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Variational Methods for Nonlinear Eigenvalue Problems Associated with Thermal Ignition

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1. INTRODUCTION

The following boundary value problem arises in the study of nonlinear heat generation (in the steady-state):

$$L(u) = \lambda f(x, u), \qquad x = (x_1, x_2, \dots, x_m) \in D, \tag{1}$$

$$B(u) \equiv \alpha(x)u + \beta(x)(\partial u/\partial \nu) = 0, \qquad x \in \partial D, \tag{2}$$

where L is the uniformly elliptic, self-adjoint, second-order operator

$$L(u) \equiv - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u, \tag{3}$$

and D is the interior of a bounded region of R^m with a smooth boundary ∂D. The coefficients $a_{ij}(x) = a_{ji}(x)$ are continuously differentiable, $a_0(x) \geqslant 0$ is continuous, and for all unit vectors $p = (p_1, p_2, \dots, p_m)$,

$$\sum_{i,j=1}^{m} p_i a_{ij}(x) p_j > 0, \qquad x \in D.$$

In Eq. (2) $\partial/\partial\nu$ is the conormal derivative:

$$\frac{\partial u}{\partial \nu} \equiv \sum_{i,j=1}^{m} n_i(x)\, a_{ij}(x) \frac{\partial u}{\partial x_j}, \tag{4}$$

where $n(x) = (n_1(x), n_2(x),..., n_m(x))$ is the outer unit normal to ∂D at a point x on the surface. The functions $\alpha(x)$ and $\beta(x)$ are assumed to be nonnegative and piecewise continuous on ∂D. The boundary ∂D will be subdivided into two disjoint parts ∂D_1 and $\partial D_2 = \partial D - \partial D_1$, where $\alpha(x) \neq 0$, $\beta(x) \equiv 0$ on ∂D_1 and $\beta(x) \neq 0$ on ∂D_2.

This paper will be concerned with estimates for the least upper bound λ_{cr} of the values of λ for which the nonlinear eigenvalue problems (1) and (2) have real positive solutions. This parameter is the critical explosion parameter for the unsteady problem, that is, for $\lambda \geqslant \lambda_{cr}$ there does not exist a stable solution of the time-dependent equations (see Wake [1] and Keller and Cohen [2]). This problem has been widely discussed. Keller and Cohen [2] gave upper and lower bounds for λ_{cr} under various requirements on the monotonicity of $f(x, u)$, $f_u(x, u)$ with u. In particular they were able to show that if:

H-0: $f(x, u)$ is continuous and positive for $x \in D$, $u \in R$;

H-1: $f(x, 0) \equiv f_0(x) > 0$, $x \in D$;

H-2: $f_u(x, u) > 0$ and is continuous for $x \in D$ and $u \in R$;

and λ_{cr} exists; then problems (1) and (2) have (positive) solutions for all λ in $0 < \lambda < \lambda_{cr}$. If we denote by $\mathbf{u}(x)$ the minimal solution of (1) and (2) (that is, $u(x) \geqslant \mathbf{u}(x)$ on D for any solution of (1) and (2)), Keller and Cohen showed that, for each $\lambda \in [0, \lambda_{cr}]$,

$$\lambda \leqslant \mu_1(\lambda), \tag{5}$$

where $\mu_1(\lambda)$ is the principal eigenvalue of the linearized system

$$L(v) = \mu f_u(x, \mathbf{u}(x))v, \qquad x \in D, \tag{6}$$

$$B(v) = 0, \qquad x \in \partial D. \tag{7}$$

If, in addition, f were concave with u, that is

H-3*a*: $f_u(x, u_1) < f_u(x, u_2)$ on D if $u_1 > u_2$;

then they were able to show that the problem has no solution for $\lambda = \lambda_{cr}$, but that

$$\lim_{\lambda \nearrow \lambda_{cr}} \mu_1(\lambda) = \lambda_{cr}. \tag{8}$$

However, if f were convex with u, that is,

H-3*b*: $f_u(x, u_1) > f_u(x, u_2)$ on D if $u_1 > u_2$;

they were unable to show in [2] that Eq. (8) held for this case.

There is some evidence that this result is true for convex f and indeed Keller and Cohen conjecture that this is so. Also Hudjaeov [3] has shown that Eq. (8) is valid in the special case when f is separable, i.e., when

$$f(x, u) = a(x)\, h(u), \tag{9}$$

where $h'(u)$ is an increasing function of u, $h(0) > 0$ and $a(x) > 0$. Recently Keller and Keener [4] have been able to show that Eq. (8) does hold for convex f satisfying

$$H\text{-}4: \qquad \lim_{u\to\infty} u^2 \frac{\partial}{\partial u}[u^{-1}f(x, u)] < 0, \qquad x \in D,$$

and that positive solutions exist for this case.

The present paper will propose a variational method of determining λ_{cr} under the hypotheses H-0, 1, 2, 3b (convex f), which will be a nonlinear analog of the well known procedures for linear eigenvalue problems. This method provides a useful device in practice as a "rough" approximation to the solution seems to lead to a "good" approximation to the critical parameter λ_{cr}.

Variational methods have been used by other authors, notably Simpson and Cohen [5] and Levinson [6, 7], for equations like (1) but not in order to obtain estimates for λ_{cr}. Levinson used the variational technique to establish the existence of solutions, whereas we shall simply find a necessary and sufficient condition for the existence of solutions, not proving the latter result. (In addition the results of [4] ensure the existence of solutions for some special cases.)

The next section will give the main results of the paper and the last section will give a specific example to illustrate the application of the method.

2. Main Results

As each nonlinear eigenvalue problem (1) and (2) has a nondiscrete spectrum, we are led to introduce a parameter c, which ensures that for some value of that parameter each value of the spectrum is achieved. In a manner similar to that for a variational method for a linear eigenvalue problem, we define a functional

$$J(u) = \int_D \sum_{i,j=1}^{m} \left(a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + a_0(x)\, u^2\right) dx + \int_{\partial D} k(x)\, u^2\, ds(x), \tag{10}$$

where the domain of J is the space of functions

$$M = \{u(x) \colon u(x) = 0 \text{ on } \partial D_1,\ u(x) \in C(\bar{D}) \cap C^1(D)\},$$

and we have defined $k(x)$ by

$$k(x)\begin{cases} = 0, & x \in \partial D_1, \\ = \alpha(x)/\beta(x), & x \in \partial D_2. \end{cases}$$

We will consider stationary values of the functional J on the domain M subject to the constraint (a normalizing condition)

$$\int_D F(x, u(x))\, dx = c \geqslant 0, \tag{11}$$

where

$$F(x, u) = 2\int_0^u f(x, t)\, dt.$$

For convenience we will denote this subset of functions by M_c, that is,

$$M_c = \left\{u(x) : u(x) \in M, \int_D F(x, u(x))\, dx = c\right\},$$

then the domain of J is M_c.

Finding the stationary value of $J(v)$ on M_c can be considered as an isoperimetric problem; if $u(x)$ gives $J(v)$ a stationary value on M_c then there exists a Lagrange multiplier λ such that the functional,

$$K(v) = J(v) - \lambda \int_D F(x, v(x))\, dx, \tag{12}$$

has a stationary value for $v = u \in M$. Conversely if λ and $u(x)$ are such that $K(v)$ has a stationary value on M, then for some c, $J(v)$ has a stationary value for $u \in M_c$ since $M_c \subseteq M$.

Before we proceed to the main result, we give a necessary preliminary result.

LEMMA. *If u_1 and u_2 are distinct positive solutions of Eqs.* (1) *and* (2) *when f is convex, and if $u_1 \geqslant u_2$ on D, then $J(u_1) > J(u_2)$, with equality only if $u_1 \equiv u_2$.*

Proof. By a well known result and Eqs. (1) and (2)

$$J(u) = \int_D uL(u)\, dx = \lambda \int_D uf(x, u)\, dx,$$

and so we have

$$J(u_1) - J(u_2) = \lambda \int_D (u_1 f(x, u_1) - u_2 f(x, u_2))\, dx.$$

Since, for distinct positive solutions we must have that $0 < \lambda < \lambda_{cr}$ and f is increasing with u on D, we conclude that the right side of the above equation is positive if $u_1 \not\equiv u_2$ and zero otherwise.

The following theorem gives the main result of this section. This result equates the existence of a minimizing function with the existence of the minimal solution to Eqs. (1) and (2). Since the latter result is known in some cases (Ref. [4]), this theorem establishes that a minimizing function exists under the additional hypothesis H-4. As a consequence to this theorem we obtain a formula for the critical parameter λ_{cr}, when it exists.

THEOREM. *Suppose that f satisfies the hypotheses H-0, 1, 2, 3b. A function $u(x, c)$ minimizes $J(u)$ on the subset M_c for $c > 0$ if and only if it is the minimal positive solution of the problem*

$$L(u) = \lambda(c) f(x, u), \qquad x \in D,$$
$$B(u) = 0, \qquad x \in \partial D,$$

where

$$g(c) = \min_{u \in M_c} J(u),$$

and

$$\lambda(c) = g(c) \Big/ \int_D u(x, c) f(x, u(x, c))\, dx.$$

Proof. We shall consider the proof in two stages.

(1). Suppose that the function $u(x, c)$ minimizes $J(u)$. Then the function $u(x, c) \in M_c$ gives a stationary value of the functional J (and hence of K) on the set M_c. We consider the varied functions $u + \epsilon v$, ϵ real and $v \in M$. In general $u + \epsilon v \notin M_c$. To ensure that the varied functions are admissible, that is in M_c, we would introduce, as in Gelfand and Fomin [8, pp. 42–45], an extra term so that the varied functions are of the form $u + \epsilon v + \epsilon_1 v_1$, where ϵ_1 is real and $v_1 \in M$. We then determine ϵ_1 in terms of ϵ so that this function is admissible in a neighborhood of $\epsilon = \epsilon_1 = 0$. This enables us to simply consider by redefining v, the varied functions $u + \epsilon v$, for all $v \in M$. We find that

$$K(u + \epsilon v) = K(u) + 2\epsilon K_1(u, v) + O(\epsilon^2),$$

where, since $\epsilon = 0$ gives a stationary value of K,

$$K_1(u, v) \equiv \int_D v(L(u) - \lambda f(x, u))\, dx + \int_{\partial D_2} \frac{vB(u)}{\beta(x)}\, ds(x) = 0. \qquad (13)$$

This is true for all $v \in M$ and so u satisfies Eq. (1) with Eq. (2) as the natural boundary condition on ∂D_2. The restriction of the space M to include only functions which vanish on ∂D_1 ensures that Eq. (2) is satisfied everywhere on ∂D.

For the minimizing function $u = u(x, c)$ we can write, from Eq. (1)

$$g(c) = J(u) = \int_D uL(u)\, dx = \lambda \int_D uf(x, u)\, dx. \tag{14}$$

Equation (14) determines λ as a function of c in accordance with the statement in the theorem.

The minimizing function will be nonnegative in D. We prove this by contradiction. Suppose that the minimizing function u has some negative values in D. Then there exists at least one negative g.l.b. for u. If such a value occurs at an internal point P of D, we can clearly surround P by a region $G \subseteq D$ for which u is negative and constant on the boundary ∂G of G, and the value of u within the region G is no greater than its value on ∂G. On ∂G,

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{m} n_i(x)\, a_{ij}(x) \frac{\partial u}{\partial x_j} = |\operatorname{grad} u| \sum_{i,j=1}^{m} n_i(x)\, a_{ij}(x)\, n_j(x) \geqslant 0.$$

However,

$$\int_G (L(u) - a_0 u)\, dx = -\int_{\partial G} \frac{\partial u}{\partial \nu}\, ds(x),$$

so

$$\lambda \int_G f(x, u)\, dx = \int_G a_0 u\, dx - \int_{\partial G} \frac{\partial u}{\partial \nu}\, ds(x).$$

Since

$$\int_G f(x, u)\, dx > 0, \qquad \int_G a_0 u\, dx \leqslant 0, \qquad \text{and} \qquad \int_{\partial G} \frac{\partial u}{\partial \nu}\, ds(x) \geqslant 0,$$

λ cannot be positive. A similar argument will show that if u has a negative g.l.b. on ∂D_2, λ must also be negative (or zero). But if λ is negative, corresponding arguments show that u has no positive l.u.b. in D or on ∂D_2, therefore u has no positive values. If this is the case,

$$F(x, u) = 2 \int_0^u f(x, t)\, dt \leqslant 0, \qquad \text{and} \qquad \int_D F(x, u)\, dx \leqslant 0.$$

Hence, $u \notin M_c$ for $c > 0$. Therefore, with the possible exception of $c = 0$, u is not an admissible function. When $c = 0$, the only solution is $u \equiv 0$. Thus, any minimizing functions are never negative.

For any other solution (that is, other than the minimal solution) u_1 of Eqs. (1) and (2) we have $u_1 \geqslant u$ on D and so by the lemma $J(u_1) > J(u)$. Hence the global minimum of J on the set M_c gives the minimal solution of Eqs. (1) and (2).

(2). Conversely, for the second part of the theorem, the parameter c has to be introduced in a more artificial way. Suppose that there exists $u \in M$ which is the minimal positive solution of Eqs. (1) and (2). The parameter c is *defined* by the condition

$$c = c(\lambda) = \int_D F(x, u)\, dx.$$

Since u is a strictly increasing function of λ (as in [2]) and F is a strictly increasing function of u, we may then consider the inverse function $\lambda = \lambda(c)$ to be determined. For the minimal solution u to exist we must have $0 \leqslant \lambda(c) \leqslant \lambda_{\text{cr}}$.

By considering the varied function $u + \epsilon v$, where $v \in M$ and ϵ is real (where, as before. we have introduced another term $\epsilon_1 v_1$, determined ϵ_1 so that the varied function is admissible, and then rewritten it as $u + \epsilon v$), we obtain an expansion for $K(u + \epsilon v)$ similar to that in Eq. (13), and the coefficient of ϵ^2 in the expansion is

$$K_2(u, v) \equiv J(v) - \lambda(c) \int_D f_u(x, u)\, v^2\, dx.$$

As before, the coefficient of ϵ is zero by the assumption on u (see Eq. (13)) and the coefficient of ϵ^2 is nonnegative, since, by the result of Keller and Cohen [2],

$$\lambda(c) \leqslant \lambda_{\text{cr}} \leqslant \mu_1(\lambda) \leqslant \left[J(v) \Big/ \int_D f_u(x, u)\, v^2\, dx \right],$$

for all $v \in M$, and equality is achieved only if v is an eigenfunction of the linearized equation. Hence the minimal solution is a minimizing function of K on M (and hence of J on M_c).

For this function,

$$J(u) = \int_D uL(u)\, dx = \lambda(c) \int_D uf(x, u)\, dx,$$

and the last expression is the quantity we have defined as $g(c)$.

COROLLARY.

$$\lambda_{cr} = \max_{c \geqslant 0} \lambda(c).$$

Proof. For each $\lambda \in [0, \lambda_{cr}]$ there exists a value of $c \geqslant 0$ such that, for this λ, $\lambda = \lambda(c)$. In particular, since $0 \leqslant \lambda(c) \leqslant \lambda_{cr}$, for all $c \geqslant 0$ we have the above formula for the critical parameter λ_{cr}.

Laetsch [9] has shown that the graph of λ against any norm $\| u \|$ is parabolic near λ_{cr}, and so we know that the graph of $\lambda(c)$ against c is smooth near λ_{cr}. In Fig. 1 we have illustrated the expected type of behavior of $\lambda(c)$ against c. The part of the curve where $\lambda(c)$ is increasing with c corresponding to the minimal solution.

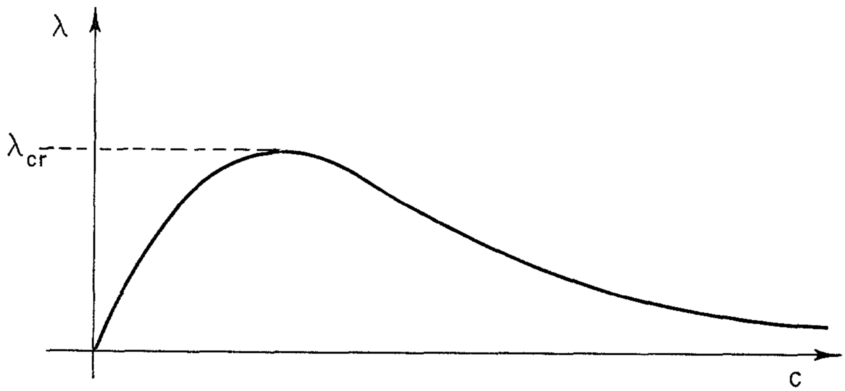

FIGURE 1

3. ILLUSTRATIVE EXAMPLE

The procedure described in the previous section is very useful for obtaining estimates for λ_{cr}. As a first guess we choose a trial function

$$u(x) = ru_1(x) + su_2(x), \tag{15}$$

where u_1, $u_2 \in M$ and r, s are arbitrary real numbers. Then we minimize $J(u)$ with respect to the parameters r and s subject to the constraint that $u \in M_c$. For these values of r and s we calculate the maximum value of the quotient $J(u)/\int_D uf(x, u)\, dx$ as c varies over the positive reals. This will give an approximate value for λ_{cr}.

To illustrate the procedure we take a simple example in one-dimension. Here we take the equations

$$u''(x) + \lambda(1 + 3u(x)^2) = 0, \qquad 0 < x < 1, \tag{16}$$

subject to the conditions $u'(0) = 0$, $u(1) = 0$, $u(x) \geqslant 0$. Hence, we have $f(x, u) \equiv f(u) = 1 + 3u^2$, $J(u) = \int_0^1 u'(x)^2\,dx$,

$$M_c = \left\{ u(x) : u(x) \in C[0, 1] \cap C^1(0, 1), u(1) = 0; 2 \int_0^1 (u + u^3)\,dx = c \right\}.$$

In accordance with Eqs. (15) and (16) we take

$$u(x) = r(1 - x) + s(1 - x^2).$$

Then, for this u, $J(u) = (r + s)^2 + (1/3)s^2 = (4/3)[s + (3/4)r]^2 + r^2/4$. If we regard one of these parameters fixed by the condition that $u \in M_c$, then we have one of four possibilities for a minimum with respect to the remaining parameter:

(i) $s = 0$, $J(u) = r^2$, $u(x) = r(1 - x)$, where r is determined so that $u \in M_c$;

(ii) $r = -s$, $J(u) = (1/3)s^2$, $u(x) = s(x - x^2)$, where s is determined so that $u \in M_c$;

(iii) $r = 0$, $J(u) = (4/3)s^2$, $u(x) = s(1 - x^2)$, where s is determined so that $u \in M_c$;

(iv) $s = (-3/4)r$, $J(u) = (1/4)r^2$, $u(x) = r[(1/4) - x + (3/4)x^2]$ where r is determined so that $u \in M_c$.

Which of the four expressions gives the lowest value for $J(u)$ depends on the value of c at a maximum of the quotient $\lambda(c) = J(u)/\int_0^1 (u + 3u^3)\,dx$. Corresponding to the four cases above we obtain:

(i) $\lambda(c) = 4r/(2 + 3r^2)$, where $r + (1/3)r^3 = c$;

(ii) $\lambda(c) = 140s/(70 + 9s^2)$, where $(1/3)s + (1/70)s^3 = c$;

(iii) $\lambda(c) = 140s/(70 + 144s^2)$. where $(4/3)s + (32/35)s^3 = c$;

(iv) $\lambda(c) = -(5/26)r^{-1}$, where $-(13/30)r^3 = c$.

We can easily find the maximum value of these expressions for $\lambda(c)$ as c varies over the positive reals. The best estimate for λ_{cr} will be given by the maximum value of the expressions (i)–(iv) above which gives rise to the smallest value for $g(c) = J(u)$. This happen to be (iii) above, which suggests $\lambda_{cr} \approx 0.697$. We obtain exactly the same estimate if we treat $u(x) = r(1 - x^2) + s(1 - x^4)$ in a similar way.

By considering the exact solution of Eq. (16), we see that λ can be determined in terms of $u(0) = U$ by

$$(2\lambda)^{1/2} = \int_0^U \frac{dw}{(U + U^3 - w - w^3)^{1/2}}.$$

The maximum value of the right hand side for positive U has been determined numerically as 1.172, which gives the maximum value of λ for a real solution as $\lambda_{cr} = 0.687$ to three significant figures. This is in good agreement with the result obtained by the method proposed in this paper, that is 0.697.

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