

Note

Burnside's Lemma: A Historical Note*

E. M. WRIGHT

University of Aberdeen, Aberdeen, United Kingdom

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Burnside himself correctly ascribed the lemma now given his name to Frobenius. We explain how the subsequent error seems to have arisen.

If F is the number of a family of graphs on n labelled points (e.g., those with q lines), F_π the number of these which are invariant under the permutation π of the labels of the n points and T the number of the corresponding unlabelled graphs, the result

$$n! T = \sum_{\pi} F_{\pi},$$

where the sum is over all the $n!$ permutations π of the symmetric group of degree n , is usually called Burnside's Lemma. It is widely used to enumerate the number T of unlabelled graphs of a particular family and is particularly useful to obtain asymptotic results (e.g., [4-8]). Many theorems are named after those who were not their discoverers but in this case there is the additional oddity that Burnside himself [1] correctly ascribes the result to Frobenius [3]. Both of them are, of course, concerned with the result in the more general form in which T is the number of equivalence classes under the operators of a group and $n!$ is replaced by the order of the group.

The mistake seems to have arisen as follows. Burnside reaches the result at the end of Section 118 of [1] and gives the ascription at the beginning of Section 119. In his second edition [2] the section corresponding to Section 119 is extensively rewritten and the ascription is omitted. We can only surmise whether this was an oversight, whether Burnside felt that the result was now commonplace or whether he found that others had discovered

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it about the same time as Frobenius. I have found no evidence for this last possibility. Most authors refer to [2], copies of which seem to be far more widespread than those of [1].

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