1. Introduction

Since its inception, Quantum Field Theory has been confronted by the occurrence of infinite quantities. In the process of computing finite and experimentally observable quantities, for example the Lamb shift, one encounters infinite quantities in the form of divergent integrals. These divergent integrals are removed and a finite answer is obtained by the process of adjusting a few parameters in the equations of motion (mass, charge, vacuum energy). It is disconcerting, however, that the adjusted parameters (the bare mass, etc.) are themselves infinite. Thus the equations of motion contain terms with infinite coefficients and they might appear to lie outside of the domain of conventional mathematics. The central objects of the theory, for example the Hamiltonian or total energy operator, $H$, and $S$, the scattering operator, are not well defined in a rigorous mathematical sense. Through the use of formal reasoning, primarily power series manipulation, one can define $H$ and $S$. While these definitions are not rigorous (the power series presumably diverge), they provide a basis for numerical calculation. If $e$ is the charge on the electron, we write $H = H(e)$ and $S = S(e)$, and we expect that $H$ and $S$, once they are rigorously defined, will be $C^\infty$ but not analytic functions of $e$ for $0 \leq e$. The formal calculations determine the Taylor series of

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$S$ and $H$ about $e = 0$; the calculations may be done by power series manipulations without knowing whether $S(e)$ or $H(e)$ exist for any $e > 0$. Even in the formal calculations, infinite quantities occur, so that

$$\left. \frac{d^n}{de^n} S(e) \right|_{e=0}$$

which is finite, appears as the difference of several infinite quantities. This extraction of the finite difference in (1.1) is due to Dyson, Schwinger, and Feynman and is one of the major accomplishments of Quantum Field Theory. Great mathematical care must be exercised in subtracting infinite quantities, but since no one is troubled by the equation

$$P \int_{-1}^{1} \frac{dx}{x} = 0,$$

it should not be a surprise that the infinities in the formal expression for (1.1) can be cancelled rigorously [2, 3, 19, 34].

For Quantum Electrodynamics, the calculations are in excellent agreement with experiment, and this is the best reason for believing that the theory is sound, in spite of the considerable mathematical difficulties which it presents, see Table I.

| TABLE I* |
|-----------------|-----------------|
| **Experimental value** | **Predicted value** |
| Lamb shift in Hydrogen, Mc/sec. | 1057.77 ± 0.10 | 1057.70 ± 0.15 |
| Lamb shift in Deuterium, Mc/sec. | 1059.00 ± 0.10 | 1058.96 ± 0.16 |
| Lamb shift in Helium, Mc/sec. | 14040.2 ± 4.5 | 14046.3 ± 3.0 |
| Anomalous magnetic moment of electron | \(0.0011609 ± 0.0000024\) \(\hbar/2mc\) | 0.0011596 \(\hbar/2mc\) |

*Source: Schweber ([31], pp. 546, 577).

The Lamb shift is the displacement between two slightly differing energy levels ($2S$ and $2P$) in the spectrum of an atom.
For strong interactions, which determine the forces in the nucleus of the atom, the expansion parameter $g$ is of the order of one or larger while the expansion of $S$ up to second order,

$$S(0) + g \frac{dS}{dg}(0) + \frac{g^2}{2!} \frac{d^2S}{dg^2}(0) = 1 + \frac{g^2}{2!} d^2S(0),$$

does not agree with experiment. The inclusion of higher order terms seems to give worse answers; these facts are consistent with the conjecture that $S(g)$ is $C^\infty$ but not analytic as a function of $g$. Partial success has been achieved by computations based on other methods, notably dispersion relations and (for weak interactions) current commutation relations, but the problem of obtaining satisfactory calculations for the non-electromagnetic interactions remains as a major problem of theoretical physics.

In this article we will deal with the mathematical consistency of Quantum Field Theory, or in other words with the problem of finding rigorously defined mathematical objects which correspond to the operators discussed in Quantum Field Theory. These operators are defined at present only through a formal power series. The first question to be answered is: how does one recognize a solution to the problem? In general terms a solution should possess as far as possible the expected properties. For example the Taylor coefficients (1.1) should agree with the standard formulae. In recent years there has been a great deal of work on the axioms of Quantum Field Theory [18, 23, 35]. The axioms are a short list of basic qualitative properties postulated for the solutions of Quantum Field Theory. The axioms are statements of basic principles of physics (Lorentz covariance, locality, positivity of the energy, etc.) supplemented by assumptions of a technical mathematical nature (e.g. the field operators are distributions in the Schwartz space $S'$). One would certainly expect a solution to satisfy most or all of the axioms; possibly it would have properties not deducible from the axioms, thus permitting a strengthening of the axioms.

The list of proposed properties, as furnished by the axioms or by perturbation theory, is invaluable for another reason. The most practical method for establishing the existence of a Quantum Field Theory seems to be the following [38]. First one modifies the laws of physics in such a way that the interactions between particles are less singular and the divergent integrals become finite; the resulting modified or approximate Quantum Field Theory will not be Lorentz invariant. The solutions of
the approximate Quantum Field Theory can be obtained by methods from partial differential equations and from the theory of unbounded operators [21, 26, 28]. As the modifications are removed from the laws of physics, the solutions of the approximate theory are expected to converge to solutions of the true theory. The sense in which the approximate solutions converge to the true solution may be conjectured from the proposed properties of the true solution.

There are two sources for the divergent integrals. The first is translation invariance together with the infinite volume of space. Consider an interaction in which three particles, for example an electron, a positron and a photon, are first created at the point $x$ and later annihilated at a point of $y$, see Fig. 1.1.

Common sense, i.e. translation invariance, tells us that the strength of this interaction depends only on $x - y$, and so the total strength after integration over both $x$ and $y$ is either zero or infinite. It is the second possibility which prevails. One approximation which removes such an infinity from the theory is to replace Euclidean space $\mathbb{R}^3$ by a large torus $T^3$. In physics this approximation is known as quantization in a box with periodic boundary conditions. Alternately one may declare that particles interact only in some large bounded region of space. In this case the strength of the interaction depicted in Fig. 1.1 is zero unless $x$ and $y$ lie in a bounded set and then the total strength after integration over $x$ and $y$ is finite. Either of these approximations will be called a space cutoff or a volume cutoff. A second type of infinity comes from the interaction of high energy particles. Suppose a photon is annihilated, creating an electron positron pair and that subsequently the electron positron pair is annihilated with the emission of a new photon, see Fig. 1.2.
If the first photon has momentum \( k \), the second one will also, by conservation of momentum and if the electron has momentum \( p \) then the positron will have momentum \( k - p \). For fixed \( p \) and \( k \) the strength of this interaction is finite, but again the integration over \( p \) is divergent. This infinity is called an ultraviolet or momentum infinity. It may be removed by declaring that high energy particles do not interact or interact weakly; such a change in the laws of physics is called a momentum cutoff. The ultraviolet divergences have a strength which depends on the number of space dimensions; one may think of

\[
\int \frac{dp}{1 + |p|}
\]

as a simplified prototype. Thus the degree of this divergence is reduced by considering problems in two- or three-dimensional space-time. The volume infinities have the general form of

\[
\int dx = \int \delta(k)^2 dk
\]

and are not made easier to handle by reducing the number of space dimensions.

2. Fock Space

In this section we describe the basic Hilbert space of Quantum Field Theory, Fock space, and we study an important class of operators on Fock space. In Quantum Field Theory, particles may be created and annihilated and the total number of particles present is not a constant independent of time. The most widely known examples are the creation of an electron positron pair while absorbing energy in the form of light, that is, while annihilating a photon, see Fig. 2.1.
and the reverse process of annihilating an electron positron pair with the emission of a photon, see Fig. 2.2. An element \( \varphi \) of

![Fig. 2.2](image)

Fock space \( \mathcal{F} \) describes a state of the quantum field, and so \( \varphi \) must be capable of describing an indefinite or infinite number of particles. The state of a single particle may be described by a function \( \varphi_1 \) defined on \( R^3 \) and taking values in \( C^s \) (complex \( s \)-space), for some integer \( s \). We interpret

\[
| \varphi_1(k) |^2
\]

as the probability density for the particle to have momentum \( k \). Consistency requires that the probability that \( k \in R^3 \) be one,

\[
| \varphi_1(k) |^2 = \int | \varphi_1(k) |^2 \, dk = 1,
\]

and so \( \varphi_1 \) is a vector of unit length in a Hilbert space \( \mathcal{H}_1 \). The vector \( \varphi_1(k) \in C^s \) describes the internal degrees of freedom of the particle, for example spin or helicity and we call \( C^s \) the spin space. More generally, \( n \) identical particles are described by a function \( \varphi_n(k_1, \ldots, k_n) \) of \( n \) variables \( k_j \in R^3 \). We require that \( \varphi_n \) be either symmetric (and the particles are called bosons) or antisymmetric (and the particles are called fermions):

\[
S \varphi_n = (n!)^{-1} \sum_\sigma \varphi_n(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = \varphi_n
\]

or

\[
A \varphi_n = (n!)^{-1} \sum_\sigma (-1)^{\text{sgn} \sigma} \varphi_n(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = \varphi_n.
\]

The effect of the symmetry or antisymmetry is that individual particles are indistinguishable. Furthermore antisymmetry leads to the Pauli exclusion principle: \( \varphi_n \) is zero whenever two or more of its arguments coincide. As before we interpret

\[
| \varphi_n(k_1, \ldots, k_n) |^2
\]
as the probability density that the particles have momenta $k_1, \ldots, k_n$; 
$\| \varphi_n \|^2 = 1$ and $\varphi_n(k) \in C^\infty \otimes \cdots \otimes C^\infty = C^\infty$. Thus $\varphi_n$ is a unit vector in a Hilbert space $\mathcal{F}_n$. Finally we set $\mathcal{F} = \sum_{n=0}^\infty \oplus \mathcal{F}_n$ and if $\varphi = \varphi_0, \varphi_1, \ldots$ is in $\mathcal{F}$ we interpret $\| \varphi_n \|^2$ as the probability that there are $n$ particles present. We must have $\| \varphi_n \|^2 \leq 1$,

$$\| \varphi \|^2 = \sum_{n=0}^\infty \| \varphi_n \|^2 = 1,$$

and we interpret (2.3) as before. $\mathcal{F}_0$ is the set of complex numbers and $| \varphi_0|^2$ is the probability that there are no particles present. We summarize this discussion with the following definition.

**Definition 2.1.** Let $\mathcal{H}$ be a Hilbert space. The boson Fock space over $\mathcal{H}$ is the Hilbert space completion $\mathcal{P}(\mathcal{H})$ of the symmetric tensor algebra over $\mathcal{H}$. The fermion Fock space over $\mathcal{H}$ is the Hilbert space completion $\mathcal{G}(\mathcal{H})$ of the alternating tensor algebra over $\mathcal{H}$.

In general we will consider simultaneously bosons and fermions and our corresponding Fock space will have the form

(2.4)  
$$\mathcal{F} = \mathcal{P}(\mathcal{H}_b) \otimes \mathcal{G}(\mathcal{H}_f).$$

The single-particle boson and fermion Hilbert spaces $\mathcal{H}_b$ and $\mathcal{H}_f$ are determined by the classical field equations which are to be quantized. A classical field is a solution of some hyperbolic partial differential equation and $\mathcal{H}_b$ and $\mathcal{H}_f$ are Hilbert spaces consisting of positive energy solutions of these classical field equations. Consider for example the Klein–Gordon equation

(2.5)  
$$0 = \left( \frac{d^2}{dt^2} - \Delta + m^2 \right) u$$

$$= \left( i \frac{d}{dt} - \sqrt{-\Delta + m^2} \right) \left( i \frac{d}{dt} + \sqrt{-\Delta + m^2} \right) u,$$

where $\Delta = \sum_i \partial^2 / \partial x_i^2$. The positive energy solutions are the solutions of the equation

(2.6)  
$$\left( i \frac{d}{dt} - \sqrt{-\Delta + m^2} \right) u = 0.$$

The solution $u = u(x, t)$ is uniquely determined by its Cauchy data.
$u(x, 0)$ and by the Fourier Transform (in the space variables) $\hat{u} = \hat{u}(\cdot, 0)$. The inner product in $\mathcal{H}_\mu$ is then

$$\langle u, v \rangle = \int \hat{u}(k) \cdot \hat{v}(k) \mu(k)^1 \, dk$$

where

$$\mu(k) = (k^2 + m^2)^{1/2}.$$  

This choice of the inner product is determined up to a scalar multiple by the fact that it is invariant under Lorentz transformations, but since Lorentz transformations are cumbersome to describe in our non-relativistic notation, we will not justify this statement. We see that the function $\varphi_1$ in (2.1) corresponds to $\mu^{1/2} \hat{u}$. For more details and a discussion of other important special cases, see [24].

In the cases with which we will be concerned the classical field equation is implied by (but is not necessarily equivalent to) a Schrödinger equation

$$i \frac{d}{dt} = H_c$$

where $H_c$ is a nonnegative pseudodifferential operator; in momentum, or Fourier transform space, $H_c$ is multiplication by a nonnegative function $\mu$. If the spin space $C^s$ has dimension $s > 1$ then $\mu$ is a matrix-valued function. A solution $\varphi(t) = \exp(-itH_c) \varphi(0)$ is also a solution of the classical field equation and for fixed $t$, $\varphi(t)$ describes the state of a single particle which at time $t = 0$ was described by the element $\varphi(0)$ of $\mathcal{H}_\mu$ or of $\mathcal{H}_r$. We extend $H_c$ to an operator $H_0$ on the full Fock space $\mathcal{F}$ by the definition

$$H_0 \varphi_n(k_1, \ldots, k_n) = \sum_{j=1}^{n} \mu(k_j) \varphi_n(k_1, \ldots, k_n).$$

$H_0$ is a sum of commuting selfadjoint operators and is thus selfadjoint. The corresponding Schrödinger equation is $i(d/dt) = H_0$ and its solution is $\varphi(t) = e^{-itH_0} \varphi(0)$.

In the dynamics determined by this Schrödinger equation the $n$-particle subspace $\mathcal{F}_n$ of $\mathcal{F}$ is invariant. We have

$$\varphi_n(t, k_1, \ldots, k_n) = \exp \left[-\sum_j i t \mu(k_j) \right] \varphi_n(k_1, \ldots, k_n) = \pi \varphi_n(k_1, \ldots, k_n),$$
where we have set
\[ \varphi_n(0) = \varphi_n(0, k_1, \ldots, k_n) = \varphi_n(k_1, \ldots, k_n). \]

Thus we see that each of the \( n \) particles described by \( \varphi_n \) moves in time according to the classical wave equation and its motion is independent of the presence or position of the remaining \( n - 1 \) particles. This independence of the motion of the individual particles is particularly transparent if the Cauchy data \( \varphi_n(0) \) factors, so that
\[ \varphi_n(0, k_1, \ldots, k_n) = \psi_1(k_1) \times \cdots \times \psi_n(k_n) \]
because then
\[ \varphi_n(t, k_1, \ldots, k_n) = \prod_{j=1}^n e^{-itu(k_j)} \psi_j(k_j). \]

For this reason, the dynamics \( e^{-itH_0} \) is called the free dynamics and the Hamiltonian \( H_0 \) is called the free Hamiltonian. What we have described so far is merely the superposition of an indefinite number of particles, each moving independently of one another.

The interesting problems occur, of course, when the particles interact and as we have already observed, operators describing interaction will not in general preserve the number of particles. Our next job is to introduce a class of operators and bilinear forms sufficiently large to include those describing interactions between particles.

Let \( B \) be a bilinear form with a distribution kernel densely defined on \( \mathcal{F}_m \times \mathcal{F}_n \). We define the quantized bilinear form \( \hat{B} \) as follows. If \( i - j \neq m - n \) or if \( i < m \) then
\[ \langle \psi_i, \hat{B}\varphi_j \rangle = 0. \] (2.11)

Otherwise \( i = m + a, j = n + a, a \geq 0 \) and we set
\[ \langle \psi_i, \hat{B}\varphi_j \rangle = c(i, j, m, n) \langle \varphi_i, B\varphi_j \rangle. \] (2.12)

In the right-hand side, \( B \) acts on \( m \) variables of \( \varphi_i \) and \( n \) variables of \( \varphi_j \) while the remaining \( a \) variables of \( \varphi_i \) and \( \varphi_j \) are held fixed; then an inner product is taken in these remaining variables. The constant is
\[ c(i, j, m, n) = [(i + 1) \cdots (i + m)(j + 1) \cdots (j + n)]^{1/2}. \] (2.13)

This definition applies when \( \mathcal{F} = \mathcal{S}(\mathcal{H}_b) \) or \( \mathcal{F} = \mathcal{D}(\mathcal{H}_f) \) but a slight modification is required if there are both bosons and fermions. Physically
$\hat{B}$ may be thought of as acting on $\mathcal{F}$ by the process of first annihilating $n$ particles and then creating $m$ new particles as in Figs. 2.1 and 2.2. This point of view leads to an integral representation for $B$, see (2.16) below.

If $B$ is a bounded operator then $\hat{B}$ is an unbounded operator and

$$ (\hat{B}\varphi_i)_i = 0 $$

if $i \neq j + m - n$ or if $j < n$ and otherwise

$$ (\hat{B}\varphi_i)_i = c(i, j, m, n) SAB\varphi_j. $$

On the right, $B$ acts on the first $n$ variables of $\varphi_j$ and produces an unsymmetrized function $B\varphi_j$ of $j + m - n$ variables; one then applies the correct symmetrization operator $S$ or $A$. Again a slight modification is required if there are both bosons and fermions. One can check easily that $H_0 = \hat{H}_0$. Formally we have

$$ \hat{B}^* = \hat{B}^*. $$

If $\delta$ is the Dirac $\delta$-function then we define

$$ a(k) = \delta(\cdot - k)^\wedge $$

and we compute

$$ (a(k) \varphi)_{n-1}(k_1, \ldots, k_{n-1}) = n^{1/2} \varphi_n(k, k_1, \ldots, k_{n-1}). \tag{2.14} $$

Thus $a(k)$ annihilates a particle of momentum $k$ if there is any such particle in the state $\varphi$ and it otherwise yields zero, up to a normalization factor $n^{1/2}$. If the spin space $C^s$ has a dimension $s > 1$ then the definition above of $a(k)$ must be replaced by

$$ a(k, u) = (\delta(\cdot - k) \otimes u)^\wedge, $$

where $u \in C^{s*}$ and $\delta(\cdot - k) \otimes u$ is a densely defined linear functional on $\mathcal{F}_1$ and a densely defined bilinear form on $\mathcal{F}_0 \times \mathcal{F}_1$. In this case (2.14) becomes

$$ (a(k, u) \varphi_{n-1})(k_1, \ldots, k_{n-1}) = n^{1/2} \langle u, \varphi_{n-1}(k, k_1, \ldots, k_{n-1}) \rangle \tag{2.14}' $$

where $\langle , \rangle$ is the inner product between $C^{s*}$ and $C^s$. The adjoint of
\(\delta(\cdot - k)\) is a bilinear form on \(\mathcal{F}_1 \times \mathcal{F}_0\) and the adjoint \(a^*(k)\) of \(a(k)\) is the improper operator

\[
(a^*(k) \varphi)_{n+1}(k_1, \ldots, k_{n+1}) = (n + 1)^{1/2} S A \delta(k_1 - k) \otimes \varphi_n(k_2, \ldots, k_{n+1}).
\]

We see that \(a^*(k)\) creates a new particle which has momentum exactly \(k\).

If \(s > 1\), replace \(\delta(\cdot - k)\) by \(\delta(\cdot - k) \otimes \mu\) as above. If \(B\) has a kernel \(b\) then formally

\[
B = \int b(k_1, \ldots, k_m, k'_1, \ldots, k'_n) a^*(k_1) \cdots a^*(k_m) a(k'_1) \cdots a(k'_n) \, dk \, dk'.
\]

One can check that formally

\[
[a(k), a^*(\ell)]_\pm = \delta(k - \ell)
\]

\[
[a(k), a(\ell)]_\pm = 0 = [a^*(k), a^*(\ell)]_\pm
\]

where

\[
[A, B]_\pm = AB \pm BA
\]

and the minus is chosen for bosons, the plus for fermions. For details, see [1, 24].

Let \(N\) be the number of particles operator defined by the equation

\[N \varphi_n = n \varphi_n.\]

One can check that \(\langle \varphi, N \varphi \rangle\) is the expected number of particles in the state \(\varphi\) and that \(N = \mathbb{1}\) if \(I\) is the identity operator on \(\mathcal{F}_1\). With \(B\) a bounded operator, there is no difficulty in checking that \(\hat{B}\) can be defined as above on the domain \(\mathcal{D}(N^{(m+n)/2})\) of the operator \(N^{(m+n)/2}\).

**Proposition 2.1.** If \((m + n)/2 \leq a + b\) then

\[
\|(N + I)^{-a} \hat{B}(N + I)^{-b}\| \leq \text{const} \|B\|
\]

and the constant depends only on \(a, b, m\) and \(n\).

The proof is elementary. Somewhat better bounds can be obtained...
if \( N \) is replaced by a larger operator, for example \( H_0 \) [9]. For fermions there are better bounds in many cases. Let \( f \in \mathcal{H}_t \) and define

\[
(2.19) \quad a(f) = \int f(k) a(k) \, dk,
\]

\[
(2.20) \quad a^*(f) = \int f(k) a^*(k) \, dk.
\]

These operators have the form \( \hat{B} \) where \( B \) is the bilinear form

\[
\lambda, \varphi_1 \to \lambda \int f(k) \varphi_1(k) \, dk
\]

or

\[
\varphi_1 , \lambda \to \lambda \int f(k) \varphi_1(k)^* \, dk
\]

on \( \mathcal{F}_0 \times \mathcal{F}_1 \) or \( \mathcal{F}_1 \times \mathcal{F}_0 \). The anticommutation relations take the form

\[
(2.21) \quad a^*(f^-) a(f) + a(f) a^*(f^-) = \|f\|_2^2
\]

and

\[
a^*(f^-) = (a(f))^*.
\]

It follows that \( a(f) \) is a bounded operator. Now consider a more general \( B \) whose kernel \( b \) is a tensor of rank 1:

\[
b = f_1 \otimes \cdots \otimes f_{m+n}.
\]

One can verify that

\[
\hat{B} = a^*(f_1) \cdots a(f_{m+n})
\]

and so \( \hat{B} \) is again a bounded operator. If \( b \) is smooth and has compact support, for example, we can write

\[
b = \sum \lambda_{i_1 \ldots i_{m+n}} e_{i_1} \otimes \cdots \otimes e_{i_{m+n}}
\]

with \( \{e_1, e_2, \ldots\} \) an orthogonal basis for \( \mathcal{H}_t \) and with \( \sum |\lambda| < \infty \); again \( \hat{B} \) is bounded.

Now suppose that there are both bosons and fermions in the Fock space and let \( \mathcal{F}_{n_b n_f} \) be the subspace of \( \mathcal{F} \) with \( n_b \) bosons and \( n_f \) fermions.
If $B$ is a bilinear form on $\mathcal{F}_{m_bm_t} \times \mathcal{F}_{n_bn_t}$ we define $\hat{B}$ as before. In particular

\begin{align*}
\langle \psi_{i_b,i_t}, \hat{B}\psi_{j_b,j_t} \rangle &= \begin{cases} 0 & \text{if } i_b - j_b \neq m_b - n_b \text{ or } i_t - j_t \neq m_t - n_t \text{ or } i_b < m_b \text{ or } i_t < m_t \\ c(i_b, j_b, m_b, n_b) c(i_t, j_t, m_t, n_t) \langle \psi_{i_b,i_t}, B\psi_{j_b,j_t} \rangle & \text{otherwise.} \end{cases}
\end{align*}

Combining the above fermion estimates with Proposition 2.1, one can prove

**Proposition 2.2.** Let the kernel $b$ of $B$ be smooth with compact support. If $(m_b + n_b)/2 < a + b$ then $(N + 1)^{-a} \hat{B}(N + 1)^{-b}$ is a bounded operator.

### 3. The Hamiltonian Operator

The renormalized Hamiltonian operator can be written as

\begin{align*}
H_{\text{ren}} &= H_0 + H_1 + \sum_{i=1}^{N} \alpha_i \hat{B}_i
\end{align*}

where $H_1$ is the interaction Hamiltonian and is a linear combination of the bilinear forms $\hat{B}$ from Section 2. The $\hat{B}_i$ are counter terms and the coefficients $\alpha_i$ are scalars, generally infinite, and given in the form of divergent integrals. The role of the counter terms is to provide adjustments in the spectrum of $H_{\text{ren}}$ and in the scattering. This point is discussed at length by Friedrichs [6]. In addition the counter terms may convert the unrenormalized Hamiltonian

\begin{align*}
H &= H_0 + H_1
\end{align*}

which is merely a bilinear form, into a self-adjoint operator. In the next section we explain why (3.1) is more tractable than (3.2) and we also explain how the counterterms are given as functions of $H_0$ and $H_1$. The interaction energy $H_1$ is determined by the correspondence principle, which means that we substitute quantized fields for classical fields in the classical expression for the interaction energy. Thus for quantum electrodynamics, $H_1$ is $\int A^\mu(x) J_\mu(x) dx$ where $A^\mu$ is the quantized electromagnetic potential and $J_\mu$ is the quantized current. For details, see [24].
In this section we consider the cutoff Hamiltonians, which are presumed to approximate $H_{\text{ren}}$ in some sense. The interaction Hamiltonians $H_i$ which occur in Quantum Field Theory are either (a) fourth-order in the fermion field operators [and in the fermion annihilation and creation operators $a(k)$ and $a^*(k)$] or (b) linear in the boson field operators and quadratic in the fermion field operators or (c) fourth-order in the boson field operators or a linear combination of (a), (b) and (c). The interactions (a)–(c) seem to occur in the physical world; weak interactions are thought to be of type (a). The interactions (b) include the electromagnetic interaction and the Yukawa interaction. The Yukawa interaction describes the coupling between nucleons (protons and neutrons) and mesons; this interaction produces the strong but short ranged forces which hold together the nucleus of the atom. The electromagnetic interaction accounts for the structure of the atom as a whole; this portion of Quantum Field Theory provides small but very accurate corrections to the ordinary nonrelativistic quantum mechanics of atomic physics. The interaction (c) may occur in the strong interaction of $\pi$-mesons and also occurs as an infinite counter term in the four-dimensional Yukawa interaction.

To study the interaction (a) we take

$$H_1 = \sum_{j=0}^{4} \int v_j(k_1, \ldots, k_4) a^*(k_1) \cdots a^*(k_j) a(k_{j+1}) \cdots a(k_4) \, dk,$$

and we assume some regularity properties on the kernels $v_j$, for example that they be smooth and have compact support. Also we require $H_1 = H_1^*$ formally, or in other words

$$v_0 = v_4^-, \quad \text{etc.}$$

Then $H_1$ is a bounded self-adjoint perturbation and so $H = H_0 + H_1$ is self-adjoint. The coefficients $\alpha_i$ in (3.1) are finite for this choice of the $v_j$ and the corresponding cutoff renormalized Hamiltonian is also self adjoint and a bounded perturbation of $H_0$. The presence of the cutoffs is reflected in the regularity of $v_j$. This Hamiltonian has been studied under both more and less stringent regularity assumptions by Y. Kato [26], Y. Kato and Mugibayashi [27], Guenin [17] and Hoegh-Krohn [20]. The main results of these authors give the existence of the asymptotic limits for large positive and negative times of the field operators.
For the interaction (b), $H_1$ is a sum of eight terms each with the form

$$\int v(k, p_1, p_2) a_0^#(k) a_t^#(p_1) a_t^#(p_2) \, dk \, dp$$

where $a^# = a$ or $a^*$ and $a_0^#$ is a boson annihilation creation operator, $a_t^#$ is a fermion annihilation creation operator. As before we make a regularity assumption on the kernels $v$ and we require that $H_1 = H_1^*$ formally. In this case $H_1$ is not a bounded operator but if we assume that the rest mass $m$ of the bosons is positive, then $H_1$ is bounded relatively to $H_0$ with relative bound zero, as one can see without difficulty from Proposition 2.2. By a theorem of Rellich, $H = H_0 + H_1$ is self-adjoint. This interaction has been studied by Y. Kato, Y. Kato and Mugibayashi, and by Lanford [26–28]. Kato and Mugibayashi study asymptotic limits for large $t$, while Lanford’s work is directed toward verifying those Wightman axioms not destroyed by the cutoffs.

The interaction (c) was studied by Jaffe [21]; see also [14, 15]. $H_1$ can again be given by (3.3) but now the operators $a^#$ are boson operators. $H_1$ is not bounded and is not bounded relative to $H_0$ and so $H = H_0 + H_1$ is a singular perturbation of $H_0$. For this reason more care must be exercised in choosing the kernels $v_j$. If we start with the correct Quantum Field Theory interaction Hamiltonian corresponding to a classical $\phi^4$ interaction and then introduce cutoffs in the manner described in Section 1, the resulting cutoff $H_1$ is a semibounded self-adjoint operator. This fact together with the regularity of the $v_j (v_j \in L_2)$ are the principle hypotheses required to prove that $H = H_0 + H_1$ is self adjoint. In addition to proving that $H$ is self adjoint, Jaffe’s work [21] is directed toward verifying the Wightman axioms not destroyed by the cutoffs.

4. Formal Perturbation Theory

We use diagrams to represent the annihilation creation operators $\hat{B}$ of Section 2. The diagrams clarify some of the algebraic operations performed on the operators $\hat{B}$ and they suggest the physical processes corresponding to the algebraic operations. If $\hat{B}$ is the operator or bilinear form defined by (2.11), (2.12), (2.16), we associate to $\hat{B}$ a diagram with $m$ lines pointing to the left and $n$ lines pointing to the right, all lines coming from a common vertex, see [6], and Fig. 4.1. This diagram is a
schematic for the formula (2.16). Note that the creators are on the left, the annihilations are on the right, as in (2.16).

\[ \begin{align*}
\text{m creating lines} & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{fig1.png}
\end{array} & \text{n annihilating lines}
\end{align*} \]

Fig. 4.1

If there are any distinct types of particles present, we use different types of lines for each type of particle. In quantum electrodynamics, it is customary to use straight lines to represent electrons and positrons and wavy lines for photons. These diagrams are related to Feynman diagrams. Feynman diagrams refer to bilinear forms \( B \) which occur in the \( S \) matrix and each diagram uniquely determines the corresponding bilinear form. The diagrams we use do not determine the form \( B \) because the diagram does not determine the kernel \( b \). Figs. 2.1 and 2.2 are the diagrams of two terms from the interaction Hamiltonian \( H_I \) of quantum electrodynamics. The complete \( H_I \) is expressed in diagrams in Fig. 4.2. The two middle terms

\[ H_I = \begin{array}{c}
\includegraphics[width=0.4\textwidth]{fig2.png}
\end{array} \]

Fig. 4.2

correspond to the emission and absorption of a photon by an electron or positron. The term to the extreme left corresponds to the simultaneous creation of an electron, a positron, and a photon. This term actually occurs and the associated kernel \( b \) is not zero. The presence of this term does not violate conservation of energy because it is the total energy \( H = H_0 + H_I \) or more properly \( H_{\text{ren}} = H_0 + H_I + \) counter terms which is conserved and not the free energy \( H_0 \). The product \( \hat{B} \hat{C} \) of two annihilation creation operators \( \hat{B} \) and \( \hat{C} \) is not in general an operator of the same form because the creation operators \( a^*(k) \) in \( \hat{C} \) occur to the right of the annihilation operators \( a(\xi) \) from \( \hat{B} \), but using the commutation relations (2.17) one can write \( \hat{B} \hat{C} \) as a linear combination of annihilation creation operators, \( \hat{B} \hat{C} = \sum_i \hat{D}_i \). Each use of the commutation relations (2.17) may introduce a \( \delta \) factor and thus an integration which reduces by two the number of variables in the kernel. Each such integration is called a contraction and the term \( \hat{D}_i \) with \( i \)
contractions in the product is represented by a new diagram obtained by joining $i$ creating legs from the diagram of $\hat{C}$ each with an annihilating leg from the diagram of $\hat{B}$, see Fig. 4.3. The last term in Fig. 4.3 refers to the annihilation of a photon and creation of an electron positron pair followed by the annihilation of this same pair and the creation of a new photon. This term has two contractions. In the first term entirely distinct electron positron pairs are created and annihilated. This term contains no contractions; in general the term in $\hat{B}\hat{C}$ without contractions is called the Wick product and is denoted by $:\hat{B}\hat{C}:$

Let

$$C = \sum_{i=1}^{N} \alpha_i \hat{B}_i$$

be the sum of counter terms in the renormalized Hamiltonian (3.1). We use perturbation theory to determine the infinite part of $C$ as a formal power series in powers of $H_1$ and for simplicity we consider only the terms of degree two or less in $H_1$, i.e., second-order perturbation theory. In physics books it is customary to give the perturbation series for the $S$ matrix, $S = \sum_{n=0}^{\infty} S^{(n)}$, with $S^{(n)}$ of degree $n$ in $H_1$, because the cross sections, which are the experimentally measured quantities, can be computed easily from $S$. For our purposes it is more useful to give the perturbation expansion of a different operator. In Section 5 we consider the problem of removing the cutoffs. When the cutoffs are removed in the renormalized Hamiltonian, we obtain a very singular formal expression which appears to have no vector other than zero in its domain. However, we construct the perturbation series for an operator mapping into the domain of $H_{\text{ren}}$ and as a result we obtain a perturbation series for elements of the domain of $H_{\text{ren}}$.

A natural operator to choose would be the resolvent $R(z) = (H_{\text{ren}} - z)^{-1}$ and its perturbation series is the second Neumann series,

$$R(z) = R_0(z) \sum_{n=0}^{\infty} \left( -(H_1 + C) R_0(z) \right)^n,$$

$$R_0(z) = (H_0 - z)^{-1},$$
derived from the expansion \((1 + a)^{-1} = \sum_{n=0}^{\infty} (-a)^n\). Instead of \(R(z)\), however, we choose the wave operators \(W_+\) and \(W_-\). According to the formal theory \(W_+\) and \(W_-\) are unitary operators intertwining \(H_0\) and \(H_{\text{ren}}\) and they are defined by the limits

\[
W_{\pm} = \lim_{t \to \pm \infty} e^{itH_{\text{ren}}} e^{-itH_0}.
\]

We will need to solve the equation

\[(4.2) \quad [H_0, X] = B\]

for the unknown \(X\). One can check without difficulty that

\[(4.3) \quad X = \hat{A} + D\]

where \(D\) is any operator commuting with \(H_0\) and \(A\) has the kernel

\[(4.4) \quad a = \frac{b}{\sum_{i=1}^{m} \mu(k_i) - \sum_{j=1}^{n} \mu(k_j)}\]

if \(b\) is the kernel of \(B\). [\(\mu(k)\) was defined in (2.8).] According to the choice made for \(D\), we will obtain a multiple of \(W_+\) or \(W_-\) or neither below [6]. Even if \(b\) is a nice function, \(a\) is not locally summable, but we take the principal value in (4.4) and \(a\) is a distribution. For nice \(b\), \(a\) is the kernel of a bounded operator and so \(\hat{A}\) is an operator. We define

\[(4.5) \quad \Gamma \hat{B} = X = \hat{A} + D.\]

We seek an intertwining operator \(T\) and the counter term \(C\) such that

\[(4.6) \quad H_{\text{ren}} T = TH_0.\]

To first order, \(T = I - \Gamma H_1\) and \(C = 0\) because

\[
H_{\text{ren}}(I - \Gamma H_1) = (I - \Gamma H_1) H_0 + H_1 - [H_0, \Gamma H_1] + \cdots
\]

\[
= (I - \Gamma H_1) H_0 + \cdots,
\]

The \(\cdots\) indicate quantities of higher order in \(H_1\), in this case of second order at least. Let \(C^{(2)}\) be the second-order contribution to \(C\). Then

\[
H_{\text{ren}}(I - \Gamma H_1 + T^{(2)}) = (I - \Gamma H_1 + T^{(2)}) H_0
\]

\[
- H_1 \Gamma H_1 + C^{(2)} + [H_0, T^{(2)}] + \cdots
\]
and

\begin{equation}
T^{(2)} = \Gamma(H_1\Gamma H_1 - C^{(2)}).
\end{equation}

We have one equation in two unknowns which determines neither \(T^{(2)}\) nor \(C^{(2)}\) uniquely, but in order that \(T^{(2)}\) be defined the right-hand side must be finite, and this is a requirement which determines the infinite part of \(C^{(2)}\). If we expand \(H_1\Gamma H_1\) as a sum of terms of the form of \(\hat{B}\) (called Wick ordered terms), we obtain a large number of finite terms and a few infinite terms. For quantum electrodynamics in four dimensional space time, the infinite terms are exactly the nine terms with two or three contractions; typical infinite diagrams are illustrated in Fig. 4.4. The term with no external legs is

Fig. 4.4

an infinite constant. For the \(\phi^4\) interaction in four dimensions the infinite terms are exactly those terms with two, three, or four contractions and typical terms are illustrated in Fig. 4.5. The vacuum

Fig. 4.5

energy diagram has both an ultraviolet and a volume divergence. The volume divergence is independent of the number of dimensions while the ultraviolet divergence is infinite only in three and four dimensions. The other diagrams have only an ultraviolet divergence and become less divergent or finite as the number of space dimensions is reduced. The higher-order perturbation theory proceeds in a similar fashion. For the Yukawa and the \(\phi^4\) interaction in two and three dimensions the higher-order perturbation is simpler because in these cases \(C^{(n)}\) for large \(n\) has no ultraviolet divergence. This means that the infinite part of \(C^{(n)}\) comes from the vacuum diagram: (the diagrams without external legs) and so \(C^{(n)}\) is an infinite constant plus a finite operator. Interactions with this
property are called *superrenormalizable* and are essentially easier to study. For the four fermion interaction in four (and apparently in three) dimensions, the higher order perturbation theory becomes more complicated because the infinite part of \( C^{(n)} \) contains diagrams with an increasing number of external legs. This type of interaction is called *unrenormalizable* because the higher order perturbation theory contains an increasing number of parameters which must be determined by experiment before predictions can be made.

In [6] Friedrichs derives a set of equations to be satisfied by the intertwining operator \( T \) and related operators. These equations may be used to generate a perturbation expansion for \( T \) and Friedrichs raises the possibility that the equations might be solved by other methods; if this were accomplished one would have a definition for \( H_{\text{ren}} \) and presumably a proof of its unitary equivalence to \( H_0 \).

### 5. Removing the Cutoffs

At present the renormalizable interactions in four dimensions seem remote and so we will discuss superrenormalizable interactions, in particular the Yukawa and \( \phi^4 \) interactions in two and three dimensions. Also we mention Nelson’s study of a model in which the fermions are quantized nonrelativistically [29]; this work has been carried forward by Cannon [4]. In addition to reducing the degree of the ultraviolet divergence, the nonrelativistic quantization has another effect. Namely the total number of fermions is conserved in time and the fermions behave like a classical field. This raises the possibility of semiquantized interactions in which one of the two interacting fields is quantized and one is classical. In Nelson’s model all cutoffs have been removed. The problems are simplified still further by assuming that the classical field is external, or in other words is given in advance and is not affected by the dynamics of the quantized field. When one of the fields is external, the equations for the remaining field are linear and the Hamiltonian is at most quadratic in the \( a \) and \( a^* \) (or in the quantized field). The external field problems have been studied to the satisfaction of most physicists, but there remain open questions of interest to the mathematical foundations.

Symanzik has made a formidable attack on the problems of this section [36, 37] and some of his work has been simplified by Ginibre [8]. Symanzik considers the \( \phi^4 \) interaction in two and three dimensions.
He derives a new perturbation theory in which the unperturbed dynamics is not the free dynamics given by $H_0$ but rather is the dynamics with an external field. As a result the $\varphi^4$ interaction appears as a less singular perturbation and it is conceivable that the series may converge. Throughout his work, Symanzik works with imaginary time ($t$ is replaced by $i\tau$) and the solutions of the external field equations are given explicitly as Wiener integrals using the Feynman Kac formula.

5.1. Removing the Ultraviolet Cutoff

Let $H_{\text{ren}}(\sigma)$ be the renormalized Hamiltonian with an ultraviolet cutoff depending on $\sigma$. Assume a fixed space cutoff in each $H_{\text{ren}}(\sigma)$ and determine the infinite part of the counter terms by the method of Section 4. Since the interaction is superrenormalizable and since we have a fixed space cutoff, $C^{(n)}$ is finite (in the limit $\sigma \to \infty$) for large $n$. For such $n$ we set $C^{(n)} = 0$. We wish to define the limit

$$H_{\text{ren}} = \lim_{\sigma \to \infty} H_{\text{ren}}(\sigma).$$

For the $\varphi^4$ interaction in two dimensions, (5.1) holds in a conventional sense: the operators $H_{\text{ren}}(\sigma)$ converge strongly on a dense domain and the limit $H_{\text{ren}}$ is essentially self adjoint on this domain [15]. For this interaction there are no ultraviolet divergences. $H_{\text{ren}}$ is bounded from below [5, 11, 30].

For the Yukawa interaction in two dimensions and the $\varphi^4$ interaction in three dimensions, the limit (5.1) can again be defined [9, 12]. For these interactions there are infinite ultraviolet divergences and so vectors in the domain of $H_{\text{ren}}$ must be constructed explicitly; there is no class of vectors which obviously lie in the domain of the limit $H_{\text{ren}}$. To exhibit explicitly vectors in the domain of $H_{\text{ren}}$ we start with the formal expansions of Section 4. These expansions appear to diverge (even for the cutoff Hamiltonian $H_{\text{ren}}(\sigma)$) and they must be truncated. In the $n$th-order contribution $T^{(n)}$ to $T$ of Section 4, we delete certain low energy regions but leave unchanged the part of the operator which acts on high energy particles. The cross over point from low to high energy depends on $n$ and increases as $n \to \infty$. Let $T_{\text{tr}}$ and $T_{\text{tr}}(\sigma)$ be the truncated operators. For the Yukawa interaction in two dimensions the precise meaning of (5.1) is

$$H_{\text{ren}}T_{\text{tr}} = \text{St. lim.} \; H_{\text{ren}}(\sigma) \; T_{\text{tr}}(\sigma)$$

(5.2)

$$T_{\text{tr}} = \text{St. lim.} \; T_{\text{tr}}(\sigma)$$

(5.3)
on a dense domain [9, 41]. A limit $H_{\text{ren}}$ can be defined without referring to the $T_{\text{tr}}(\sigma)$. Let

$$\mathcal{D}(H_{\text{ren}}) = \{ \varphi: \varphi = \lim \varphi_\sigma, \lim H_{\text{ren}}(\sigma) \varphi_\sigma \text{ exists} \}$$

and for $\varphi \in \mathcal{D}(H_{\text{ren}})$ set $H_{\text{ren}}\varphi = \lim H_{\text{ren}}(\sigma) \varphi_\sigma$. With this definition, $H_{\text{ren}}$ is a well-defined symmetric operator. This type of convergence is called graph convergence; some general properties of graph convergence are studied in [14], given (5.2) and (5.3).

For the $\varphi^4$ interaction in three dimensions a new problem, called wave function renormalization, arises. One can check that the pure creation part of $1/H_1$ does not have an $L_2$ kernel. As a consequence $1/H_1$ and $T_{\text{tr}}$ map out of the Fock space $\mathcal{F}$ into some space of sequences $\varphi_0, \varphi_1, \ldots$ of measurable functions not necessarily in $L_2$. It develops that although the vectors in the range of $T_{\text{tr}}$ have an infinite Fock space norm, the ratios

$$\| T_{\text{tr}}p \| / \| T_{\text{tr}}\psi \|$$

are finite in the sense that the limits

$$\lim_{\sigma \to \infty} \| T_{\text{tr}}(\sigma) \varphi \| / \| T_{\text{tr}}(\sigma) \psi \|$$

exist and are finite; these limits can be used to define a pre-Hilbert space on the range of $T_{\text{tr}}$. The limit (5.1) exists in a weak sense. A similar infinite wave function renormalization occurs in the interaction of a boson field with an external source, provided the source is sufficiently singular.

For the Yukawa interaction in two dimensions $H_{\text{ren}}$ is semibounded [10, 11] but it is not known whether this is true for the $\varphi^4$ interaction in three dimensions.

5.2. Removing the Space Cutoff

We consider the $\varphi^4$ interaction in two dimensions. Let $H_{\text{ren}}(V)$ be the Hamiltonian with a space cutoff of volume $V$ but no momentum cutoff. We include a finite constant $c(V)$ in $H_{\text{ren}}(V)$,

$$H_{\text{ren}}(V) = H_0 + H_1(V) + c(V),$$

chosen so that 0 is the infimum of the spectrum of $H_{\text{ren}}(V)$. We have the following estimate [16]:

$$c(V) \leq \text{const } V.$$
We note that this estimate agrees with nth order perturbation theory for each n but that we do not use perturbation theory to set the value of the linearly divergent constant \( c(V) \). The reason for not using perturbation theory is that the nth-order contribution \( c^{(n)}(V) \) to \( c(V) \) is linearly divergent for each \( n \),

\[ c^{(n)}(V) = Vc_n, \]

and the sum \( \sum_n c_n \) of the coefficients appears to diverge. In this sense the space cutoff serves as a test for the renormalizable ultraviolet divergences in four dimensions. In particular we conclude that if the ultraviolet cutoff in four dimensions can be removed, it will probably require methods less explicit than those used for two and three dimensions.

The first step in removing the space cutoff is to consider the \( H_{\text{ren}}(V) \) dynamics in the Heisenberg picture; in the Heisenberg picture the operators move in time while the states are constant, independent of time. Now in the Heisenberg picture influence propagates at a finite speed. This fact is true not only for the dynamics in which all cutoffs have been removed (in this case it is well known on the level of formal perturbation theory) but moreover it is true for the space cutoff \( H_{\text{ren}}(V) \) dynamics, so that the time \( t \) operator \( A(t) \) is independent of \( V \) for large \( V \) provided \( A(0) \) is localized in a bounded region of space. The latter statement was observed formally by Guenin [17] and stated as a theorem by Segal [33]. The hypothesis of the theorem, that \( H_{\text{ren}}(V) \) be self adjoint, has been proved [14, 15]. The main step in the proof of the theorem is an application of Trotter's product formula

\[
(5.6) \quad \exp[-iH_{\text{ren}}(V)] = \lim_{n \to \infty} \{\exp[-i(t/n) H_0] \exp[-i(t/n)(H_1 + c(V))]\}^n.
\]

The limiting Hamiltonian \( H_{\text{ren}} \) appears not to exist as an operator on \( \mathcal{F} \). This assertion agrees with perturbation theory because the pure creation part of \( T H_1 \) does not have an \( L_2 \) kernel, due to translation invariance. In the framework of axiomatic field theory the assertion is essentially Haag's theorem [38]. In order to define \( H_{\text{ren}} \), we will need to pass to a new Hilbert space. The new Hilbert space will not be obtained in closed form (cf. Subsection 5.1) but will come from a limiting procedure using methods of Jaffe and Powers [22], see [16].

We have not discussed the S matrix and the asymptotic fields in this section in spite of their fundamental importance because they have not been shown to exist in any model with either cutoff removed, but we
mention that the work of Lehmann, Symanzik and Zimmermann (described in [31]) of Haag and Ruelle (described in [23,40]) and of T. Kato and his students (described in [25]) will probably play an important role.

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