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Maximum packings with odd cycles

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Abstract

We show how to obtain maximum packings of $K_{2kg}+v$ with k-cycles when $k\geqslant 3$ is odd, g a positive integer, and v even with $0\leqslant v<2k$. Moreover, under certain conditions on v, we obtain maximum packings of K_{2kg+v}.

1. Introduction

Let G be a graph and $k\geqslant 3$ an integer. A (*partial*) *k-cycle design* (*or decomposition*) *of G* is a pair (P,B), where P is the vertex set of G and B is a (partial) partition of the edges of G into edge disjoint k-cycles. A k-cycle design of K_n is also known as a *k-cycle system of order n*. If t is a nonnegative integer, the graph $G+t$ is defined as follows: add to G an independent set of t new vertices, each adjacent to every vertex in G. A *k-cycle system of order w with a hole of size v* is a partition of the graph $K_{w-v}+v$ into k-cycles. A partial k-cycle decomposition of G is also known as a *packing* of G. The *leave* of a packing of G is the set of edges of G that are not used in the packing. We define the *deficiency* of a packing to be the cardinality of the leave of the packing. A *maximum packing* is a packing with minimum deficiency. Hence, a k-cycle design is a maximum packing with zero deficiency. Some necessary conditions for the existence of a k-cycle system of order n are:

(1) if $n>1$ then $n\geqslant k$,
(2) n is odd (so each vertex in K_n would have even degree), and
(3) $2k\,|\,n(n-1)$ (since $|E(K_n)|=n(n-1)/2$).

For a fixed k, these necessary conditions require that n lies in certain congruence classes (called *fibers*) modulo $2k$. If n does not lie in these fibers, then only partial k-cycle designs of order n are possible. There is strong evidence now that the above necessary conditions are sufficient for the existence of k-cycle decompositions of K_n. However, despite being considered for the past 25 years, the necessary conditions have been proven to be sufficient in the following cases only:

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(a) $k=p^e$ for some prime p,
(b) $k=2p^e$ for some prime p [1], and
(c) $k<51$ [2, 7].

Furthermore, a k-cycle decomposition of K_n is known to exist if
(d) $n\equiv 1 \pmod{2k}$ [6, 8] and if
(e) $n\equiv k \pmod{2k}$ and k is odd [5, 9].

It is shown in [3, 12] that when k is even, the necessary conditions are sufficient for the existence of k-cycle systems of order n if they are sufficient for the existence of k-cycle systems of order m where $k+1\leqslant m\leqslant 3k$. Hoffman et al. [4] showed the corresponding result for k odd. Other than for Steiner triple systems ($k=3$) [11], and for 5-cycle systems [10], little is known about maximum packings of K_n with k-cycles. In this paper, we obtain a companion result to the results by Hoffman et al. [4] to obtain a maximum k-cycle packing of $K_{2kg}+v$ when $k\geqslant 3$ is odd, g a positive integer, and v even with $0\leqslant v<2k$. Moreover, under certain conditions on v, we obtain maximum packings of K_{2kg+v}.

2. Preliminaries

The following lemma gives us a lower bound on the deficiency of a partial k-cycle design of order $2m$.

Lemma 2.1. *If $P=(V,B)$ is a partial k-cycle design of K_{2m} with deficiency d, then $d\geqslant m(2m-1)-k\lfloor 2m(m-1)/k\rfloor$.*

Proof. Clearly, $d\geqslant m$, since K_{2m} is $(2m-1)$-regular; hence the graph induced by the leave of P is a spanning subgraph of odd degrees. The smallest such subgraph is a 1-factor (with m edges). Now, the number of k-cycles in P is $|B|=(m(2m-1)-d)/k$. Therefore,

$$\begin{aligned} d&=m(2m-1)-k|B|\\ &=m(2m-1)-k((m(2m-1)-d)/k)\\ &\geqslant m(2m-1)-k\lfloor (m(2m-1)-m)/k\rfloor \quad \text{(since } d\geqslant m)\\ &=m(2m-1)-k\lfloor 2m(m-1)/k\rfloor. \qquad \square \end{aligned}$$

Let $\theta(2m,k)=m(2m-1)-k\lfloor 2m(m-1)/k\rfloor$. We say that $2m$ is *k-nice* if there exists a packing of K_{2m} with k-cycles with deficiency exactly $\theta(2m,k)$.

Throughout this paper, $k=2l+1\geqslant 3$ is a fixed odd integer. We say that a graph G is *partitionable* if the edges of G can be partitioned into k-cycles. Our goal in the next few sections is to show that certain graphs $G+t$ are partitionable. To do that, we will need to use difference methods. Here we borrow some definitions from [4].

If x is an integer, define $|x|_k$ as follows: find $y \equiv x \pmod k$, with $-l \leqslant y \leqslant l$; then $|x|_k = |y|$. Let $Z_k \times Z_2$ be the vertex set of the complete graph K_{2k}, and let e be an edge. If e is incident with $(x,0)$ and $(y,0)$, then e is called an edge of *left pure difference* $|y-x|_k$. If e is incident with $(x,1)$ and $(y,1)$, then e is an edge *of right pure difference* $|y-x|_k$. Finally, if e is incident with $(x,0)$ and $(y,1)$, then e is an edge of mixed difference $y-x$. Note that the pure differences are in the set $D=\{1,2,\dots,l\}$, while mixed differences are elements of Z_k.

For $A \subseteq D$, $C \subseteq D$, and $B \subseteq Z_k$, we define $\langle A,B,C\rangle$ to be the spanning subgraph of K_{2k} whose edges are all those with left pure difference in A, or mixed difference in B, or right pure difference in C. Note that $\langle A,B,C\rangle$ is isomorphic to $\langle C,-B,A\rangle$, where $-B=\{-b \mid b \in B\}$ as well as to $\langle A,B+x,C\rangle$ for any $x \in Z_k$, where $B+x=\{b+x \mid b \in B\}$.

3. Designs with a hole

The following lemmas, showing that certain graphs $\langle A,B,C\rangle + t$ are partitionable, can be found in [4]. Throughout, $s \leqslant l$ is a nonnegative integer.

Lemma 3.1. *If s is odd, then $\langle \emptyset, \{0,1,\dots,k-2s-1\}, \emptyset\rangle + s$ is partitionable.*

Lemma 3.2. *If s is even (and $0 \leqslant s \leqslant l$), then $\langle \{l\}, \{0,1,\dots,k-2s-2\}, \emptyset\rangle + s$ is partitionable.*

Lemma 3.3. *Let $S \subseteq \{2m-1 \mid 1 \leqslant m \leqslant l\}$. Then $G=\langle \{l\}, S \cup (S+1), \emptyset\rangle$ is partitionable.*

As in [4], for $1 \leqslant i \leqslant 3$, we say that a graph G is of *type i* if it is isomorphic to the partitionable graph of Lemma 3.*i* above.

Remark 3.4. We note that if l is odd and M is a perfect matching in K_{2k}, then $M+l$ is isomorphic to $\langle \emptyset, \{0\}, \emptyset\rangle + l$ and is hence of type 1. Similarly, if l is even and C is k-cycle in K_{2k}, then $C+l$ is isomorphic to $\langle \{l\}, \emptyset, \emptyset\rangle + l$ and is hence of type 2.

The preceding lemmas are used in [4] to prove the following essential theorem.

Theorem 3.5. *Let $k=2l+1$ be a positive integer. For v a positive odd integer, write $v=ql+r$, where $1 \leqslant r \leqslant l$. If $q \leqslant k+2r-1$, then there is a k-cycle system of order $2k+v$ with a hole of size v.*

Theorem 3.5 fails if v is even, since for v even, the vertices outside the hole have odd degrees. Hence, if L is the graph induced by the leave of a k-cycle packing of $K_{2k}+v$ with v even, then $d_L(u)$ is odd for all vertices u outside the hole. Hence, a maximum k-cycle packing of $K_{2k}+v$ has deficiency $\geqslant k$ (the size of a 1-factor in K_{2k}). In the next theorem, we exhibit a k-cycle packing of $K_{2k}+v$ with deficiency k. Before we proceed, we will need yet another lemma. This one is due to Stern and Lenz [13]. A proof can be found in both [4] and [13].

Lemma 3.6. *Let G be a graph, and let H be the graph whose vertices are* $V(G)\times Z_2$, *and* (v,i), (w,j) *are adjacent in H if and only if either* $i=j$ *and v, w are adjacent in G, or* $v=w$ *and* $i\neq j$. *Then H has chromatic index* $d=\Delta(H)$, *the maximum degree of H.*

Theorem 3.7. *Let* $k=2l+1$ *be a positive odd integer with* $k\geqslant 1$. *Let* $v<4k$ *be a non-negative* even *integer. For* $v\neq 0$, *write* $v=ql+r$, *where* $1\leqslant r\leqslant l$. *If* $q\leqslant k+2r-2$, *then there exists a k-cycle decomposition of* $(K_{2k}-F)+v$, *where F is a 1-factor in* K_{2k}.

Proof. It is well known that K_k is partitionable (has a Hamiltonian decomposition); let $P_1, P_2, \ldots, P_l$ be the k-cycles in such a partition. We may assume that Z_k is the vertex set and that P_l consists of the edges of pure difference l.

It suffices to show that for some $f\in Z_k$, $G=\langle D, Z_k\backslash\{f\}, D\rangle+v$ is partitionable. We will do so by partitioning the edges of G into partitionable subgraphs.

Case 1: *l is even (hence r is even).* First partition $G=G_1\cup(G_2+ql)$, where $G_1=\langle\emptyset,\{0,1,\ldots,k-2r-2\},\{l\}\rangle+r$ and $G_2=\langle D,\{k-2r,\ldots,k-1\}\rangle, D\backslash\{l\}\rangle$. G_1 is of type 2 with $s=r$, and hence is partitionable. Now $G_2=\langle\{l\},\ \{k-2r,\ldots,k-1\},\emptyset\rangle\cup\langle D\backslash\{l\},\emptyset,D\backslash\{l\}\rangle$. Of these two graphs, the first is of type 3, and hence is partitionable. The second is partitionable into the k-cycles $P_i\times\{0,1\}$, $1\leqslant i\leqslant l-1$. Thus G_2 is partitionable, and a simple calculation shows that there are $k+2r-2$ k-cycles in the partition. Choose q of these cycles, $C_1, C_2, \ldots, C_q$, and partition $G_2=G_3\cup G_4$, where G_4 consists of the q cycles $C_1, C_2, \ldots, C_q$ and G_3 consists of the remaining edges of G_2. Now G_3 is partitionable, so it remains to partition G_4+ql. But this last graph can be partitioned into q graphs of type 2 with $s=l$, and is hence partitionable.

Case 2: *l is odd, r is even (hence q is even).* First suppose $q\leqslant 2r$. Partition $G=G_1\cup(G_2+ql)$, where $G_1=\langle\emptyset,\{0,1,\ldots,k-2r-2\},\{l\}\rangle+r$ and $G_2=\langle D,\{k-2r,\ldots,k-1\}, D\backslash\{l\}\rangle$. G_1 is of type 2 with $s=r$, and is hence partitionable. Now

$$\begin{aligned}G_2&=\langle\emptyset,\{k-2r,\ldots,k-2r+q-1\},\emptyset\rangle\\&\quad\cup\langle\{l\},\{k-2r+q,\ldots,k-1\},\emptyset\rangle\\&\quad\cup\langle D\backslash\{l\},\emptyset,D\backslash\{l\}\rangle.\end{aligned}$$

Of these three graphs, the second is type 3, and hence partitionable, and the third can be partitioned into $P_i\times\{0,1\}$, $1\leqslant i\leqslant l-1$. It remains only to partition $\langle\emptyset,\{k-2r,\ldots,k-2r+q-1\},\emptyset\rangle+ql$. But this can be partitioned into q graphs of type 1 with $s=l$.

Now suppose $q\geqslant 2r+2$. First partition $G=G_1\cup(G_2+ql)$, where $G_1=\langle\emptyset,\{1,2,\ldots,k-2r\},\emptyset\rangle+r$ and $G_2=\langle D,\{k-2r+2,\ldots,k\},D\rangle$. Let $P=\bigcup P_i$, $1\leqslant i\leqslant q/2-r$, let H be the graph with edges $P\times\{\varepsilon\}$, $\varepsilon=0,1$, and $\langle\emptyset,\{k-2r+2,\ldots,k\},\emptyset\rangle$. Now H has chromatic index $q=\Delta(H)$, with $\langle\emptyset,\{i\},\emptyset\}$ a color class for each $k-2r+2\leqslant i\leqslant k-1$, and the remaining color classes given by Lemma 3.6. Hence,

$H+ql$ is the union of q graphs of type 1, and so is partitionable. All that is left are the k-cycles $P_i \times \{\varepsilon\}$, $q/2-r+1 \leqslant i \leqslant l$, $\varepsilon=0,1$.

Case 3: *both l and r are odd (hence q is odd).* First suppose that $q \leqslant 2r-1$. Partition $G=G_1 \cup (G_2+ql)$, where $G_1=\langle \emptyset, \{0,1,\dots,k-2r-1\}, \emptyset\rangle + r$ and $G_2=\langle D, \{k-2r+1,\dots,k-1\}, D\rangle$. G_1 is of type 1 with $s=r$, and is hence partitionable.

$$\begin{aligned} G_2 = & \langle \emptyset, \{k-2r+1,\dots,k-2r+q\}, \emptyset\rangle \\ & \cup \langle \{l\}, \{k-2r+q+1,\dots,k-1\}, \emptyset\rangle \\ & \cup \langle D \backslash \{l\}, \emptyset, D\rangle. \end{aligned}$$

Of these three graphs, the second is of type 3 and hence is partitionable. The third can be partitioned into $P_i \times \{0\}$, $1 \leqslant i \leqslant l-1$, $P_i \times \{1\}$, $1 \leqslant i \leqslant l$. It remains only to partition $\langle \emptyset, \{k-2r+1,\dots,k-2r+q\}, \emptyset\rangle + ql$. But this can be partitioned into q graphs of type 1 with $s=l$.

Now suppose $q \geqslant 2r+1$. Partition $G=G_1 \cup (G_2+ql)$, where $G_1=\langle \emptyset, \{1,2,\dots,k-2r\}, \emptyset\rangle + r$ and $G_2=\langle D, \{k-2r+2,\dots,k\}, D\rangle$. G_1 is of type 1 and is hence partitionable. Let P be the union of P_i, $1 \leqslant i \leqslant -r+(q-1)/2$, let H be the graph with edges $P \times \{\varepsilon\}$, $\varepsilon=0,1$, and $\langle \emptyset, \{k-2r+2,\dots,k\}, \emptyset\rangle$. Continue as in case 2.

We note that the hypothesis $q \leqslant k+2r-2$ is automatically satisfied when $v<4k$ except for the cases

(i) $k=3$ and $v \in \{6,8,10\}$, and

(ii) $k=5$ and $v=18$. □

4. Filling in the fibers

A *group divisible k-cycle system with g groups of size x* is a partition of the complete g-partite graph K_x^g with g parts (holes or groups) of size x into k-cycles.

The following results can be found in both [4] and [7].

Theorem 4.1. *If k is odd and $g \geqslant 3$, there is a group divisible k-cycle system with g groups of size $2k$ (a k-cycle decomposition of K_{2k}^g).*

Theorem 4.2. *If k, g are odd, there is a group divisible k-cycle system with g groups of size k (a k-cycle decomposition of K_k^g).*

Finally, here is our main theorem. As in the previous section, $v<4k$ is a nonnegative even integer, and for $v \neq 0$, write $v=ql+r$, $1 \leqslant r \leqslant l$. (Recall $k=2l+1$.)

Theorem 4.3. *If $q \leqslant k+2r-2$ and $g \neq 2$ is a positive integer, then $(K_{2kg}-F)+v$ is partitionable for some F, a 1-factor in K_{2kg}.*

Proof. If $g=1$, then this reduces to Theorem 3.7. Hence, assume $g \geqslant 3$. Let $Z_k \times Z_{2g}$ be the vertex set of K_{2kg}, and let $V=\{\infty_1, \ldots, \infty_v\}$ be the set of added points. Let $H=\{h_0, h_1, \ldots, h_{g-1}\}$ where $h_i=\{2i, 2i+1\}$. For $i=0,1,\ldots,g-1$, let G_i denote the complete graph with vertex set $Z_k \times h_i$. By Theorem 4.1, there exists a k-cycle decomposition of K_{2k}^{g}, the complete k-partite graph with holes of size $2k$ (the holes being $G_0, G_1, \ldots, G_{g-1}$). By Theorem 3.7, there exists a k-cycle decomposition of $(G_i - F_i)+v$, where F_i is a 1-factor in G_i, for $i=0,1,\ldots,g-1$. Hence, we have a k-cycle decomposition of $(K_{2kg}-F)+v$, where $F=\bigcup_{i=0}^{g-1} F_i$. This decomposition is a maximum packing of $K_{2kg}+v$ since its leave is a 1-factor in K_{2kg}. □

Corollary 4.4. *Under the hypotheses of Theorem* 4.3, *if v is k-nice, then $2kg+v$ is k-nice.*

Proof. Let $P_1=(V_1, B_1)$ be a packing of K_v with deficiency $\theta(v,k)$ and $P_2=(V_2, B_2)$ be the packing of $K_{2kg}+v$ obtained by Theorem 4.3. Then $P=(V_1 \cup V_2, B_1 \cup B_2)$ is a packing of K_{2kg+v}, with deficiency $d=kg+\theta(v,k)$. But

$$\begin{aligned}
\theta(2kg+v,k) &= (kg+v/2)(2kg+v-1)-k\lfloor (2kg+v)(kg-1+v/2)/k \rfloor \\
&= 2k^2g^2+2kgv-kg+(v^2-v)/2-k\lfloor 2kg^2+2gv-2g+(v^2-2v)/2k \rfloor \\
&= kg+((v^2-v)/2)-k\lfloor (v^2-2v)/2k \rfloor \\
&= kg+\theta(v,k) \\
&= d.
\end{aligned}$$

Hence, $2kg+v$ is k-nice. □

Corollary 4.5. *Suppose that $v<2k$ is a nonnegative even integer, and $n \equiv v \pmod{2k}$. If v is k-nice, then n is k-nice.*

Proof. The hypothesis $q \leqslant k+2r-2$ in Corollary 4.4 is automatically satisfied when $v<2k$. Hence, Corollary 4.4 handles all the cases except for $n=4k+v$. But this last case follows from Theorem 3.7 and Corollary 4.4 with a new v equal to the old v plus $2k$ except for possibly when $k=3$ and $n \in \{12, 14, 16\}$, and $k=5$ and $n=28$. However, the existence of maximum packings of K_{12}, K_{14}, and K_{16} with 3-cycles with deficiencies 6, 7, and 9, respectively, is shown in [11]. Similarly, a 5-cycle packing of K_{28} with deficiency 18 can be found in [10]. □

Corollary 4.6. *Under the hypotheses of Theorem* 4.3, *if $v<1+\sqrt{1+2k}$ then $2kg+v$ is k-nice.*

Proof. For v even, $\theta(v,k)=(v(v-1)/2)-k\lfloor (v^2-2v)/2k \rfloor$; hence, if $(v^2-2v)<2k$, then $\theta(v,k)=v(v-1)/2=|E(K_v)|$. But, $v<1+\sqrt{1+2k}$ implies $v^2-2v<2k$. □

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