Constructing Bad Noetherian Local Domains Using Derivations

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The purpose of this paper is to present a new, relatively uncomplicated construction of commutative noetherian local domains $R$ with certain bad properties, such as the integral closure of $R$ not being finitely generated as an $R$-module, even when $R$ supports a derivation leaving no ideals invariant. Such examples are constructed having arbitrary embedding dimension and multiplicity.

The first example of a 1-dimensional noetherian local domain $R$ for which the integral closure $S$ is not a finitely generated $R$-module was constructed by Akizuki [1, Section 3]. (See also [5, Appendix, Example 3].) If $R$ is an arbitrary noetherian domain containing the rational numbers and $C = \{ x \in R \mid xS \subseteq R \}$ is the conductor, then Seidenberg [6, p. 169] has shown that $C$ is invariant under any derivation $\delta$ of $R$. Hence, if $R$ is $\delta$-simple (i.e., $R$ contains no $\delta$-invariant ideals other than 0 and $R$), then either $R = S$ or $C = 0$, in which case $S$ is not a finitely generated $R$-module. Vasconcelos [7, p. 230] asked whether it is possible to have a $\delta$-simple 1-dimensional noetherian local domain that is not integrally closed. The first example of such a ring was constructed by Lequain [4, Example 2.2]. Of course, if a 1-dimensional noetherian local domain $R$ is not integrally closed, then it is not regular and so its embedding dimension (namely $\dim_{K/M}(M/M^2)$) is greater than one. In this setting the multiplicity is $\dim(M^n/M^{n+1})$ for $n \gg 0$, and recently de Souza Doering and Lequain [2, 478]
Proposition 1] have constructed an example of a 1-dimensional noetherian local domain that is \( \delta \)-simple and has arbitrary embedding dimension greater than one and multiplicity greater than any chosen integer greater than the embedding dimension. In the present paper we construct, for each pair of positive integers \( m < s \), examples of \( \delta \)-simple 1-dimensional noetherian local domains \( R \) with \( \text{emb.dim}(R) = m + 1 \) while \( \text{mult}(R) = s \).

The method of construction is of interest in its own right. Previous examples were constructed by carefully choosing, within a power series algebra or the \( p \)-adic integers, an infinitely generated subalgebra \( R \); the computations then required to show that \( R \) has the desired properties were long and technical. Our method, which was suggested by the construction in [3, Section 3], is to define a subring of a power series algebra by using a derivation; the properties required are then demonstrated using this derivation. We do not construct the most general examples at the outset, but in Section I describe a version of our results that leads quite easily to examples of the type constructed by Akizuki and Lequain. Our most general construction is presented in Section II, and in the final section we indicate a modification that produces 2-dimensional examples.

All the rings considered in this paper are assumed to be commutative and associative with unit.

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I. THE BASIC CONSTRUCTION

PROPOSITION 1. Let \( A \) be a discrete valuation ring with maximal ideal \( xA \) and quotient field \( K \), let \( B \) be a local subring of \( A \), and let \( C \) be a subring of \( B \) with \( x \in C \). Assume that \( C + xA = A \) and \( C + xB = B \), and that \( C \cap xA = xC \). Let \( d_1, \ldots, d_m : B \to K \) be derivations such that \( d_i(C) = 0 \) for all \( i \), and assume that there exist elements \( z_1, \ldots, z_m \in B \cap xA \) such that \( d_i(z_j) = \delta_{ij} \) for all \( i, j \). Set

\[
R = \{ b \in B \mid d_i(b) \in A \text{ for } i = 1, \ldots, m \}.
\]

Then:

(a) \( R \) is a ring, \( C \subseteq R \subseteq B \subseteq R^\prime \subseteq \text{quotient field of } R \), and \( B \) is integral over \( R \).
(b) \( R \) is local with maximal ideal \( M = R \cap xA \), and \( M \) is generated by \( \{x, z_1, \ldots, z_m\} \).

(c) If \( \dim(B) = 1 \), then \( R \) is noetherian and \( \dim(R) = 1 \).

(d) Neither \( B \) nor the integral closure of \( R \) is finitely generated as an \( R \)-module.

Proof. Note from \( C + xA = A \) that \( B + xA = A \), whence

\[ B/(B \cap xA) \cong A/xA. \]

Thus \( B \cap xA \) is the maximal ideal of \( B \).

(a) That \( R \) is a subring of \( B \) is clear, as are the first, second, and fourth inclusions. Given \( b \in B \), we have

\[ d_1(b), \ldots, d_m(b) \in x^n A \]

for a suitable \( n \in \mathbb{N} \). Since \( x \in C \), each \( d_i(x) = 0 \), and so \( d_i(x^n b) = x^n d_i(b) \in A \). Thus \( x^n b \in R \) and so \( b \in R_x \). This proves the third inclusion.

Any \( b \in B \) may be written in the form \( b = c + b' \) for some \( c \in C \), \( b' \in xB \). As \( c \in R \), to see that \( b \) is integral over \( R \) we need only show that \( b' \) is integral over \( R \). Hence, there is no loss of generality in assuming that \( b \in xB \), whence \( b \in xA \). As above,

\[ d_1(b), \ldots, d_m(b) \in x^n A \]

for a suitable \( n \in \mathbb{N} \). Now \( b^n \in x^n A \), and so

\[ d_i(b^{n+1}) = (n + 1) b^n d_i(b) \in A \]

for each \( i \). Therefore \( b^{n+1} \in R \) and \( b \) is integral over \( R \).

(b) Since \( C \subseteq R \), we have \( R + xA = A \), whence \( R/M \cong A/xA \). Thus \( M \) is a maximal ideal of \( R \). Given \( r \in R - M \), we see that \( r \in B - xA \), and so \( r \) is a unit in both \( B \) and \( A \). As \( d_i(r) \in A \) for each \( i \), we obtain \( d_i(r^{-1}) \in A \) for each \( i \), and hence \( r^{-1} \in R \). Therefore \( R \) is local.

Now each \( z \in B \cap xA \cap R = M \). We claim that \( M = z_1 C + \cdots + z_m C + xR \), whence \( M \) is generated by \( \{x, z_1, \ldots, z_m\} \).

Consider \( r \in M \). For \( i = 1, \ldots, m \), we have \( d_i(r) \in A = C + xA \), whence there exists \( c_i \in C \) such that \( d_i(r) - c_i \in xA \). As \( d_i(c_i z) = 0 \) for \( i \neq j \), we obtain

\[ d_i(r - c_1 z_1 - \cdots - c_m z_m) = d_i(r) - c_i \in xA \]

for all \( i \). Replacing \( r \) by \( r - c_1 z_1 - \cdots - c_m z_m \), we may assume that \( d_i(r) \in xA \) for all \( i \).

Since \( C + xB = B \), we have \( r = u + xv \) for some \( u \in C \), \( v \in B \). Then \( xd_i(v) = d_i(r) \in xA \) for each \( i \), whence each \( d_i(v) \in A \) and so \( v \in R \). In
addition, \( r, xv \in xA \) and so \( u \in C \cap xA = xC \). Therefore \( r \in xR \), and the claim is proved.

(c) Since \( B \) is integral over \( R \), we have \( \dim(R) = 1 \). Hence, \( M \) is the only nonzero prime ideal of \( R \), and since it is finitely generated, \( R \) is noetherian.

(d) Extend \( d_i \) to the quotient field of \( R \) (via the quotient rule). If either \( B \) or the integral closure of \( R \) is finitely generated as an \( R \)-module, then \( B \subseteq u_1 R + \cdots + u_n R \) for some \( u_i \) in the quotient field of \( R \). Since \( d_i \) is defined on this field,

\[
d_i(B) = d_i(u_1)R + \cdots + d_i(u_n)R + u_1d_i(R) + \cdots + u_n d_i(R)
\leq d_i(u_1)A + \cdots + d_i(u_n)A + u_1 A + \cdots + u_n A \subseteq x^{-k} A
\]

for some \( k \in \mathbb{N} \), and hence \( d_i(x^{k+1}B) \subseteq xA \).

As \( B = C + xB \), we see that \( B = C + x^{k+1}B \), and thus \( d_i(B) = d_i(x^{k+1}B) \subseteq xA \). However, this contradicts the assumption that \( d_i(z_1) = 1 \).

EXAMPLE A. Given an arbitrary field \( F \), there exists a 1-dimensional noetherian local domain \( R \supseteq F \) such that the integral closure of \( R \) is not a finitely generated \( R \)-module.

Proof: Let \( x \) be an analytic indeterminate, let \( A = F[[x]] \) and \( C = F[x] \), and let \( K \) be the quotient field of \( A \). Obviously \( C + xA = A \) and \( C \cap xA = xC \).

Choose an element \( z \in xA \) which is transcendental over \( C \), and set \( B = F(x, z) \cap A \). (Here \( F(x, z) \) of course denotes the subfield of \( K \) generated by \( F \cup \{ x, z \} \).) Observe that \( xB = B \cap xA \), whence \( xB \) is a maximal ideal of \( B \) and \( C + xB = B \). Observe also that \( B \) is local. As

\[
\bigcap_{n=1}^{\infty} x^n B \subseteq \bigcap_{n=1}^{\infty} x^n A = 0,
\]

it follows that \( B \) is a discrete valuation ring, and so \( \dim(B) = 1 \).

Let \( d : B \to K \) be the restriction of the derivation \( \partial / \partial z \) on \( F(x, z) \). (This derivation exists because \( z \) is transcendental over \( C \).) Then \( d(C) = 0 \) and \( d(z) = 1 \). Note also that \( z \in B \cap xA \).

Now if \( R = \{ b \in B | d(b) \in A \} \), the desired properties follow immediately from Proposition 1.

Let \( \delta \) be a derivation on a ring \( R \). A \( \delta \)-ideal of \( R \) is any ideal \( I \) such that \( \delta(I) \subseteq I \). The ring \( R \) is said to be \( \delta \)-simple if \( R \neq 0 \) and 0, \( R \) are the only \( \delta \)-ideals of \( R \). In order to produce an example that is \( \delta \)-simple, we need considerable freedom to define \( \delta \). For this reason, we operate in characteristic zero with a field \( F \) of infinite transcendence degree over \( Q \).
EXAMPLE B. There exists a 1-dimensional noetherian local domain $R \supseteq \mathbb{Q}$ such that the integral closure of $R$ is not a finitely generated $R$-module (in particular, $R$ is not regular), and there is a derivation $\delta$ on $R$ such that $R$ is $\delta$-simple.

Proof. Let $F = \mathbb{Q}(y_1, y_2, \ldots)$ for some independent indeterminates $y_i$, and let $x$ be an analytic indeterminate. Since $F[x]$ is countable, there exist $m_i \in \{\pm 1\}$ for $i = 1, 2, \ldots$ such that the element

$$z = \sum_{i=1}^{\infty} m_i y_i x^i$$

is transcendental over $F[x]$. Now construct $A, C, K, B, d, R$ as in Example A. Then $R$ is a 1-dimensional noetherian local domain whose integral closure is not finitely generated as an $R$-module.

Define a derivation $\delta$ on $F$ such that $\delta(y_i) = -(i+1) m_i m_{i+1} y_{i+1}$ for all $i$, and extend $\delta$ to a derivation on $K$ according to the rule

$$\delta \left( \sum a_i x^i \right) = \sum \delta(a_i) x^i + \sum i a_i x^{i-1}$$

(for $a_i \in F$). Then $\delta(x) = 1$ and $\delta(z) = m_1 y_1$. Observe that $\delta(A) \subseteq A$ and $\delta(F[x, z]) \subseteq F[x, z]$, whence $\delta(B) \subseteq B$.

Since $d$ vanishes on $F[x]$, we see that $d\delta$ and $\delta d$ both vanish on $F[x] \cup \{z\}$, and hence the derivation $d\delta - \delta d$ vanishes on $F[x, z]$. Thus $d\delta - \delta d$ vanishes on $F(x, z)$ and hence on $B$. Now $\delta(R) \subseteq \delta(B) \subseteq B$ and $d\delta(R) = \delta d(R) \subseteq \delta(A) \subseteq A$,

and therefore $\delta(R) \subseteq R$.

As $R$ is 1-dimensional and $x$ lies in its maximal ideal, every nonzero ideal of $R$ contains a power of $x$. Since $\delta(x) = 1$, we conclude that $R$ must be $\delta$-simple. \hfill \blacksquare

David Jordan has pointed out to us a nice alternate version of this example. For his version, replace $F$ by an arbitrary field of characteristic zero, choose $z = e^x - 1$, and let $\delta: K \rightarrow K$ be the derivation $d/dx$. Here the derivation $d\delta - \delta d$ does not vanish on $F(x, z)$, but the derivation $d\delta - \delta d - d$ does. Hence, $d\delta$ and $\delta d + d$ agree on $B$, from which one obtains $\delta(R) \subseteq R$.

Proposition 2. In the situation of Proposition 1, $\dim_{R/M}(M/M^2) = m + 1$.

Proof. By Proposition 1, $M/M^2$ is spanned by the cosets $x + M^2$, $z_1 + M^2$, ..., $z_m + M^2$. Note that since $M^2 \subseteq x^2 A$, we must have $x \notin M^2$. 


For $i = 1, ..., m$, observe that $d_i(M) \subseteq A$ and $d_i(M^2) \subseteq Md_i(M) \subseteq xA$, whence $d_i$ induces an $(R/M)$-linear map $d_i^*: M/M^2 \rightarrow A/xA$. If

$$\alpha(x + M^2) + \beta_1(z_1 + M^2) + \cdots + \beta_m(z_m + M^2) = 0$$

for some $\alpha, \beta_i \in R/M$, then on applying the maps $d_i^*$ we obtain

$$\beta_i(1 + xA) = 0$$

and so $\beta_i = 0$, for $i = 1, ..., m$. Then $\alpha = 0$ because $x + M^2$ is nonzero.

Therefore the cosets $x + M^2$, $z_1 + M^2$, ..., $z_m + M^2$ are linearly independent over $R/M$, and hence they form a basis for $M/M^2$. Thus

$$\dim_{R/M}(M/M^2) = m + 1.$$
this case, conclusion (b) becomes that $M$ is generated by $N \cup \{z_1, \ldots, z_m\}$, and conclusions (a), (c), (d) follow as before. If we assume in addition that $N \cap (B \cap xA)^2 = N^2$, Proposition 2 can be modified to show that
\[
\dim_{R/M}(M/M^2) = m + \dim_{C/N}(N/N^2).
\]
We have not attempted a corresponding version of Proposition 3.

**Example C.** Let $m$ be any positive integer. Then there exists a 1-dimensional noetherian local domain $R \ni \mathbb{Q}$ such that $\text{emb.dim}(R) = \text{mult}(R) = m + 1$ and there is a derivation $\delta$ on $R$ such that $R$ is $\delta$-simple.

**Proof.** Let $F = \mathbb{Q}(\{y_{ki} | k = 1, \ldots, m; i = 1, 2, \ldots\})$ for some independent indeterminates $y_{ki}$, and let $x$ be an analytic indeterminate. Since $F[x]$ is countable, there exist $m_{ki} \in \{ \pm 1 \}$ such that the elements
\[
z_k = \sum_{i=1}^{x} m_{ki} y_{ki} x^i \quad \text{(for } k = 1, \ldots, m)\]
are algebraically independent over $F[x]$.

As in Example A, we set $A = F[[x]]$ and $C = F[x]$, while $K$ denotes the quotient field of $A$. Then $C + xA = A$ and $C \cap xA = xC$. Next, set
\[
B = F(x, z_1, \ldots, z_m) \cap A,
\]
and note that $xB = B \cap xA$. Then $B$ is a local subring of $A$ with maximal ideal $xB$, and $C + xB = B$. As in Example A, we see that $B$ is a discrete valuation ring, and so $\dim(B) = 1$. Observe that each $z_k \in B \cap xA = xB$.

For $i = 1, \ldots, m$, let $d_i : B \to K$ be the restriction of the derivation $\partial/\partial z_i$ on $F(x, z_1, \ldots, z_m)$. Now set
\[
R = \{b \in B | d_i(b) \in A \text{ for } i = 1, \ldots, m\}.
\]
In view of Propositions 1, 2, 3, we see that $R$ is a 1-dimensional noetherian local domain with $\text{emb.dim}(R) = \text{mult}(R) = m + 1$.

Define a derivation $\delta$ on $F$ such that $\delta(y_{ki}) = -(i + 1) m_{ki} m_{k,i+1} y_{k,i+1}$ for all $k, i$, and extend $\delta$ to a derivation on $K$ according to the rule
\[
\delta \left( \sum \alpha_i x^i \right) = \sum \delta(\alpha_i) x^i + \sum i \alpha_i x^{i-1}
\]
(for $\alpha_i \in F$). Then $\delta(x) = 1$ and $\delta(z_k) = m_{ki} y_{ki}$ for all $k$. As in Example B, we find that $\delta(R) \subseteq R$. Since $R$ is 1-dimensional, $x$ lies in its maximal ideal, and $\delta(x) = 1$, we conclude that $R$ is $\delta$-simple. \[\square\]
A version of Example C suggested by Jordan consists of replacing $F$ by an arbitrary field of characteristic zero, choosing $z_k = e^{x^k} - 1$ for $k = 1, \ldots, m$, and again letting $\delta: K \to K$ be the derivation $d/dx$. (That $z_1, \ldots, z_m$ are algebraically independent over $F[x]$ is a consequence of the Lindemann-Weierstrass Theorem.) The key to obtaining $\delta(R) \subseteq R$ here lies in showing that the derivations

$$d, \delta - \delta d_i - ix^i \frac{1}{d_x} F(x, z_1, \ldots, z_m) \to K$$

all vanish.

In [2, Proposition 1], de Souza Doering and Lequain construct examples similar to Example C. For any integers $u_1, \ldots, u_n \geq 2$, they construct a 1-dimensional noetherian local domain $R \supseteq \mathbb{Q}$ such that $\text{emb.dim}(R) = n + 1$ and $\text{mult}(R) = u_1 u_2 \cdots u_n$, and there is a derivation $\delta$ on $R$ such that $R$ is $\delta$-simple. Except in the case that $n = 1$ and $u_1 = 2$, their examples satisfy $\text{mult}(R) > \text{emb.dim}(R)$.

II. ARBITRARY EMBEDDING DIMENSION AND MULTIPLICITY

In this section we produce, for arbitrary positive integers $m$ and $t$, examples of 1-dimensional $\delta$-simple noetherian local domains with $\text{emb.dim}(R) = m + 1$ and $\text{mult}(R) = m + t$. The embedding dimension is fixed at $m + 1$ by using $m$ derivations in the definition of $R$. However, if we wish to increase the multiplicity then we cannot require that $M^2 = xM$ as in Proposition 3. The increase in multiplicity is generated by using higher powers of the derivations. The following proposition generalizes Proposition 1 to this setting.

**Proposition 4.** Let $A \supseteq \mathbb{Q}$ be a discrete valuation ring with maximal ideal $xA$ and quotient field $K$, let $B$ be a local subring of $A$, and let $C$ be a subring of $B$ with $\mathbb{Q}[x] \subseteq C$. Assume that $C + xA = A$ and $C + xB = B$, and that $C \cap xA = xC$. Let $d_1, \ldots, d_m$ be $C$-linear derivations from $K$ to itself, and assume that there exist elements $z_1, \ldots, z_m \in B \cap xA$ such that $d_i(z_j) = \delta_{ij}$ for all $i, j$. Let $t_1, \ldots, t_m \in \mathbb{N}$, and set

$$R = \{ b \in B \mid d_i(b) \in A \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, t_i \}.$$  

Then:

(a) $R$ is a ring, $C \subseteq R \subseteq B \subseteq R_x \subseteq (quotient \ field \ of \ R)$, and $B$ is integral over $R$.

(b) $R$ is local with maximal ideal $M = R \cap xA$, and $M$ is generated by $\{x, z_1, \ldots, z_m\}$.  

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(c) If $\dim(B) = 1$, then $R$ is noetherian and $\dim(R) = 1$.

(d) Neither $B$ nor the integral closure of $R$ is finitely generated as an $R$-module.

Proof. As in the proof of Proposition 1, note that $B \cap xA$ is the maximal ideal of $B$. Set $t = \max\{t_1, \ldots, t_m\}$.

(a) That $R$ is a subring of $B$ is clear, as are the first, second, and fourth inclusions. Given $b \in B$, we have

$$\{d_i(b) | i = 1, \ldots, m; j = 1, \ldots, t \} \subseteq x^nA$$

for a suitable $n \in \mathbb{N}$. Then $d_i(x^n b) \in A$ for $i = 1, \ldots, m$ and $j = 1, \ldots, t$, whence $x^n b \in R$ and so $b \in R$. This proves the third inclusion.

Any $b \in B$ may be written in the form $b = c + b'$ for some $c \in C$, $b' \in xB$. As $c \in R$, to see that $b$ is integral over $R$ we need only show that $b'$ is integral over $R$. Hence, there is no loss of generality in assuming that $b \in xR$, whence $b \in xA$. As above, there exists $n \in \mathbb{N}$ such that $d_i(b) \in x^{-n}A$ for $i = 1, \ldots, m$ and $j = 1, \ldots, t$. By induction on $j$, each $d_i(b^{n+t})$ is a $\mathbb{Z}$-linear combination of terms

$$b^{p(0)}d_i(b)^{p(1)}d_j^2(b)^{p(2)} \cdots d_j^{p(j)}(b)^{p(j)}$$

where $p(0), \ldots, p(j)$ are nonnegative integers whose sum is $nt + t$ and $p(0) \geq nt + t - j$. Since $p(0) \geq nt$ and $b \in xA$, we have $b^{p(0)} \in x^nA$. On the other hand, since $p(1) + \cdots + p(j) \leq t$ and $d_i(b), \ldots, d_j(b) \in x^{-n}A$, we have

$$d_i(b)^{p(1)}d_j^2(b)^{p(2)} \cdots d_j^{p(j)}(b)^{p(j)} \in x^{-n}A,$$

and hence $d_i(b^{n+t}) \in A$ for $i = 1, \ldots, m$ and $j = 1, \ldots, t$. Therefore $b^{n+t} \in R$ and $b$ is integral over $R$.

(b) As in the proof of Proposition 1, $M$ is a maximal ideal of $R$. Given $r \in R - M$, we see that $r$ is a unit in both $B$ and $A$. For $i = 1, \ldots, m$ and $j = 1, \ldots, t$, we see by induction on $j$ that $d_i(r^{-1})$ is a $\mathbb{Z}$-linear combination of terms

$$r^{-p(0)}d_i(r)^{p(1)}d_j^2(r)^{p(2)} \cdots d_j^{p(j)}(r)^{p(j)}$$

where $p(0), \ldots, p(j)$ are nonnegative integers such that $-p(0) + p(1) + p(2) + \cdots + p(j) = -1$. Since $d_i(r) \in A$ for $j = 1, \ldots, t$, it follows that $d_i(r^{-1}) \in A$ for $j = 1, \ldots, t$. Hence, $r^{-1} \in R$, and therefore $R$ is local.

Now we show that any element $r \in M$ lies in $xR + z_1R + \cdots + z_mR$. Let $s = t_1$. Since $d_i^s(r) \in A = C + xA$, there exists $c \in C$ such that $d_i^s(r) - s! c \in xA$, and hence $d_i^s(r - cz_1^1) \in xA$. Thus, replacing $r$ by $r - cz_1^1$,
we may assume that $d_{i_1}^t(r) \in xA$. Next, there exists $c' \in C$ such that
\[ d_{i_1}^{t-1}(r) - (s-1)! c' \in xA, \] and hence
\[ d_{i_1}^{t-1}(r - c'z_{i_1}^{t-1}) = d_{i_1}^{t-1}(r) - (s-1)! c' \in xA. \]
\[ d_{i_1}^t(r - c'z_{i_1}^{t-1}) = d_{i_1}^t(r) \in xA. \]

Thus we may now assume that $d_{i_1}^{t-1}(r), d_{i_1}^t(r) \in xA$.

Continuing in this manner, we see that there is no loss of generality in assuming that $d_{i_j}^t(r) \in xA$ for $j = 1, \ldots, t_1$. Similarly, we may also assume that $d_{i_i}^t(r) \in xA$ for $i = 1, \ldots, m$ and $j = 1, \ldots, t_i$. Since $C + xB = B$, we have $r = u + xv$ for some $u \in C$, $v \in B$. Then
\[ xd_{i_j}^t(v) = d_{i_1}^t(r) \in xA \]
for $i = 1, \ldots, m$ and $j = 1, \ldots, t_i$, whence these $d_{i_j}^t(v) \in A$ and so $v \in R$. In addition, $r, xv \in xA$ and so $u \in C \cap xA = xC$. Therefore $r \in xR$, proving that $M$ is indeed generated as claimed.

(c) This is clear.
(d) Use the same proof as Proposition 1(d).

PROPOSITION 5. In the situation of Proposition 4, $\dim_{R/M}(M/M^2) = m + 1$.

Proof. Use the same proof as Proposition 2.

In the 1-dimensional case of Proposition 4, varying the parameters $t_1, \ldots, t_m$ will vary the multiplicity of $R$. The easiest way to control the multiplicity is to arrange $M^{t+1} = xM^t$ for some $t$, and we do this by using the first $t$ powers of one of the derivations.

PROPOSITION 6. Let $A \supset \mathbb{Q}$ be a discrete valuation ring with maximal ideal $xA$ and quotient field $K$, let $B$ be a 1-dimensional local subring of $A$, and let $C$ be a subring of $B$ with $\mathbb{Q}[x] \subseteq C$. Assume that $C + xA = A$ and $C + xB = B$, and that $C \cap xA = xC$. Let $d_1, \ldots, d_m$ be $C$-linear derivations from $K$ to itself, and assume that there exist elements $z_1, \ldots, z_m \in B \cap xA$ such that $d_i(z_j) = \delta_{ij}$ for all $i, j$. Let $t \in \mathbb{N}$, and set
\[ R = \{ b \in B \mid d_{i_1}^t(b) \in A \text{ for } j = 1, \ldots, t \text{ and } d_{i_1}^t(b) \in A \text{ for } i = 2, \ldots, m \}. \]
Then:

(a) $R$ is a 1-dimensional noetherian local domain with maximal ideal $M = R \cap xA$, the integral closure of $R$ is not a finitely generated $R$-module, and $\operatorname{emb.dim}(R) = m + 1$. 
(b) Assume that $z_i \in x_iA$ and $z_{i+1} \in x_iB$, while $z_i, z_j \in xB$ for all $i, j$. Then $\text{mult}(R) = m + t$.

Proof. (a) Propositions 4 and 5.

(b) By Proposition 4, $M = xR + z_1R + \cdots + z_mR$. For $k = 1, 2, \ldots$, observe that $M^k \subseteq x^kA$, whence $x^{k-1} \notin M^k$.

For $i, j = 1, \ldots, m$, we have $z_i z_j = x b_i$ for some $b_i \in B$. Since $z_i, z_j \in xA$, it follows that $b_i \in xA$. For $k = 1, \ldots, m$, observe that
\[
x d_k(b_i) = d_k(z_i z_j) = d_k(z_i) z_j + z_i d_k(z_j) \in xA,
\]
whence $d_k(b_i) \in A$. If $j \neq 1$, then
\[
x d^2_k(b_i) = d^2_k(z_i z_j) = d^2_k(z_i) z_j = 0
\]
aid so $d^2_k(b_i) = 0$. Thus $b_i \in R$ in this case, and then $b_i \in R \cap xA = M$. Therefore
\[
z_i z_j \in xM
\]
for all $i = 1, \ldots, m$ and $j = 2, \ldots, m$.

In addition, $z_{i+1}^* = x b$ for some $b \in B$, and $b \in xA$ because $z_i \in xA$. For $i = 2, \ldots, m$, observe that $x d_i(b) = d_i(z_{i+1}^{-1}) = 0$, whence $d_i(b) = 0$. For $j = 1, \ldots, t$, we have
\[
x d_i^t(b) = d_i^t(z_{i+1}^+) = (t + 1) t \cdots (t + 2 - j) z_{i+1}^+ \in \sum_{i=1}^t aA,
\]
whence $d_i^t(b) \in A$. Thus $b \in R \cap xA = M$, and therefore
\[
z_{i+1}^+ \in x^i M \subseteq xM^i.
\]

We now claim that
\[
M^k = \sum_{j=0}^k x^j z_1^j R + \sum_{i=2}^m x^{k-1} z_i R
\]
for $k = 1, 2, \ldots$. For $k = 1$, this was observed above. If this formula holds for $M^1, \ldots, M^k$, then
\[
M^{k+1} = \sum_{j=0}^k x^j z_1^j M + \sum_{i=2}^m x^{k-1} z_i M
\]
\[
= \sum_{j=0}^{k+1} x^{k+1-j} z_1^j R + \sum_{i=2}^m x^{k} z_i R
\]
\[
+ \sum_{i=2}^m \sum_{j=1}^k x^{k-j} z_1^j z_i R + \sum_{i=2}^m \sum_{j=2}^m x^{k-1} z_i z_j R.
\]
For $i, p = 2, \ldots, m$, we have $z_iz_p \in xM$, and so
\[ x^{k-1}z_iz_p \in x^kM = x^{k+1}R + x^kz_1R + \cdots + x^kz_mR. \]
For $i = 2, \ldots, m$ and $j = 1, \ldots, k$, we have $z_iz_j \in xM$ and hence
\[ x^k \cdot jz_iz_j \in x^k \cdot jz_j^{-1}xM \subseteq x^{k+1-j}M^j. \]
Consequently,
\[
\begin{align*}
&= \sum_{q=0}^{j} x^{k+1-\frac{j}{q}}qz_q^qR + \sum_{i=2}^{m} x^{k-1}z_iR.
\end{align*}
\]
Therefore
\[
M^{k+1} = \sum_{j=0}^{k+1} x^{k+1-j}z_1^jR + \sum_{i=2}^{m} x^kz_iR,
\]
completing the induction step.

From these formulas, and the fact that $z_1^{t+1} \in xM'$, it follows that
\[
M^{t+1} = xM' + z_1^{t+1}R = xM'.
\]
Hence, $M^{t+n} = x^nM' \cong M'$ for $n = 1, 2, \ldots$, and so
\[
\text{mult}(R) = \dim_{R/M} (M'/M^{t+1}).
\]
We have seen that $M'$ is generated by the $m + t$ elements
\[
\{ x^{t-j}z_j^t | j = 0, \ldots, t \} \cup \{ x^{t-1}z_i | i = 2, \ldots, m \}.
\]
Thus it only remains to show that the cosets of these elements in $M'/M^{t+1}$ are linearly independent over $R/M$. Consider any relation
\[
\sum_{j=0}^{t} \alpha_j(x^{t-j}z_j^t + M^{t+1}) + \sum_{i=2}^{m} \beta_i(x^{t-1}z_i + M^{t+1}) = 0,
\]
where the $\alpha_j, \beta_i \in R/M$.

For $i = 2, \ldots, m$, observe that $d_i$ induces an $(R/M)$-linear map
\[
d_i^*: M'/M^{t+1} \to x^{t-1}A/x'A.
\]
such that \( d_i^*(x^{-i}z_1 + M' + 1) = 0 \) for \( j = 0, \ldots, t \) and \( d_i^*(x^{-i}z_k + M' + 1) = 0 \) for \( k \neq i \), while

\[
d_i^*(x^{-i}z_i + M' + 1) = x^{i-1} + x^i A \neq 0.
\]

Applying \( d_i^* \) to (\( \dagger \)), we conclude that \( \beta_i = 0 \).

For \( j = 0, \ldots, t \), we infer by induction that \( d_i^!(M') \) is contained in the product of \( M'^{-j} \) with the subring of \( A \) generated by \( d_i!(M) \). Thus \( d_i^!(M') \subseteq x^{-j}A \), and similarly \( d_i^!(M^{-1}) \subseteq x^{i-1}A \). Hence, \( d_i! \) induces an \( (R/M) \)-linear map

\[
(d_i^!)^*: M'/M^{-1} \to x^{i-1}A/x^{i-1-i}A
\]

such that \( (d_i^!)^*(x^{-j}z_1 + M' + 1) = 0 \) for \( p = 0, \ldots, j - 1 \) while

\[
(d_i^!)^*(x^{-j}z_i + M' + 1) = j! x^{i-1} + x^{i-1-i}A \neq 0.
\]

Applying the maps \( (d_i^!)^* \), \( (d_{i-1}^!)^* \), \ldots, \( (d_0^!)^* \) to (\( \dagger \)), we conclude that \( \alpha_i, \alpha_{i-1}, \ldots, \alpha_0 = 0 \).

Therefore the cosets of the elements

\[
\{ x^{-j}z_j | j = 0, \ldots, t \} \cup \{ x^{-i}z_i | i = 2, \ldots, m \}
\]

in \( M'/M'^{t+1} \) are linearly independent over \( R/M \), as desired. \( \blacksquare \)

For use in our final example, we recall the well-known result that any derivation on a field of characteristic zero extends to a derivation on any extension field [8, Chapter II, Sect. 17, Corollary 3].

**Example D.** Let \( m, t \) be any positive integers. Then there exists a 1-dimensional noetherian local domain \( R \supseteq \mathbb{Q} \) such that \( \text{emb.dim}(R) = m + 1 \) and \( \text{mult}(R) = m + t \), and there is a derivation \( \delta \) on \( R \) such that \( R \) is \( \delta \)-simple.

**Proof.** Let \( F = \mathbb{Q}(\{ y_{ki} | k = 1, \ldots, m; i = 1, 2, \ldots \}) \) for some independent indeterminates \( y_{ki} \), and let \( x \) be an analytic indeterminate. Since \( F[x] \) is countable, there exist \( m_{ki} \in \{ \pm 1 \} \) such that the elements

\[
z_1 = \sum_{i=t}^\infty m_{1i} y_{1i} x^i \quad \text{and} \quad z_k = \sum_{i=1}^\infty m_{ki} y_{ki} x^i \quad \text{(for } k = 2, \ldots, m) \]

are algebraically independent over \( F[x] \).

As in Example A, we set \( A = F[[x]] \) and \( C = F[x] \), while \( K \) denotes the quotient field of \( A \). Then \( C + xA = A \) and \( C \cap xA = xC \). Next, set

\[
B = F(x, z_1, \ldots, z_m) \cap A,
\]
and note that \( xB = B \cap xA \). Then \( B \) is a local subring of \( A \) with maximal ideal \( xB \), and \( C + xB = B \). As in Example A, we see that \( B \) is a discrete valuation ring, and so \( \dim(B) = 1 \). Observe that each \( z_k \in B \cap xA = xB \).

For \( i = 1, \ldots, m \), let \( d_i \) denote the derivation \( \partial/\partial z_i \) on \( F(x, z_1, \ldots, z_m) \). Since \( F \) has characteristic zero, each \( d_i \) extends to a derivation \( K \to K \). Now set

\[
R = \{ b \in B \mid d'_j(b) \in A \text{ for } j = 1, \ldots, t \text{ and } d_i(b) \in A \text{ for } i = 2, \ldots, m \}.
\]

In view of Proposition 6, we see that \( R \) is a 1-dimensional noetherian local domain with \( \embdim(R) = m + 1 \) and \( \mult(R) = m + t \).

Define a derivation \( \delta \) on \( F \) such that \( \delta(y_{ki}) = -(i+1)\,m_k m_{k,i+1} y_{k,i+1} \) for all \( k, i \), and extend \( \delta \) to a derivation on \( K \) according to the rule

\[
\delta\left( \sum a_i x^i \right) = \sum \delta(a_i) x^i + \sum i a_i x^{i-1}
\]

(for \( a_i \in F \)). Then \( \delta(x) = 1 \) and \( \delta(z_1) = m_{1,1} y_1 x^1 \) while \( \delta(z_k) = m_{k,1} y_{k1} \) for \( k = 2, \ldots, m \). As in Example B, we find that \( \delta(R) \subseteq R \). Since \( R \) is 1-dimensional, \( x \) lies in its maximal ideal, and \( \delta(x) = 1 \), we conclude that \( R \) is \( \delta \)-simple.

To construct a version of Example D using Jordan's idea, replace \( F \) by an arbitrary field of characteristic zero, choose

\[
z_1 = e^x - \sum_{i=0}^{i-1} x^i / i!
\]

and \( z_k = e^x - 1 \) for \( k = 2, \ldots, m \), and let \( \delta: K \to K \) be the derivation \( d/dx \). One checks that \( d_i \delta \) and \( \delta d_i + ix^{i-1}d_i \) agree on \( B \) (for \( i = 1, \ldots, m \)), and that \( d'_j \delta \) and \( (\delta + j) d'_j \) agree on \( B \) (for \( j = 1, \ldots, t \)).

### III. Dimension Two

Establishing that a ring \( R \) is noetherian in dimension one is relatively easy: it is enough to check that prime ideals are finitely generated, and for 1-dimensional local domains there is only one prime ideal to check. With a little further work, the method of Proposition 1 may also be used to construct examples of 2-dimensional noetherian local rings. The hypothesis that \( C \cap xA = xC \) must be relaxed to the requirement that \( C \cap xA \) be a finitely generated ideal of \( C \) (since otherwise there will not be any 2-dimensional rings \( B \) fitting the hypotheses). Then Proposition 1 remains valid with one change: \( M \) is no longer generated by \( \{ x, z_1, \ldots, z_m \} \), but \( M \) is still finitely generated.
In order to maintain a hold on the height one primes, we require that the ring $B$ be a unique factorization domain. The method we use to show that height one prime ideals of $R$ are finitely generated is adapted from the proof of [3, Proposition 3.3].

**Proposition 7.** Let $A$ be a discrete valuation ring with maximal ideal $xA$ and quotient field $K$, let $B$ be a local subring of $A$, and let $C$ be a subring of $B$ with $x \in C$. Assume that $C + xA = A$ and $C + xB = B$, and that $C \cap xA$ is a finitely generated ideal of $C$. Let $d_1, \ldots, d_m : B \to K$ be derivations such that $d_i(C) = 0$ for all $i$, and assume that there exist elements $z_1, \ldots, z_m \in B \cap xA$ such that $d_i(z_j) = \delta_{ij}$ for all $i, j$. Set

$$R = \{ b \in B | d_i(b) \in A \text{ for } i = 1, \ldots, m \}.$$ 

Then:

(a) $R$ is a ring, $C \subseteq R \subseteq B \subseteq \text{R}_x \subseteq \text{(quotient field of } R\text{)}$, and $B$ is integral over $R$.

(b) $R$ is local with maximal ideal $M = R \cap xA$, and $M$ is finitely generated.

(c) Neither $B$ nor the integral closure of $R$ is finitely generated as an $R$-module.

Now assume further that $\dim(B) = 2$ and that $B$ is a unique factorization domain. Then:

(d) $R$ is noetherian and $\dim(R) = 2$.

(e) If $P$ is any height 1 prime ideal of $R$, then $R_P = B_Q$ for some height 1 prime ideal $Q$ of $B$, and so $R_P$ is a discrete valuation ring.

**Proof.** (a) (b) (c) The proof of Proposition 1 may be used, with one modification: under the present hypotheses the proof of (b) shows that $M$ is generated by $(C \cap xA) \cup \{ z_1, \ldots, z_m \}$.

Now assume that $B$ is a 2-dimensional unique factorization domain. Note from $C + xA = A$ and $x \in C$ that $C + x^nA = A$ for all $n \in \mathbb{N}$. Let $| \cdot |$ be the valuation on $K$ induced from $A$, so that

$$|x| = \sup \{ n \in \mathbb{Z} | x \in x^nA \}$$

for all $\alpha \in K$.

(d) We have $\dim(R) = 2$ because $B$ is integral over $R$, and the only height 2 prime ideal of $R$, namely $M$, is finitely generated by (b). Hence, we need only show that any height 1 prime ideal $P$ of $R$ is finitely generated. Since $B$ is integral over $R$, we have $P = R \cap Q$ for some height 1 prime
ideal $Q$ in $B$. As $B$ is a unique factorization domain, $Q = qB$ for some non-zero $q \in Q$. Set

$$N = \{ b \in B | qb \in R \},$$

and note that $P = qN$. Hence, it suffices to show that $N$ is a finitely generated $R$-module.

For any $b \in N$, we have $qb \in R$ and so $d_i(qb) \in A$ for $i = 1, \ldots, m$, whence

$$qd_i(b) \in A + d_i(q) A$$

and so $d_i(b) \in q^{-1}A + q^{-1}d_i(q) A$. Thus each of the sets $|d_i(N)|$ is bounded below. Set $N_1 = N$ and

$$N_i = \{ a \in N | d_i(a), \ldots, d_{i-1}(a) \in A \}$$

for $i = 2, \ldots, m$, and choose elements $e_i \in N_i$ for $i = 1, \ldots, m$ such that $|d_i(e_i)|$ is minimal in $|d_i(N_i)|$.

We claim that $N = C e_1 + \cdots + C e_m + (N \cap R)$. Given any $b \in N$, we have $|d_i(b)| \geq |d_i(e_i)|$, whence $d_i(b) = a_i d_i(e_i)$ for some $a_i \in A$. Now $d_i(e_i) \in x^{-n}A$ for some $n \in \mathbb{N}$. As $C + x^n A = A$, there exists $c_i \in C$ such that $c_i - a_i \in x^n A$. Then

$$d_i(b - c_i e_i) = (a_i - c_i) d_i(e_i) \in A,$$

and so $b - c_i e_i \in N_2$. Now $|d_2(b - c_1 e_1)| \geq |d_2(e_2)|$. Arguing as above, there exists $c_2 \in C$ such that

$$d_2(b - c_1 e_1 - c_2 e_2) \in A.$$

Since $b - c_1 e_1$ and $e_2$ both lie in $N_2$, we also have

$$d_1(b - c_1 e_1 - c_2 e_2) \in A,$$

and so $b - c_1 e_1 - c_2 e_2 \in N_3$. Continuing in this manner, we obtain $c_1, \ldots, c_m \in C$ such that

$$d_i(b - c_1 e_1 - \cdots - c_m e_m) \in A$$

for all $i = 1, \ldots, m$, whence $b - c_1 e_1 - \cdots - c_m e_m \in N \cap R$. This completes the proof of the claim.

This claim having been proved, it suffices to show that $N \cap R$ is a finitely generated ideal of $R$. 
For any positive integer \( k \), note that \( M^k \) is finitely generated, and that \( R/M^k \) is noetherian (since its only prime ideal, namely \( M/M^k \), is finitely generated). Hence, any ideal of \( R \) containing \( M^k \) is finitely generated. Thus it suffices to show that \( N \cap R \) contains a power of \( M \). As \( M \) is finitely generated, we need only show that \( N \cap R \) contains a power of each element of \( M \).

Given any \( s \in M \), we have \( s \in xA \), and so there is some \( k \in \mathbb{N} \) such that \( s^{-1}d_i(s) \) and \( s^k d_i(q) \) lie in \( A \) for \( i = 1, \ldots, m \). It follows that \( d_i(s^k q) \in A \) for all \( i \), whence \( s^k q \in R \) and so \( s^k \in N \cap R \), as desired.

(e) Since \( B \) is integral over \( R \), there is a height 1 prime ideal \( Q \) of \( B \) such that \( Q \cap R = P \), and we note that \( R_P \subseteq B_Q \). As \( Q \) is principal, \( B_Q \) is a discrete valuation ring.

Since \( \text{ht}(P) = 1 \), there exists an element \( s \in M - P \), and \( s^{-1} \in R_P \). Consider any \( b \in B \). As \( s \in xA \), there is some \( k \in \mathbb{N} \) such that \( s^{-1}d_i(s) b \) and \( s^k d_i(b) \) lie in \( A \) for \( i = 1, \ldots, m \). It follows that \( d_i(s^k b) \in A \) for all \( i \), whence \( s^k b \in R \) and so \( b \in R_P \). Thus \( B \subseteq R_P \).

Given any \( b \in B - Q \), we find as above that \( s^k b \in R \) for some \( k \in \mathbb{N} \). Since \( s \notin P \), we have \( s \notin Q \) and so \( s^k b \notin Q \), whence \( s^k b \notin P \). Consequently, \( (s^k b)^{-1} \in R_P \), and thus \( b^{-1} \in R_P \). Therefore \( B_Q = R_P \).

One way to find rings \( A \supseteq B \supseteq C \) satisfying the hypotheses of Proposition 7 is as follows: let \( A = F[[x]] \) for some field \( F \), choose elements \( y, z \in xA \) which are algebraically independent over \( F(x) \), let \( C \) be the localization of \( F[x, y] \) at the maximal ideal \( F[x, y] \cap xA \), let \( D = F(x, z) \cap A \), let \( B \) be the localization of \( D[y] \) at the maximal ideal \( D[y] \cap xA \), and let \( d : B \to K \) be the restriction of the derivation \( \partial/\partial z \) on \( F(x, y, z) \).

We close with the following question. Suppose that \( A \supseteq B \) are domains with the same quotient field \( K \), and that \( d_1, \ldots, d_m \) are derivations from \( B \) to \( K \). Under what circumstances must the ring

\[
R = \{ b \in B \mid d_i(b) \in A \text{ for } i = 1, \ldots, m \}
\]

be noetherian? That this is not always the case may be seen from the example \( A = B = \mathbb{Q}[x, y] \) and \( d_1 = y^{-1} \partial/\partial x \), in which case \( R = \mathbb{Q} + yA \).

REFERENCES

BAD LOCAL DOMAINS