Symmetric Function Means and Permanents

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ABSTRACT

We define a function using permanents which generalizes the symmetric function means and show that it is monotonic. The function is conjectured to be superadditive. A special case of the conjecture is proved.

We say that a vector is positive if each component is a positive number. The notation $x \geqslant y$ for vectors x, y means that $x_i \geqslant y_i$ for all i. The transpose of x is denoted by x'.

Let $x = (x_1, \dots, x_n)'$ be a positive vector. We denote by $e_{r,n}(x)$ the rth elementary symmetric function in $x_1, \dots, x_n$. Thus

$$e_{r,n}(x) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

We set $e_{0,n}(x) = 1$.

Several inequalities are available in the literature for ratios of elementary symmetric functions [4–6, 10]. Let

$$M_{r,n}(x) = \frac{e_{r,n}(x)}{e_{r-1,n}(x)}, \qquad r = 1, 2, \dots, n.$$

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The expression

$$T_{r,n}(x) = \frac{\binom{n}{r-1}}{\binom{n}{r}} M_{r,n}(x)$$

is called a symmetric function mean or a Marcus-Lopes mean. For convenience we will work with $M_{r,n}(x)$ instead of its normalized version given above.

A well-known result of Marcus and Lopes ([5]; see also [4, 10]) asserts that for any two positive vectors x, y,

$$M_{r,n}(x+y) \geqslant M_{r,n}(x) + M_{r,n}(y). \tag{1}$$

If A is an $n \times n$ matrix, then recall that the permanent of A is defined as

$$\sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation is over all permutations of $1, 2, \dots, n$. We refer to [7–9] for a wealth of information on permanents.

If $a_1, \dots, a_n$ are vectors in R^n, then we will denote the permanent of the $n \times n$ matrix $(a_1, \dots, a_n)$ by just $[a_1, \dots, a_n]$.

Let e denote the vector of appropriate size of all ones. Observe that

$$e_{r,n}(x) = \frac{1}{r!(n-r)!}\Big[\underbrace{x, \dots, x}_{r}, \underbrace{e, \dots, e}_{n-r}\Big].$$

This motivates the following definition. Let $c, b_1, b_2, \dots$ be positive vectors in R^n which will be held fixed throughout. For any positive vector x in R^n and for $1 \leqslant r \leqslant n$, define

$$S_{r,n}(x) = \frac{\Big[\underbrace{x, \dots, x}_{r}, b_1, \dots, b_{n-r}\Big]}{\Big[\underbrace{x, \dots, x}_{r-1}, b_1, \dots, b_{n-r}, c\Big]}. \tag{2}$$

If $c = b_i = e$ for all i, then

$$S_{r,n}(x) = \frac{r}{n-r+1} M_{r,n}(x).$$

The definition (2) is not as artificial as it may appear at first sight. The idea of using permanents to construct generalizations of elementary symmetric functions has proved to be useful in other contexts. For examples we refer to the survey paper [3].

The main result of this paper is that the function defined in (2) is nondecreasing in each component of x. We first prove the following.

LEMMA. *Let $1 \leqslant r \leqslant n$, and let d be a positive vector in R^n. Then*

$$r\Big[\underbrace{x,\dots,x}_{r-1}, b_1,\dots,b_{n-r}, c\Big]\Big[\underbrace{x,\dots,x}_{r-1}, b_1,\dots,b_{n-r}, d\Big] \geqslant (r-1)\Big[\underbrace{x,\dots,x}_{r}, b_1,\dots,b_{n-r}\Big]\Big[\underbrace{x,\dots,x}_{r-2}, b_1,\dots,b_{n-r}, cd\Big]. \tag{3}$$

Proof. First let $n = r$. Then we must prove

$$r\Big[\underbrace{x,\dots,x}_{r-1}, c\Big]\Big[\underbrace{x,\dots,x}_{r-1}, d\Big] \geqslant (r-1)\Big[\underbrace{x,\dots,x}_{r}\Big]\Big[\underbrace{x,\dots,x}_{r-2}, c, d\Big].$$

This is equivalent to

$$r(r-1)!(r-1)!\Big(\sum_i c_i \prod_{k\neq i} x_k\Big)\Big(\sum_j d_j \prod_{k\neq j} x_k\Big) \geqslant (r-1)r!(r-2)!\Big(\prod_k x_k\Big)\sum_{i\neq j} c_i d_j \prod_{k\neq i,j} x_k.$$

The above inequality reduces to

$$\Big(\sum_i \frac{c_i}{x_i}\Big)\Big(\sum_j \frac{d_j}{x_j}\Big) \geqslant \prod_{i\neq j} \frac{c_i}{x_i}\frac{d_j}{x_j},$$

which is obvious.

We now proceed by induction on n. Assume therefore that the result is true for $n = m - 1$, and consider the case $n = m$.

Since both sides of (3) are linear in d, it is sufficient to prove (3) when d has only one nonzero coordinate. By symmetry, it is sufficient to consider $d = (1, 0, \ldots, 0)'$.

We will use the following notation. If z is a vector, then $\hat{z}$ will denote the vector obtained by deleting the first coordinate of z. If $(a_1, \ldots, a_n)$ is an $n \times n$ matrix, then $[a_1, \ldots, a_n](i)$ will denote the permanent of the matrix obtained by deleting a_i. Also, let b_{1i} denote the first component of b_i, $i = 1, 2, \ldots, n - r$.

Expanding the permanent along the first row, we get

$$\begin{aligned}
\Big[\underbrace{x, \ldots, x}_{r-1}, b_1, \ldots, b_{m-r}, c\Big] &= (r-1)x_1\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r-1}, \hat{b}_1, \ldots, \hat{b}_{m-r}, \hat{c}\Big](1) \\
&\quad + \sum_{i=1}^{m-r} b_{1i}\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r-1}, \hat{b}_1, \ldots, \hat{b}_{m-r}, \hat{c}\Big](r-1+i) \\
&\quad + c_1\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r-1}, \hat{b}_1, \ldots, \hat{b}_{m-r}, \hat{c}\Big](m).
\end{aligned} \tag{4}$$

Similarly

$$\begin{aligned}
\Big[\underbrace{x, \ldots, x}_{r}, b_1, \ldots, b_{m-r}\Big] &= r\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r}, \hat{b}_1, \ldots, \hat{b}_{m-r}\Big](1) \\
&\quad + \sum_{i=1}^{m-r} b_{1i}\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r}, \hat{b}_1, \ldots, \hat{b}_{m-r}\Big](r+i).
\end{aligned} \tag{5}$$

By the induction assumption,

$$\begin{aligned}
r\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r-1}, \hat{b}_1, \ldots, \hat{b}_{m-r}, \hat{c}\Big](r-1+i)\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r-1}, \hat{b}_1, \ldots, \hat{b}_{m-r}\Big] \\
\geqslant (r-1)\Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r}, \hat{b}_1, \ldots, \hat{b}_{m-r}\Big](r+i) \\
\times \Big[\underbrace{\hat{x}, \ldots, \hat{x}}_{r-2}, \hat{b}_1, \ldots, \hat{b}_{m-r}, \hat{c}\Big].
\end{aligned} \tag{6}$$

The inequality (3) for $d = (1, 0, \ldots, 0)'$ follows by combining (4), (5), (6). ■

We now have the following.

THEOREM. *Let $1 \leqslant r \leqslant n$. If x, y are positive vectors such that $x \geqslant y$, then $S_{r,n} \geqslant S_{r,n}(y)$.*

Proof. The result will be proved if we show that the derivative of $S_{r,n}(z)$ with respect to z_i at $z \geqslant 0$ is nonnegative, $i = 1, 2, \ldots, n$. The derivative of $S_{r,n}(z)$ with respect to z_1 is

$$\Big[\underbrace{z, \ldots, z}_{r-1}, b_1, \ldots, b_{n-r}, c\Big]^{-2}$$

times

$$\begin{aligned} & r\Big[\underbrace{z, \ldots, z}_{r-1}, b_1, \ldots, b_{n-r}, c\Big]\Big[\underbrace{z, \ldots, z}_{r-1}, b_1, \ldots, b_{n-r}, e_1\Big] \\ & \quad -(r-1)\Big[\underbrace{z, \ldots, z}_{r}, b_1, \ldots, b_{n-r}\Big]\Big[\underbrace{z, \ldots, z}_{r-2}, b_1, \ldots, b_{n-r}, c, e_1\Big], \end{aligned} \tag{7}$$

where $e_1 = (1, 0, \ldots, 0)'$.

The expression in (7) is nonnegative for $z \geqslant 0$ by the lemma. Similarly, the derivative of $S_{r,n}(z)$ with respect to z_i, $i = 2, \ldots, n$, can be shown to be nonnegative at $z \geqslant 0$, and the proof is complete. ■

In view of (1), we conjecture that for positive vectors x, y,

$$S_{r,n}(x + y) \geqslant S_{r,n}(x) + S_{r,n}(y). \tag{8}$$

For $r = 1$, (8) is trivial. We now prove (8) for $r = 2$. Thus we must show

$$\begin{aligned} & \frac{[x + y, x + y, b_1, \ldots, b_{n-2}]}{[x + y, b_1, \ldots, b_{n-2}, c]} \\ & \qquad \geqslant \frac{[x, x, b_1, \ldots, b_{n-2}]}{[x, b_1, \ldots, b_{n-2}, c]} + \frac{[y, y, b_1, \ldots, b_{n-2}]}{[y, b_1, \ldots, b_{n-2}, c]}. \end{aligned} \tag{9}$$

We first state the following results.

(i) *Alexandroff's inequality*: For positive vectors x, y,

$$[x, y, b_1, \dots, b_{n-2}]^2 \geqslant [x, x, b_1, \dots, b_{n-2}][y, y, b_1, \dots, b_{n-2}].$$

(ii) Let $u_1, \dots, u_n$ be linearly independent vectors in R^n, and let $T = ((t_{ij}))$ be the $n \times n$ matrix defined by

$$t_{ij} = [u_i, u_j, b_1, \dots, b_{n-2}], \qquad i \neq j,$$

and $t_{ii} = 0$. Then T is a symmetric, nonsingular matrix with exactly one positive eigenvalue.

We refer to [12] for a proof of (i). The proof of (ii) can be given as follows. Let $e_1, \dots, e_n$ be the standard basis for R^n, and let $Q = ((q_{ij}))$ be the $n \times n$ matrix defined by

$$q_{ij} = [e_i, e_j, b_1, \dots, b_{n-2}], \qquad i \neq j,$$

and $q_{ii} = 0$. It can be deduced using Alexandroff's inequality (see [12]) that Q is nonsingular and has exactly one positive eigenvalue. Now

$$T = \begin{pmatrix} u_1' \\ \vdots \\ u_n' \end{pmatrix} Q(u_1, \dots, u_n).$$

Thus T is nonsingular, and it follows by Sylvester's law of inertia that T has exactly one positive eigenvalue.

For vectors u, v let us introduce the notation

$$\langle u, v \rangle = [u, v, b_1, \dots, b_{n-2}].$$

After some simplification, (9) reduces to

$$2\langle x, y\rangle\langle x, c\rangle\langle y, c\rangle \geqslant \langle x, x\rangle\langle y, c\rangle^2 + \langle y, y\rangle\langle x, c\rangle^2. \qquad (10)$$

First suppose that x, y, c are linearly independent. By (ii) the matrix

$$T = \begin{pmatrix} \langle x, x\rangle & \langle x, y\rangle & \langle x, c\rangle \\ \langle y, x\rangle & \langle y, y\rangle & \langle y, c\rangle \\ \langle c, x\rangle & \langle c, y\rangle & \langle c, c\rangle \end{pmatrix}$$

has one positive and two negative eigenvalues. It follows by the interlacing principle that any 2×2 principal submatrix of T has at most one positive eigenvalue. However, the diagonal entries of T are positive. Hence every 2×2 principal submatrix of T must have one positive and one negative eigenvalue and therefore has negative determinant. Thus

$$t_{11}t_{22} < t_{12}^2. \tag{11}$$

Also, the determinant of T must be positive, and hence

$$2t_{12}t_{23}t_{13} - t_{22}t_{13}^2 - t_{11}t_{23}^2 + t_{33}(t_{11}t_{22} - t_{12}^2) > 0. \tag{12}$$

Since $t_{33} > 0$, it follows from (11), (12) that

$$2t_{12}t_{23}t_{13} > t_{22}t_{13}^2 + t_{11}t_{23}^2,$$

which is the same as (10) with strict inequality. If the assumption of linear independence of x, y, c is removed, then (10) holds by a continuity argument. This completes the proof of (8) for $r = 2$.

We note that (9) implies Alexandroff's inequality. To see this, just set $x = c$ in (10).

A referee has pointed out that the conjecture (8) is easily verified for $r = n$. To see this observe that

$$[\underbrace{x, \ldots, x}_{n}] = n! \prod_{i=1}^{n} x_i$$

and

$$[\underbrace{x, \ldots, x}_{n-1}, b] = (n-1)! \sum_{i=1}^{n} b_i \prod_{j \neq i} x_j.$$

Thus

$$S_{nn}(x) = \frac{n}{\sum_{i=1}^{n}(b_i/x_i)},$$

which is the harmonic mean of the numbers x_i/b_i, $i = 1, \dots, n$, and is superadditive by (1).

The conjecture (8) remains open for $2 < r < n$. Observe that the conjecture is basically a permanental inequality. Certain inequalities for the permanent, such as those in [11], could prove to be useful in solving the problem.

Symmetric function means have been defined for tuples of positive definite matrices, and this has resulted in several fascinating inequalities and open problems (see [1, 2] and the references contained therein). It may be worthwhile to try to generalize (2) in a similar way.

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