0^\infty(B_R)$. Since $\widetilde{u}$ has the expansion in spherical harmonics

$$u(x) = \sum_{l=k}^\infty f_l(r)\phi_l(\sigma),$$

where $f_l(r) \in C_0^\infty(B_R)$, satisfying $f_l(r) = O(r^l)$ and $f_l'(r) = O(r^{l-1})$ as $r \to 0$, we have,

$$\begin{aligned}\int_{B_R}|\nabla\widetilde{u}|^2-\frac{(n-2+2k)^2}{4}\int_{B_R}\frac{\widetilde{u}^2}{|x|^2}&=\sum_{l=k}^{\infty}\left[\int_{B_R}\left|f_l'(r)\right|^2dx+l(n+l-2)\int_{B_R}\frac{f_l^2(r)}{|x|^2}dx-\frac{(n-2+2k)^2}{4}\int_{B_R}\frac{f_l^2(r)}{|x|^2}dx\right]\\&=\sum_{l=k}^{\infty}\left[\int_{B_R}\left|f_l'(r)\right|^2dx+(l-k)(n+l+k-2)\int_{B_R}\frac{f_l^2(r)}{|x|^2}dx-\frac{(n-2)^2}{4}\int_{B_R}\frac{f_l^2(r)}{|x|^2}dx\right]\\&\geqslant\sum_{l=k}^{\infty}\left[\int_{B_R}\left|f_l'(r)\right|^2dx-\frac{(n-2)^2}{4}\int_{B_R}\frac{f_l^2(r)}{|x|^2}dx\right].\end{aligned}$$

To get the last inequality above, we use the fact $(l-k)(n+l+k-2)\geqslant 0$ since $l\geqslant k\geqslant 1$. Recalling the Filippas–Tertikas improvement of Hardy inequality (cf. [5,14])

$$\int_{B_R}\left|f_l'(r)\right|^2dx-\frac{(n-2)^2}{4}\int_{B_R}\frac{f_l^2(r)}{|x|^2}dx\geqslant\frac{1}{4}\sum_{i=1}^{\infty}\int_{B_R}\frac{f_l^2(r)}{|x|^2}X_1^2\left(\frac{|x|}{R}\right)\cdot\dots\cdot X_i^2\left(\frac{|x|}{R}\right),$$

we have

$$\begin{aligned}\int_{B_R}|\nabla\widetilde{u}|^2-\frac{(n-2+2k)^2}{4}\int_{B_R}\frac{\widetilde{u}^2}{|x|^2}&\geqslant\sum_{l=k}^{\infty}\left[\int_{B_R}\left|f_l'(r)\right|^2dx-\frac{(n-2)^2}{4}\int_{B_R}\frac{f_l^2(r)}{|x|^2}dx\right]\\&\geqslant\frac{1}{4}\sum_{l=k}^{\infty}\int_{B_R}\frac{f_l^2(r)}{|x|^2}X_1^2\left(\frac{|x|}{R}\right)\cdot\dots\cdot X_i^2\left(\frac{|x|}{R}\right)\\&=\frac{1}{4}\int_{B_R}\frac{\widetilde{u}^2(r)}{|x|^2}X_1^2\left(\frac{|x|}{R}\right)\cdot\dots\cdot X_i^2\left(\frac{|x|}{R}\right).\end{aligned}$$

The desired result follows. □

Acknowledgment

The authors thank the referee for his/her careful reading and very useful comments which improved the final version of this paper.

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