Computing Orbits of Minimal Parabolic $k$-subgroups Acting on Symmetric $k$-varieties

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In this paper we present an algorithm to compute the orbits of a minimal parabolic $k$-subgroup acting on a symmetric $k$-variety and most of the combinatorial structure of the orbit decomposition. This algorithm can be implemented in LiE, GAP4, Magma, Maple or in a separate program. These orbits are essential in the study of symmetric $k$-varieties and their representations. In a similar way to the special case of a Borel subgroup acting on the symmetric variety, (see A. G. Helminck. Computing $B$-orbits on $G/H$. J. Symb. Comput., 21, 169–209, 1996.) one can use the associated twisted involutions in the restricted Weyl group to describe these orbits (see A. G. Helminck and S. P. Wang. On rationality properties of involutions of reductive groups. Adv. Math., 99, 26–96, 1993). However, the orbit structure in this case is much more complicated than the special case of orbits of a Borel subgroup. We will first modify the characterization of the orbits of minimal parabolic $k$-subgroups acting on the symmetric $k$-varieties given in Helminck and Wang (1993), to illuminate the similarity to the one for orbits of a Borel subgroup acting on a symmetric variety in Helminck (1996). Using this characterization we show how the algorithm in Helminck (1996) can be adjusted and extended to compute these twisted involutions as well.

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1. Introduction

In the last two decades much of the algebraic/combinatorial structure of Lie groups and their representations has been implemented in several excellent computer algebra packages, such as LiE, GAP4, Magma and Maple. The structure of reductive symmetric spaces or more generally symmetric $k$-varieties is very similar to that of the underlying Lie group, with a few additional complications. However, except in the case of symmetric $k$-varieties over algebraically closed fields, hardly any of this structure has been implemented in a computer algebra package. This is partly due to the fact that there were no suitable algorithms available. In this paper we open up this area of mathematics to symbolic computation by giving an algorithm to compute the orbits of a minimal parabolic $k$-subgroup acting on a symmetric $k$-variety. A description of these orbits and their closures determines most of the fine structure of these symmetric $k$-varieties and their representations. After completely implementing the algorithm in this paper we hope to evolve the program to a general symbolic computation package for computations related to symmetric $k$-varieties.

The symmetric $k$-varieties are defined as follows. Let $G$ be a reductive algebraic group and $H$ the fixed point group of an involution $\theta$. Then the spherical homogeneous space $G/H$ is called a symmetric variety. They occur in many problems in representation
theory, including the study of Harish-Chandra modules with $k = \mathbb{C}$ (see for example Vogan, 1983), geometry (see De Concini and Procesi, 1983; Abeasis, 1988; Helminck and Schwarz, 1998) and singularity theory (see Lusztig and Vogan, 1983; Hirzebruch and Slodowy, 1990). When $G$ and $\theta$ are defined over a field $k$ which is not necessarily algebraically closed, then the variety $G_k/H_k$ is called a symmetric $k$-variety. Here $G_k$ (resp. $H_k$) denotes the set of $k$-rational points of $G$ (resp. $H$). These also occur in several areas including the cohomology of arithmetic subgroups (see Tong and Wang, 1989) and representation theory. It is in this last area that these symmetric $k$-varieties are best known. The representation theory and Plancherel formulas of symmetric $k$-varieties over the real numbers (also called reductive symmetric spaces) has been studied extensively in the last few decades. Most of the early work was done by Harish-Chandra, who gave the Plancherel formulas for the Riemannian symmetric spaces and the groups case. Expanding on Harish-Chandra’s ideas the representation theory for the general reductive symmetric spaces has been carried out by a number of mathematicians including Flensted-Jensen, Oshima, Sekiguchi, Matsuki, Brylinski, Delorme, Schlichtkrul and van den Ban (see Flensted-Jensen, 1980; Oshima and Sekiguchi, 1980; Oshima and Matsuki, 1984; Ban, 1988; Brylinski and Delorme, 1992; Ban and Schlichtkrull, 1997). Once the work on the real symmetric $k$-varieties and their representations has been completed, a natural next case to study will be the $p$-adic symmetric $k$-varieties and their representations. Some people have already started on this with a number of interesting results (see Jacquet et al., 1993; Rader and Rallis, 1996; Helminck and Helminck, 1999, 2000).

Another case which has been studied is the representations associated with symmetric $k$-varieties defined over a finite field (see Lusztig, 1990; Grojnowski, 1992).

The one essential tool in all of the above studies of symmetric $k$-varieties and their representations is a description of the geometry of the orbits of a minimal parabolic $k$-subgroup acting on the symmetric $k$-variety. A description of these orbits naturally leads to a description of most of the fine structure of these symmetric $k$-varieties including the restricted root system of the symmetric $k$-variety including multiplicities etc. Besides representation theory, these orbits are also of importance in a number of other areas, including geometry (see De Concini and Procesi, 1983; Lusztig and Vogan, 1983) and the cohomology of arithmetic subgroups (see Tong and Wang, 1989).

The orbits of a minimal parabolic $k$-subgroup acting on the symmetric $k$-variety have been studied for various base fields by many mathematicians. For $k = \mathbb{C}$ a description was given by Springer (1984). For $k = \mathbb{R}$ descriptions were given by Matsuki (1979) and Rossmann (1979). For arbitrary fields of characteristic not 2 a description was given by Helminck and Wang (1993). In practice it is quite difficult and cumbersome to actually compute these orbits and their closures. Fortunately most of the geometry of these orbits and their closures can be described combinatorially and therefore most of this work could be done by a computer. For $k$ algebraically closed (of characteristic not 2) we gave in Helminck (1996) an efficient algorithm to compute the orbits and their closures. In this paper we show how the algorithm in Helminck (1996) can be modified and extended to compute the orbits for a number of base fields, including $k = \mathbb{R}$. Although the structure of symmetric $k$-varieties and the related $k$-groups is much more complicated than that of symmetric varieties over algebraically closed fields the algorithm to compute the above orbits is similar to the one for symmetric varieties over algebraically closed fields given in Helminck (1996). Most of the code developed for that situation can be modified to compute the orbits in this case as well.

Essential in the algorithm to compute the orbits in Helminck (1996) was the description
of the image and fibers of the natural map \( \varphi_k : V_k \rightarrow I \), where \( V_k \) denotes the set of orbits of a minimal parabolic \( k \)-subgroup acting on the symmetric \( k \)-variety and \( I \) the set of twisted involutions in the Weyl group. Most of the combinatorial structure of the orbits in \( V_k \) depended only on properties of the associated twisted involutions in \( \varphi_k(V_k) \subset I \). For arbitrary base fields we can define this map again and also in this case most of the combinatorial structure can be adapted from the twisted involutions to the orbits and their closures. As can be expected there are a few additional difficulties in describing the twisted involutions contained in \( \varphi_k(V_k) \), but the main problems occur when one tries to compute the orbits in \( V_k \). For most base fields the fibers of \( \varphi_k \) are infinite, which makes it impossible to compute \( V_k \). For a number of base fields, including local fields and finite fields there are finitely many orbits and a computation of the orbits becomes feasible. The algorithm to compute the orbits in \( V_k \) necessarily also depends on a classification of the \( k \)-involutions involved. These have been fully classified for \( k = k \) and \( k = \mathbb{R} \). For these fields the algorithm will be able to compute the orbits. For \( k \) a \( p \)-adic field, a finite field or a number field, partial classifications of the \( k \)-involutions exist, which actually enable us to compute the twisted involutions and the corresponding combinatorial structure in these cases. For \( k = Q_p \) a classification of the \( k \)-involutions is nearly complete, which will enable us to compute the orbits in \( V_k \) in that case as well.

For \( k = \mathbb{R} \) one can derive a method to compute the orbits from each of the different characterizations of the orbits in Matsuki (1979), Rossmann (1979) and Helminck and Wang (1993), but not all of these methods are precise enough to lead to an efficient algorithm. The description of the orbit decomposition in Helminck and Wang (1993) contains more combinatorial data than the other characterizations of the orbits and consequently the algorithm derived from this will be much more efficient. Another advantage is that the description of the orbits in Helminck and Wang (1993) holds not only for the real numbers but for general base fields of characteristic not 2. Consequently, at the same time we get an algorithm for orbits of minimal parabolic \( k \)-subgroups for general symmetric \( k \)-varieties. In this paper we will also derive a number of additional results about these orbits which are useful for the actual computation. Using these results we can modify the algorithm in Helminck (1996) to compute the orbits for a number of other base fields as well. This algorithm can be implemented again on a computer using existing symbolic manipulation programs or by writing an independent program. Probably the easiest would be to write an extension to the program \( \text{LiE} \). This would also give a relatively efficient implementation. An example of this is given in 8.11. To obtain a more efficient implementation one will need to write an independent program. Note that most of the remaining structure of a symmetric variety, like the restricted root system with multiplicities and signatures etc., will also follow from the above structure.

A brief outline of this paper is as follows. In Section 2 we introduce notation and review a few generalities about the orbits of a minimal parabolic \( k \)-subgroup \( P \) acting on the symmetric \( k \)-variety \( G_k/H_k \). We also discuss the action of the Weyl group \( W \) on these orbits. In Section 3 we discuss the connection of these orbits with twisted involutions in the Weyl group. In particular, we analyze the natural maps \( \varphi_k : V_k \rightarrow I \subset W \) and \( \varphi : V \rightarrow I \), where \( V_k \) is the set of orbits \( P_k \setminus G_k/H_k \), \( V \) the set of orbits \( P \setminus G/H \) and \( I \) the set of twisted involutions in the Weyl group \( W \). The image \( \varphi_k(V_k) \) is contained in \( \varphi(V) \) and the combinatorial structure from \( \varphi(V) \) carries over to \( \varphi_k(V_k) \). We will use these maps \( \varphi \) and \( \varphi_k \) to classify the sets of orbits \( V \) and \( V_k \) for a number of base fields. For \( k \) algebraically closed, the real numbers, the \( p \)-adic numbers, a finite field or a number field we will compute \( \varphi(V) \). For \( k = k \) and \( k = \mathbb{R} \) we will compute \( \varphi_k(V_k) \) and \( V_k \) as
well. Most of these results are from Helminck and Wang (1993) and Brion and Helminck (1999).

In Section 4 we discuss the \( \Gamma \)-, \( \theta \)- and \((\Gamma, \theta)\)-indices related to the \( k \)-structure and the \( k \)-involutions. In Section 5 we introduce the \( \theta \)-singular and \( (\theta, k) \)-singular involutions in the restricted Weyl group and prove a number of results about these. These involutions will give us a set of representatives for the \( W \)-orbits in \( \varphi(V) \) and \( \varphi_k(V_k) \). For this see Sections 6 and 8. In Section 6 we also look at the Weyl group \( W(A) \) of a \( \theta \)-stable maximal \( k \)-split torus \( A \) containing a maximal \((\theta, k)\)-split torus. In this case we can show that there is a bijective correspondence between the twisted \( W(A) \)-orbits in \( \varphi(V) \) and the \( W(A) \)-conjugacy classes of \( \theta \)-singular involutions in the Weyl group \( W(A) \).

In Section 7 we show how the algorithm in Helminck (1996) can be modified to compute \( \varphi(V) \) for \( k \) algebraically closed, the real numbers, a \( p \)-adic field, a finite field or a number field. This computation only depends on the \((\Gamma, \theta)\)-index of the \( k \)-involution.

Finally in Section 8 we show how the algorithm in Section 7 can be extended to compute \( \varphi_k(V_k) \) and \( V_k \) as well, provided one has a classification for the \( k \)-involutions for that field. This algorithm can be used to compute \( \varphi_k(V_k) \) and \( V_k \) for \( k = \mathbb{R} \). A discussion of this case is also included.

2. Preliminaries and Recollections

In this section we set the notations and discuss the relation between the orbits of minimal parabolic \( k \)-subgroup acting on a symmetric \( k \)-variety and the \( H_k \)-conjugacy classes of \( \theta \)-stable maximal \( k \)-split tori. For this we will rephrase the characterization of these orbits in Helminck and Wang (1993) and Brion and Helminck (1999) by giving another characterization of the orbits, which is geared more toward the conjugacy classes of \( \theta \)-stable maximal \( k \)-split tori. We will also prove a number of additional results. Our basic reference for reductive groups will be the papers of Borel and Tits (1965, 1972) and also the books of Borel (1991), Humphreys (1975) and Springer (1981). We shall follow their notations and terminology.

2.1. Notations

Given an algebraic group \( G \), the identity component is denoted by \( G^0 \). We use \( L(G) \) (resp. \( g \), the corresponding lower case German letter) for the Lie algebra of \( G \). If \( H \) is a subset of \( G \), then we write \( N_G(H) \) (resp. \( Z_G(H) \)) for the normalizer (resp. centralizer) of \( H \) in \( G \). We write \( Z(G) \) for the center of \( G \). The commutator subgroup of \( G \) is denoted by \( D(G) \) or \([G, G]\).

An algebraic group defined over \( k \) shall also be called an algebraic \( k \)-group. For an extension \( K \) of \( k \), the set of \( K \)-rational points of \( G \) is denoted by \( G_K \) or \( G(K) \).

If \( G \) is a reductive \( k \)-group and \( A \) a torus of \( G \) then we denote by \( X^*(A) \) (resp. \( X_*(A) \)) the group of characters of \( A \) (resp. one-parameter subgroups of \( A \)) and by \( \Phi(A) = \Phi(G, A) \) the set of the roots of \( A \) in \( G \). The group \( X^*(A) \) can be put in duality with \( X_*(A) \) by a pairing \( \langle \cdot, \cdot \rangle \) defined as follows: if \( \chi \in X^*(A) \), \( \lambda \in X_*(A) \), then \( \chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle} \) for all \( t \in k^* \). Let \( W(A) = W(G, A) = N_G(A)/Z_G(A) \) denote the Weyl group of \( G \) relative to \( A \). If \( \alpha \in \Phi(G, A) \), then let \( U_\alpha \) denote the unipotent subgroup of \( G \) corresponding to \( \alpha \). If \( A \) is a maximal torus, then \( U_\alpha \) is one-dimensional. Given a quasi-closed subset \( \psi \) of \( \Phi(G, A) \), the group \( G^*_\psi \) (resp. \( G^{*\psi} \)) is defined in Borel and Tits (1965, 3.8). If \( G^*_\psi \) is unipotent, \( \psi \) is said to be unipotent and often one writes \( U_\psi \) for \( G^{*\psi} \).
2.2. Throughout the paper $G$ will denote a connected reductive algebraic $k$-group, $\theta$ an involution of $G$ defined over $k$, $G_\theta = \{ g \in G \mid \theta(g) = g \}$ the set of fixed points of $\theta$ and $H$ a $k$-open subgroup of $G_\theta$. The involution $\theta$ is also called a $k$-involution of $G$. The variety $G/H$ is called a symmetric variety and the variety $G_k/H_k$ is called a symmetric $k$-variety.

Given $g, x \in G$, the \textit{twisted action} associated with $\theta$ is given by $(g, x) \mapsto g \ast x = gx\theta(g)^{-1}$. Let $Q = \{ g^{-1}\theta(g) \mid g \in G \}$ and $Q' = \{ g \in G \mid \theta(g) = g^{-1} \}$. The set $Q$ is contained in $Q'$. Both $Q$ and $Q'$ are invariant under the twisted action associated to $\theta$. There are only a finite number of twisted $G$-orbits in $Q'$ and each such orbit is closed (see Richardson, 1982). In particular, $Q$ is a connected closed $k$-subvariety of $G$. Define a morphism $\tau : G \to G$ by

$$\tau(x) = x\theta(x^{-1}), \quad (x \in G).$$

(1)

The image $\tau(G) = Q$ is a closed $k$-subvariety of $G$ and $\tau$ induces an isomorphism of the coset space $G/G_\theta$ onto $\tau(G)$. Note that $\tau(x) = \tau(y)$ if and only if $y^{-1}x \in G_\theta$ and $\theta(\tau(x)) = \tau(x)^{-1}$ for $x \in G$.

2.3. If $T \subset G$ is a torus and $\sigma \in \text{Aut}(G, T)$ an involution, then we write $T^+ = (T \cap G_\sigma)^0$ and $T^- = \{ x \in T \mid \sigma(x) = x^{-1} \}^0$. It is easy to verify that the product map

$$\mu : T^+_\sigma \times T^-_\sigma \to T, \quad \mu(t_1, t_2) = t_1 t_2,$$

is a separable isogeny. In particular $T = T^+_\sigma T^-_\sigma$ and $T^+_\sigma \cap T^-_\sigma$ is a finite group. (In fact it is an elementary Abelian 2-group.) The automorphisms of $\Phi(G, T)$ and $W(G, T)$ induced by $\sigma$ will also be denoted by $\sigma$. If $\sigma = \theta$ we reserve the notation $T^+$ and $T^-$ for $T^+_\theta$ and $T^-_\theta$ respectively. For other involutions of $T$, we shall keep the subscript.

Recall from Helminck (1988) that a torus $A$ is called $\theta$-split if $\theta(a) = a^{-1}$ for every $a \in A$. If $A$ is a maximal $\theta$-split torus of $G$, then $\Phi(G, A)$ is a root system with Weyl group $W(A) = N_G(A)/Z_G(A)$ (see Richardson, 1982). This is the root system associated with the symmetric variety $G/H$. To the symmetric $k$-variety $G_k/H_k$ one can also associate a natural root system. To see this we consider the following tori.

\textbf{Definition 2.4.} A $k$-torus $A$ of $G$ is called $(\theta, k)$-split if it is both $\theta$-split and $k$-split.

Consider a maximal $(\theta, k)$-split torus $A$ in $G$. In Helminck and Wang (1993, 5.9) it was shown that $\Phi(G, A)$ is a root system and $N_{G_k}(A)/Z_{G_k}(A)$ is the Weyl group of this root system. We can also obtain this root system by restricting the root system of $G_k$. Namely let $A^0 \supset A$ be a $\theta$-stable maximal $k$-split torus of $G$. Then $A = (A^0)_\theta$ and $\Phi(G, A)$ can be identified with $\Phi_\theta = \{ \alpha | A \neq 0 \mid \alpha \in \Phi(G, A^0) \}$.

2.5. $P_k$-orbits on $G_k/H_k$

Let $P$ be a minimal parabolic $k$-subgroup of $G$. There are several ways in which one can characterize the double cosets $P_k \backslash G_k/H_k$. One can characterize them as the $P_k$-orbits on the symmetric $k$-variety $G_k/H_k$ (using the $\theta$-twisted action), one can take the $H_k$-orbits on the flag variety $G_k/P_k$ or one can consider the $P_k \times H_k$-orbits on $G_k$. All these characterizations are essentially the same. For more details see Helminck and Wang (1993). We will use the $P_k \times H_k$-orbits on $G_k$ to characterize $P_k \backslash G_k/H_k$. In this subsection we will briefly review this characterization.
Let \( A \) be a \( \theta \)-stable maximal \( k \)-split torus of \( P \), \( N = N_G(A) \), \( Z = Z_G(A) \) and \( W = W(A) = N_G(A)/Z_G(A) \) the corresponding Weyl group. As in Helminck and Wang (1993, 6.7) set \( V_k = \{ x \in G_k \mid \tau(x) \in N_k \} \). The group \( Z_k \times H_k \) acts on \( V_k \) by \((x, z) \cdot y = xyz^{-1}, (x, z) \in Z_k \times H_k, y \in V_k \). Let \( V_k \) be the set of \((Z_k \times H_k)\)-orbits on \( V_k \). Similarly let \( V = \{ x \in G \mid \tau(x) \in N \} \). Then \( Z \times H \) acts on \( V \) by \((x, z) \cdot y = xyz^{-1}, (x, z) \in Z \times H, y \in V \). Denote the set of \((Z \times H)\)-orbits on \( V \) by \( V \). If \( v \in V_k \) (resp. \( V \)), we let \( x(v) \in V_k \) (resp. \( V \)) be a representative of the orbit \( v \) in \( V_k \) (resp. \( V \)). These sets \( V_k \) (resp. \( V \)) are essential in the study of orbits of minimal parabolic subgroups on the symmetric \( k \)-variety \( G_k/H_k \). The inclusion map \( V_k \to G_k \) induces a bijection of the set \( V_k \) of \((Z_k \times H_k)\)-orbits on \( V_k \) onto the set of \((P_k \times H_k)\)-orbits on \( G_k \) (see Helminck and Wang, 1993). Similarly the set \( V \) is associated with the set of \( H \times P \) orbits on \( G \) (see Brion and Helminck, 1999). The set \( V \) is finite, but the set \( V_k \) is in general infinite. In a number of cases one can show that there are only finitely many \((P_k \times H_k)\)-orbits on \( G_k \). If \( k \) is algebraically closed, the finiteness of \( V_k \) was proved by Springer (1984). The finiteness of the orbit decomposition for \( k = \mathbb{R} \) was discussed by Wolf (1974), Rossmann (1979) and Matsuki (1979). For general local fields this result can be found in Helminck and Wang (1993). An example that in most cases the set \( V_k \) is infinite can be found in Helminck and Wang (1993, example 6.12).

2.6. \( W \)-action on \( V \) and \( V_k \)

The Weyl group \( W \) acts on both \( V \) and \( V_k \). The action of \( W \) on \( V \) (resp. \( V_k \)) is defined as follows. Let \( v \in V \) and let \( x = x(v) \). If \( n \in N \), then \( nx \in V \) and its image in \( V \) depends only on the image of \( n \) in \( W \). We thus obtain a (left) action of \( W \) on \( V \), denoted by \((w, v) \to w \cdot v \).\( \{w \in W, v \in V \}\).

To get another description of \( V_k \) and \( V \) we first define the following.

**Definition 2.7.** A torus \( A \) of \( G \) is called a quasi-\( k \)-split torus if \( A \) is conjugate under \( G \) with a \( k \)-split torus of \( G \).

Since all maximal \( k \)-split tori of \( G \) are conjugate, also all maximal quasi-\( k \)-split tori of \( G \) are conjugate. If \( A \) is a maximal quasi-\( k \)-split torus of \( G \), then \( \Phi(G, A) \) is a root system in \((X^*(A), X_0)\) in the sense of Borel and Tits (1965, Section 2.1) where \( X_0 \) is the set of characters of \( A \) which are trivial on \((A \cap [G, G])^0\); moreover the Weyl group of \( \Phi(G, A) \) is \( N_G(A)/Z_G(A) \).

2.8. Let \( A \) be the variety of maximal quasi-\( k \)-split tori of \( G \). This is an affine variety, isomorphic to \( G/N_G(A) \), on which \( \theta \) acts. Let \( A^0 \) be the fixed point set of \( \theta \), i.e. the set of \( \theta \)-stable maximal quasi-\( k \)-split tori. It is an affine variety on which \( H \) acts by conjugation.

Similarly let \( A_k \) denote the set of maximal \( k \)-split tori of \( G \) and let \( A^0_k \) be the fixed point set of \( \theta \), i.e. the set of \( \theta \)-stable maximal \( k \)-split tori. The group \( H_k \) acts on \( A^0_k \) by conjugation.

If \( x \in V_k \), then \( x^{-1}Ax \) is again a maximal \( k \)-split torus and conversely any \( \theta \)-stable maximal \( k \)-split torus in \( A^0_k \) can be written as \( x^{-1}Ax \) for some \( x \in V_k \). Similarly any \( \theta \)-stable maximal quasi-\( k \)-split torus is of the form \( x^{-1}Ax \) for some \( x \in V \).

If \( v \in V_k \) (resp. \( v \in V \)), then \( x(v)^{-1}Ax(v) \in A^0_k \) (resp. \( A^0 \)). This determines maps of \( V_k \), resp. \( V \) to the orbit sets \( A^0_k/H_k \) resp. \( A^0/H \). It is easy to check that these maps are independent of the choice of the representative \( x(v) \) for \( v \) and they are constant on
Proof. We explicit in the set of twisted involutions in $W$ that enables us to port the natural combinatorial structure on the image (the proposition below).

**Proposition 2.9.** Let $G$, $A^0$, $A^\theta$, $\gamma$ and $\gamma_k$ be as above. Then we have the following:

(i) $\gamma_k : V_k/W \to A^\theta_k/H_k$ is bijective.
(ii) $\gamma : V/W \to A^\theta/H$ is bijective.

**Proof.** (i). Since any $\theta$-stable maximal $k$-split torus is of the form $x^{-1}Ax$ for some $x \in V_k$, the map $\gamma_k$ is surjective.

Let $v_1, v_2 \in V_k$ with $\gamma_k(v_1) = \gamma_k(v_2)$. Let $x_1 = x(v_1), x_2 = x(v_2) \in V_k$ be representatives of $v_1, v_2$ and let $A_1 = x_1^{-1}Ax_1, A_2 = x_2^{-1}Ax_2$. Since $\gamma_k(v_1) = \gamma_k(v_2)$ there exists $h \in H_k$ such that $hA_2h^{-1} = hA_2h^{-1} = x_1^{-1}Ax_1 = A_1$. So we may assume that $A_1 = A_2$. However, then by Helminck and Wang (1993, 6.10) there exists $n \in N_G(A_1)$ such that $P_knx_1h = P_kx_2h$. If $w \in W(A_1)$ is the image of $n$ in $W(A_1)$, then $w \cdot v_1 = v_2$, which proves (i).

The proof of (ii) follows using a similar argument as in (i). $\square$

2.10. Let $A^\theta_0$ denote the set of $\theta$-stable quasi-$k$-split tori of $G$, which are $H$-conjugate to a $\theta$-stable maximal $k$-split torus. Then $A^\theta_0/H \subset A^\theta/H$ can be identified with the set of $H$-conjugacy classes of $\theta$-stable maximal $k$-split tori of $G$. There is a natural map

$$\zeta : A^\theta_0/H_k \to A^\theta/H,$$

sending the $H_k$-conjugacy class of a $\theta$-stable maximal $k$-split torus onto its $H$-conjugacy class. Then $A^\theta_0/H$ is precisely the image of $\zeta$. On the other hand the inclusion map $V_k \to V$ induces a map $\eta : V_k \to V$, where $\eta$ maps the orbit $Z_kH_k$ onto $Z_\theta H$. This map is $W$-equivariant. Denote the corresponding orbit map by $\delta : V_k/W \to V/W$ and write $V_0$ for the image of $\eta$ in $V$. Denote the restriction of $\gamma$ to $V_0/W$ by $\gamma_0$. Then $\gamma_0$ maps $V_0/W$ onto $A^\theta_0/H$. This all leads to the following diagram:

$$\begin{array}{ccc}
V_k/W & \xrightarrow{\gamma_k} & A^\theta_k/H_k \\
\downarrow{\delta} & & \downarrow{\zeta} \\
V_0/W & \xrightarrow{\gamma_0} & A^\theta_0/H
\end{array}$$

Since $\gamma_0$ and $\gamma_k$ are bijections, it follows that there is a bijection between the fibers of $\delta$ and $\zeta$. For a $\theta$-stable maximal $k$-split torus $A$, the fiber $\zeta^{-1}(A)$ consists of all $\theta$-stable maximal $k$-split tori, which are $H$-conjugate to $A$, but not $H_k$-conjugate. Evidence that these fibers can be infinite can be seen from the example given in Helminck and Wang (1993, 6.12).

### 3. Twisted Involutions

Throughout this section let $A$ be a fixed $\theta$-stable maximal $k$-split torus and let $W = W(A)$ be the Weyl group of $A$ with respect to $G$.

Another way to characterize the $W$-orbits in $V_k$ and $V$ is by characterizing the image and fibers of the natural map from $V$ into $W$, induced by the map $\tau|V : V \to N_G(A)$. This map also enables us to port the natural combinatorial structure on the image (containing the in the set of twisted involutions in $W$) to $V$ and $V_k$. Similar to the case of orbits of a
Borel subgroup acting on a symmetric variety as in Helminck (1996) and Richardson and Springer (1990) this will enable us to describe most of the combinatorial structure involved.

3.1. Recall that an element \( a \in W \) is a **twisted involution** if \( \theta(a) = a^{-1} \) (see Springer, 1984, Section 3, or Helminck and Wang, 1993, Section 7). Let

\[
\mathcal{I} = \mathcal{I}_\theta = \mathcal{I}(W, \theta) = \{ w \in W \mid \theta(w) = w^{-1} \},
\]

be the set of twisted involutions in \( W \). If \( v \in V \), then \( \varphi(v) = \tau(x(v))Z_G(A) \in W \) is a twisted involution. The element \( \varphi(v) \in \mathcal{I} \) is independent of the choice of representative \( x(v) \in V \) for \( v \). So this defines a map \( \varphi : V \to \mathcal{I} \).

We can define a map \( \varphi_k : V_k \to \mathcal{I} \) in a similar manner. Namely if \( v \in V_k \), then let \( \varphi_k(v) = \tau(x(v))Z_G(A) \in W \). Again this is a twisted involution. From the above observations we get the following relation between \( \varphi_k \) and \( \varphi \):

\[
\varphi_k = \varphi \circ \eta.
\]

The maps \( \varphi \) (resp. \( \varphi_k \)) play an important role in the study of the Bruhat order on \( V \) (resp. \( V_k \)). For more details, see Richardson and Springer (1990) and Brion and Helminck (1999).

3.2. The Weyl group \( W \) also acts on \( \mathcal{I} \). This action comes from the **twisted action of \( W \)** on (the set) \( W \), which is defined as follows: if \( w, w_1 \in W \), then \( w \ast w_1 = w w_1 \theta(w)^{-1} \). If \( w_1 \in W \), then \( W \ast w_1 = \{ w \ast w_1 \mid w \in W \} \) is the **twisted \( W \)-orbit of \( w_1 \)**. Now \( \mathcal{I} \) is stable under the twisted action, so that we get a twisted action of \( W \) on \( \mathcal{I} \). The images of \( \varphi \) and \( \varphi_k \) in \( \mathcal{I} \) are unions of twisted \( W \)-orbits, as follows from the following result.

**Lemma 3.3.** Let \( w \in W \) and \( v \in V \) (resp. \( V_k \)). Then \( \varphi(w \ast v) = w \ast \varphi(v) \) (resp. \( \varphi_k(w \ast v) = w \ast \varphi_k(v) \)).

**Proof.** The proof is immediate from the above observations. \( \square \)

3.4. From this result it follows now that the maps \( \varphi : V \to \mathcal{I} \) (resp. \( \varphi_k : V_k \to \mathcal{I} \)) are equivariant with respect to the action of \( W \) on \( V \) (resp. \( V_k \)) and the twisted action of \( W \) on \( \mathcal{I} \). So there are natural orbit maps \( \phi : V/W \to \mathcal{I}/W \) and \( \phi_k : V_k/W \to \mathcal{I}/W \). From (4) and Subsection 2.10 we get the following relation between \( \phi \) and \( \phi_k \).

\[
\phi_k = \phi \circ \delta.
\]

Since \( \gamma \) and \( \gamma_k \) are one-to-one, we also get embeddings of \( A_0^k/H \) and \( A_0^k/H_k \) into \( \mathcal{I}/W \). This indicates that the \( W \)-orbits of twisted involutions can be used as an invariant to characterize the conjugacy classes in \( A_0^k/H \) and \( A_0^k/H_k \). In fact, in Proposition 6.11 we will show that we can use conjugacy classes of involutions in the Weyl group \( W \), instead of \( W \)-orbits of twisted involutions.

**Remark 3.5.** If \( k \) is algebraically closed, then it follows from Richardson and Springer (1990, 2.5) that the map \( \phi_k \) is one-to-one. So in this case the classes in \( \varphi_k(V_k)/W \) completely characterize the \( H_k \)-conjugacy classes of \( \theta \)-stable maximal \( k \)-split tori. The map \( \phi_k \) is also one-to-one in a number of other cases, like the standard pairs \( (G_k, H_k) \) for \( k = \mathbb{R} \). For a further discussion of this see Helminck (1997, 1999).
An example that the map $\phi_k$ is not always one-to-one can be found in Example 5.8.

3.6. In this paper we will need several properties of the twisted involutions. In the remainder of this section we will collect some of their properties from Helminck and Wang (1993) and Helminck (1996) and prove some additional results. A first description of these twisted involutions, in the case that $\theta(\Phi^+) = \Phi^+$, was given in Springer (1984). For $k = \overline{k}$ there exists a $\theta$-stable Borel subgroup (see Steinberg, 1968), so then this condition is satisfied. However, if $k \neq \overline{k}$ then $G$ does not necessarily have $\theta$-stable minimal parabolic $k$-subgroups. One can easily generalize the description in Springer (1984) and give a similar description of the twisted involutions, when $\theta(\Phi^+) \neq \Phi^+$. This follows essentially from the results in Helminck and Wang (1993), although some of these results are not explicitly stated there. In the following we will review this characterization and prove a few additional results. First we need a few facts about real, complex and imaginary roots.

3.7. In the remainder of this section, let $A$ be a $\theta$-stable maximal quasi-$k$-split torus of $G$, $\Phi = \Phi(A)$ the root system of $A$ with respect to $G$, $\Phi^+$ a set of positive roots of $\Phi$, $\Delta$ the corresponding basis, $W = W(A)$ the Weyl group of $A$ and $\Sigma = \{s_\alpha \mid \alpha \in \Delta\}$. The Weyl group $W$ is generated by $\Sigma$. Let $E = X^*_A(\mathbb{Z}) \otimes \mathbb{R}$. If $\sigma \in \text{Aut}(\Phi)$, then we denote the eigenspace of $\sigma$ for the eigenvalue $\xi$, by $E(\sigma, \xi)$. For a subset $\Pi$ of $\Delta$ denote the subset of $\Phi$ consisting of integral combinations of $\Pi$ by $\Phi^\Pi$. Then $\Phi^\Pi$ is a subsystem of $\Phi$ with Weyl group $W^\Pi$. Let $w_0^\Pi$ denote the longest element of $W^\Pi$ with respect to $\Pi$.

The involution $\theta$ of $G$ induces an automorphism of $W$, also denoted by $\theta$, given by

$$\theta(w) = \theta \circ w \circ \theta, \quad w \in W.$$  

If $s_\alpha$ is the reflection defined by $\alpha$, then $\theta(s_\alpha) = s_{\theta(\alpha)}$, $\alpha \in \Phi$.

3.8. The roots of $\Phi$ can be divided in three subsets, related to the action of $\theta$, as follows:

(a) $\theta(\alpha) \neq \pm \alpha$. Then $\alpha$ is called complex (relative to $\theta$).
(b) $\theta(\alpha) = -\alpha$. Then $\alpha$ is called real (relative to $\theta$).
(c) $\theta(\alpha) = \alpha$. Then $\alpha$ is called imaginary (relative to $\theta$).

These definitions carry over to the Weyl group $W$ and the set $V$. We define first real, complex and imaginary elements for the Weyl group, which then will imply similar definitions for $V$.

DEFINITION 3.9. Given $w \in \mathcal{I}_\theta$, an element $\alpha \in \Phi$ is called complex (resp. real, imaginary) relative to $w$ if $w\theta\alpha \neq \pm \alpha$ (resp. $w\theta\alpha = -\alpha$, $w\theta\alpha = \alpha$). We use the following notation:

$$C'(w, \theta) = \{\alpha \in \Phi^+ \mid -\alpha \neq w\theta\alpha < 0\}$$
$$C''(w, \theta) = \{\alpha \in \Phi^+ \mid \alpha \neq w\theta\alpha > 0\}$$
$$R(w, \theta) = \{\alpha \in \Phi^+ \mid -\alpha = w\theta\alpha\}$$
$$I(w, \theta) = \{\alpha \in \Phi^+ \mid \alpha = w\theta\alpha\}.$$  

We will omit $\theta$ from this notation if there is no ambiguity as to which involution we consider.
3.10. To get a similar description of the twisted involutions as in Springer (1984), we pass to another involution, which leaves $\Phi^+$ invariant. Let $w_0 \in W$ such that
\[ \theta(\Phi^+) = w_0(\Phi^+), \]
and let $\theta' = \theta w_0$. Instead of working with $\theta$ we can work again with $\theta'$ and $w'$. In a similar way as in Helminck (1996, 4.6–4.8) we get the following results for $w_0$, $\theta'$ and the sets of twisted involutions $I_\theta$ and $I_{\theta'}$.

**Proposition 3.11.** Let $\Phi$, $\Phi^+$, $\theta$, $w_0$ and $\theta'$ be as above. Then we have the following properties:

1. $w_0 \in I_\theta$.
2. $\theta'(\Phi^+) = \Phi^+$.
3. $\theta'$ is an involution of $\Phi$.
4. $I_{\theta'} = I_\theta \cdot w_0$.

For the real, complex and imaginary roots we again have the following lemma.

**Lemma 3.12.** If $w \in I_\theta$ and $w' = w w_0$, then $w' \theta' = w \theta$. In particular, we have
\[ I(w, \theta) = I(w', \theta'), \quad R(w, \theta) = R(w', \theta') \]
\[ C'(w, \theta) = C'(w', \theta'), \quad C''(w, \theta) = C''(w', \theta'). \]

We also have again the following characterization of twisted involutions.

**Proposition 3.13.** (Helminck and Wang, 1993, 7.9) If $w \in I_\theta$ and $w' = w w_0 \in I_{\theta'}$, then there exist $s_1, \ldots, s_h \in \Sigma$ and a $\theta'$-stable subset $\Pi$ of $\Delta$ satisfying the following conditions:

1. $w' = s_1 \ldots s_h w_0^0 \theta'(s_1) \ldots \theta'(s_1)$ and $l(w') = 2k + l(w_0^0)$.
2. $w_0^0 \theta' \alpha = -\alpha$, $\alpha \in \Phi^+_1$ (i.e. $\Phi^+_1 \subset R(w_0^0, \theta')$).

Moreover if $w' = t_1 \ldots t_m w_0^0 \theta'(t_m) \ldots \theta'(t_1)$, where $t_1, \ldots, t_m \in \Sigma$ and $\Lambda$ a $\theta'$-stable subset of $\Delta$ satisfying conditions (i) and (ii), then $m = h$, $s_1 \ldots s_h \Pi = t_1 \ldots t_h \Lambda$ and
\[ s_1 \ldots s_h \theta'(s_h) \ldots \theta'(s_1) = t_1 \ldots t_h \theta'(t_h) \ldots \theta'(t_1). \]

3.14. In the case of orbits of a Borel subgroup acting on a symmetric variety as in Helminck (1996) the involution $\theta'$ could be lifted to an involution of $G$ conjugate to $\theta$. In this case $\theta'$ can again be lifted to an involution of $G$ conjugate to $\theta$, but this involution is mostly not a $k$-involution. To solve this question we will have to look at the following class of parabolic subgroups of $G$.

**Definition 3.15.** A parabolic subgroup $P$ in $G$ is called a quasi parabolic $k$-subgroup if there exist $x \in V$ such that $x P x^{-1}$ is a parabolic $k$-subgroup.

Using a similar argument as in Helminck and Wang (1993, 2.4) one can show that every quasi parabolic $k$-subgroup of $G$ contains a $\theta$-stable maximal quasi-$k$-split torus of $G$. 

Let \( P \) be a quasi parabolic \( k \)-subgroup of \( G \), \( A \subset P \) a \( \theta \)-stable maximal quasi-\( k \)-split torus, \( W = W(A) \), \( \Phi = \Phi(A) \), and \( \mathcal{I} = \mathcal{I}_0 \) the set of twisted involutions in \( W \). Let \( w \in \mathcal{I} \) and \( \xi = w\theta \). Consequently \( \xi \) is an involution of \( \Phi \). In the following we show when \( \xi \) can be lifted to an involution of \( G \) and when that involution is conjugate to \( \theta \). Using a similar argument as in Helminck (1996, 5.1) we get the following results.

**Lemma 3.16.** Let \( w \in W \) and \( n \in N_G(A) \) a representative. Then \( \text{Int}(n)\theta \) is an involution of \( G \) if and only if \( \theta(n) = n^{-1}z \), with \( z \in Z(G) \).

**Lemma 3.17.** Let \( V, V, \varphi : V \to \mathcal{I} \) be as above. Let \( w \in \mathcal{I} \), \( n \in N_G(A) \) a representative of \( w \) and \( \xi = \text{Int}(n)\theta \). Assume that \( \theta(n) = n^{-1}z \), with \( z \in Z(G) \). Then \( \xi \) is conjugate to \( \theta \) if and only if \( n \in \tau(V)Z(G) \).

**Proof.** Assume first that \( x \in G \) such that \( \xi = \text{Int}(x)\theta \text{Int}(x^{-1}) \). Then \( \xi = \text{Int}(n)\theta = \text{Int}(x\theta(x)^{-1}) \), so \( n \in \tau(V)Z(G) \). Conversely if \( n \in \tau(V)Z(G) \), then let \( x \in V \) and \( z \in Z(G) \) such that \( n = x\theta(x)^{-1}z \). Then \( \xi = \text{Int}(n)\theta = \text{Int}(x)\theta \text{Int}(x^{-1}) \) is conjugate to \( \theta \). \( \square \)

Combining the above results we get the following lemma.

**Lemma 3.18.** Let \( V, V, \varphi : V \to \mathcal{I} \) be as above and let \( w \in \mathcal{I} \). Then the following are equivalent:

1. There exists a representative \( n \in N_G(A) \) for \( w \), such that \( \xi = \text{Int}(n)\theta \) is an involution of \( G \) conjugate to \( \theta \).
2. \( w \in \varphi(V) \subset \mathcal{I} \).

**Proposition 3.19.** Let \( P \) and \( A \) be as above. As in (5), let \( w_0 \in W = W(A) \) be such that \( \theta(\Phi^+) = w_0(\Phi^+) \) and let \( \theta' = \theta w_0 = w_0^{-1}\theta \). We can solve now the question of when the involution \( \theta' \) can be lifted to a conjugate of \( \theta \). By Lemma 3.18 it suffices to show that \( w_0 \in \varphi(V) \).

This is equivalent to the following proposition.

**Proposition 3.20.** Let \( V, V, \varphi : V \to \mathcal{I} \) and \( w_0 \) be as above. Then we have the following:

1. \( w_0 \in \varphi(V) \) if and only if \( G \) contains a \( \theta \)-stable quasi parabolic \( k \)-subgroup.
2. \( w_0 \in \varphi_k(V_k) \) if and only if \( G \) contains a \( \theta \)-stable parabolic \( k \)-subgroup.

**Proof.** Assume first \( w_0 \in \varphi(V) \). Let \( v_0 \in V \) be such that \( \varphi(v_0) = w_0 \) and let \( x_0 = x(v_0) \in V \) be a representative of \( v_0 \). Then \( \tau(x_0) \) is a representative of \( w_0 \) in \( N_G(A) \). Since \( \theta(\Phi^+) = w_0(\Phi^+) \), we have \( \theta(P) = \tau(x_0)P\tau(x_0)^{-1} \). But then \( P = \tau(x_0)^{-1}P\theta(x_0) \) is a \( \theta \)-stable quasi parabolic \( k \)-subgroup of \( G \).

Conversely, assume \( P_0 \subset G \) is a \( \theta \)-stable quasi parabolic \( k \)-subgroup. By Helminck (1997, 3.11) there exists \( x \in V \) such that \( P_0 = xPx^{-1} \). Since \( \theta(P_0) = P_0 \) it follows that

\[
\theta(P) = \theta(x)^{-1}xPx^{-1}\theta(x) = \tau(\theta(x)^{-1})P\tau(\theta(x)).
\]

Now \( \tau(\theta(x)^{-1}) \in N_G(A) \). Let \( w \in W \) be the corresponding Weyl group element. Then \( \theta(\Phi^+) = w(\Phi^+) \), so \( w = w_0 \in \varphi(V) \). This shows (3.20). The proof of (3.20) follows with a similar argument replacing \( V \) and \( V \) by \( V_k \) and \( V_k \). \( \square \)
Remark 3.21. From the work found in Helminck (1997, 3.11) there always exists a \( \theta \)-stable quasiparabolic \( k \)-subgroup, but \( \theta \)-stable minimal parabolic \( k \)-subgroups of \( G \) do not necessarily exist as can be seen from Example 5.8. See Helminck and Wang (1993, Section 3) for a discussion of \( \theta \)-stable parabolic \( k \)-subgroups.

If \( \xi \in \text{Aut}(G) \) with \( \xi(A) = A \), then by abuse of notation we will write \( \xi|\Phi \) for the action of \( \xi \) on \( \Phi \). Summarizing the above results we get now the following result.

Corollary 3.22. Let \( w_0, \theta' \) be as above. There exists a representative \( n \in N_G(A) \) of \( w_0 \), such that \( \xi = \text{Int}(n)\theta \) is an involution of \( G \) conjugate to \( \theta \) satisfying \( \xi|\Phi = \theta' \). The involution \( \theta' \in \text{Aut}(\Phi) \) can be lifted to a \( k \)-involution if and only if \( G \) has a \( \theta \)-stable parabolic \( k \)-subgroup.

3.23. Relation between \( \theta \) and \( \theta' \)

Now we have shown that \( \theta' \) can be lifted to \( G \) and that it is conjugate to \( \theta' \), we can show that corresponding orbit decompositions for \( V \) and \( \varphi(V) \) are again similar. This all can be seen as follows.

Let \( P, A, V, \mathcal{I} \) and \( \varphi : V \to \mathcal{I} \) be as above. Write \( \Phi = \Phi(A) \) and \( W = W(A) \). Take \( w_0 \in W \) such that \( \theta(\Phi^\vee) = w_0(\Phi^\vee) \), \( n_0 = x_0\theta(x_0)^{-1} \in N_G(A) \cap \tau(G) \) a representative of \( w_0^{-1}, \theta' = \text{Int}(n_0)\theta \) and \( H' = x_0Hx_0^{-1} \). Then \( H' \) is a closed reductive subgroup of \( G \) satisfying

\[
G_{\theta'}^0 \subset H' \subset G_{\theta'}.
\]

Denote the actions of \( \theta \) and \( \theta' \) on \( \Phi \) also by \( \theta \) and \( \theta' \). Then \( \theta' = \theta w_0 = w_0^{-1}\theta \). As for \( \theta \) let \( \tau' : G \to G \) be the map defined by \( \tau'(x) = x\theta'(x)^{-1} \), \( \mathcal{V}' = \{ x \in G \mid \tau'(x) \in N_G(A) \} \), \( \mathcal{V}' \) the set of \( (Z_G(A) \times H') \)-orbits in \( \mathcal{V} \), \( \mathcal{I}_{\theta'} \) the set of twisted involutions of \( W \) with respect to \( \theta \) and \( \varphi' : \mathcal{V}' \to \mathcal{I}_{\theta'} \) as in Subsection 3.1.

Let \( \iota_{w_0} : \mathcal{I} \to \mathcal{I}_{\theta'} \) be the right translation by \( w_0 \) and let \( \delta : V \to V' \) be the map induced by the map \( g \mapsto gw_0 \) from \( V \) to \( V' \). Then \( \varphi' \circ \delta = \iota_{w_0} \circ \varphi \). So we obtained the following relation between the sets \( V, \mathcal{V}, \mathcal{I} \) and \( \mathcal{I}_{\theta'} \).

Lemma 3.24. Let \( V, \mathcal{V}', \mathcal{I}, \mathcal{I}_{\theta'} \), \( x_0, n_0 \) and \( w_0 \) be as above. Then we have the following:

(i) \( \mathcal{V}' = V \cdot x_0^{-1} \) and \( \tau'(\mathcal{V}') = \tau(V) \cdot n_0^{-1} \).

(ii) \( \mathcal{I}_{\theta'} = \mathcal{I}_{\theta} \cdot w_0 \).

3.25. \( W \)-action on \( \mathcal{I}_{\theta'} \)

As in 3.2 there is also an action of the Weyl group \( W \) on \( \mathcal{I}_{\theta'} \). Namely, if \( w \in W \) and \( a' \in \mathcal{I}_{\theta'} \), then define an action \( w \cdot a' = wa'\theta'(w)^{-1} \). Since \( \theta' = \theta w_0 \) and \( a' = aw_0 \) for some \( a \in \mathcal{I}_{\theta} \), we get

\[
w \cdot a' = waw_0\theta'w^{-1}\theta' = waw_0w_0^{-1}\theta w^{-1}\theta w_0 = wa\theta w^{-1}\theta w_0 = (w \cdot a)w_0.
\]

This means that the isomorphism \( \iota_{w_0} \) is equivariant with respect to the actions of \( W \) on \( \mathcal{I} \) and \( \mathcal{I}_{\theta'} \). This leads to the following result.
Proposition 3.26. Let $V, V', \mathcal{I}, \mathcal{I}_\theta, n_0$ and $w_0$ be as above. Then we have the following:

(i) The map $\iota_{w_0} : \mathcal{I} \to \mathcal{I}_\theta$ induces an isomorphism between $\mathcal{I}/W$ and $\mathcal{I}_\theta/W$.

(ii) $\varphi'(V')/W \simeq \varphi(V)/W$.

Remark 3.27. In the case that $G$ contains a $\theta$-stable parabolic $k$-subgroup we can get the same results for $V_k$ and $\varphi_k(V_k)$.

The following result is useful in the study of twisted involutions and will be used in what follows.

Lemma 3.28. Let $a \in \mathcal{I}_\theta$, $a' = aw_0 \in \mathcal{I}_\theta$ and $w \in W$. Then $E(a'a', -1) = E(a\theta, -1)$ and $w(E(a\theta, -1)) = E((w * a)\theta, -1)$.

Proof. The first statement follows from Lemma 3.12 and the second statement is immediate from the definition of the twisted $W$-action on $\mathcal{I}_\theta$. □

4. $(\Gamma, \theta)$-index

The computation of $\varphi(V)$ and $V$ will depend on the $(\Gamma, \theta)$-index of the $k$-involution $\theta$. In this section we briefly review some results about these from Helminck (2000a).

4.1. Let $G$ be a reductive $k$-group and $\theta \in \text{Aut}(G)$ a $k$-involution. Let $A$ be a $\theta$-stable maximal $k$-split torus, $T \supset A$ a $\theta$-stable maximal $k$-torus of $G$, $X = X^*(T)$ and $\Phi = \Phi(T)$. There is a finite Galois extension $K/k$ such that $T$ splits over $K$. Denote the Galois group $\text{Gal}(K/k)$ of $K/k$ by $\Gamma$ and let $\Gamma_0 \subset \text{Aut}(X, \Phi)$ be the subgroup corresponding to the action of $\Gamma$ on $(X, \Phi)$. Similarly let $\Theta = \{1, -\theta\} \subset \text{Aut}(X, \Phi)$ be the subgroup generated by $\theta$. In the following $E \subset \text{Aut}(X, \Phi)$ will be one of $\Gamma_0$, $\Theta$ or $\Gamma_\theta$.

4.2. For $\sigma \in E$ and $\chi \in X$ we will also write $\chi^\sigma$ or $\sigma(\chi)$ for the element $\sigma \chi \in X$. Write $W = W(\Phi)$ for the Weyl group of $\Phi$. For any closed subsystem $\Phi_1$ of $\Phi$ let $W(\Phi_1)$ denote the finite group generated by the $s_\alpha$ for $\alpha \in \Phi_1$. Now define the following:

$$X_0 = X_0(E) = \left\{ \chi \in X \mid \sum_{\sigma \in E} \chi^\sigma = 0 \right\}. \tag{6}$$

This is a co-torsion-free submodule of $X$, invariant under the action of $E$. Let $\Phi_0 = \Phi \cap X_0$. This is a closed subsystem of $\Phi$ invariant under the action of $E$. Denote the Weyl group of $\Phi_0$ by $W_0$ and identify it with $W(\Phi_0)$. Put $W_0 = \{w \in W \mid w(X_0) = X_0\}$, $X_0 = X/X_0(\mathcal{E})$ and let $\pi$ be the natural projection from $X$ to $X_0$. If we take $S = \{t \in T \mid \chi(t) = e$ for all $\chi \in X_0\}$ and $Y = X^*(S)$, then $Y$ may be identified with $X_0 = X/X_0$. Let $\Phi_0 = \pi(\Phi - \Phi_0(E))$ denote the set of restricted roots of $\Phi$ relative to $E$.

Remark 4.3. In the case that $E = \Gamma$, then $X_0$ is the annihilator of a maximal $k$-split torus $T$ of $T$. Similarly in the case that $E = \Theta$, then $X_0$ is the annihilator of a $\theta$-split torus $T$ of $T$. In both these cases $\Phi_0$ is the root system of $\Phi(S)$ with Weyl group $W_0$.

An order on $(X, \Phi)$ related to the action of $E$ can be defined as follows.
Definition 4.4. A linear order $\succ$ on $X$ which satisfies

\[
\text{if } \chi \succ 0 \quad \text{and} \quad \chi \notin X_0, \quad \text{then} \quad \chi^\sigma \succ 0 \quad \text{for all } \sigma \in \mathcal{E},
\]

is called an $\mathcal{E}$-linear order. A fundamental system of $\Phi$ with respect to a $\mathcal{E}$-linear order is called a $\mathcal{E}$-fundamental system of $\Phi$.

A $\mathcal{E}$-linear order on $X$ induces linear orders on $Y = X/X_0$ and $X_0$, and conversely, given linear orders on $X_0$ and $Y$, these uniquely determine a $\mathcal{E}$-linear order on $X$, which induces the given linear orders (i.e. if $\chi \notin X_0$, then define $\chi \succ 0$ if and only if $\pi(\chi) \succ 0$).

4.5. Fix a $\mathcal{E}$-linear order $\succ$ on $X$, let $\Delta$ be a $\mathcal{E}$-fundamental system of $\Phi$ and let $\Delta_0$ be a fundamental system of $\Phi_0$ with respect to the induced order on $X_0$. Define $\Delta_{\mathcal{E}} = \pi(\Delta - \Delta_0)$. This is called a restricted fundamental system of $\Phi$ relative to $S$.

There is also a natural (Weyl) group associated with the set of restricted roots, which is related to $W^\mathcal{E}/W_0$. Since $W_0$ is a normal subgroup of $W^\mathcal{E}$, every $w \in W^\mathcal{E}$ induces an automorphism of $X_\mathcal{E} = X/X_0 = Y$. Denote the induced automorphism by $\pi(w)$. Then $\pi(w\chi) = \pi(w)\pi(\chi)$ ($\chi \in X$). Define $W_E = \{\pi(w) \mid w \in W^\mathcal{E}\}$. We call this the restricted Weyl group, with respect to the action of $\mathcal{E}$ on $X$. It is not necessarily a Weyl group in the sense of Bourbaki (1968, Chapter VI, No. 1). However, in the case that $S$ is a maximal $k$-split, $\theta$-split or $(\theta, k)$-split torus, then $\Phi_\mathcal{E}$ is actually a root system with Weyl group $W_E$.

4.6. There is a natural action of $\mathcal{E}$ on the set of $\mathcal{E}$-fundamental systems of $\Phi$, this set. If $\Delta$ is a $\mathcal{E}$-fundamental system of $\Phi$, and $\sigma \in \mathcal{E}$, then the $\mathcal{E}$-fundamental system $\Delta^\sigma = \{\alpha^\sigma \mid \alpha \in \Delta\}$ gives the same restricted basis as $\Delta$, i.e. $\Delta^\sigma = \Delta$. Let $w_\sigma \in W_0$ be the unique element $w_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma \Delta$ and define a new action of $\mathcal{E}$ on $(X, \Phi)$ as follows:

\[
\chi^{[\sigma]} = w_\sigma^{-1}\chi^\sigma, \quad \chi \in X, \quad \sigma \in \mathcal{E}.
\]

It is easily verified that $\chi \rightarrow \chi^{[\sigma]}$ is an automorphism of the triple $(X, \Phi, \Delta)$ and that $\chi^{[\sigma][\tau]} = \chi^{[\sigma \tau]}$ for all $\sigma, \tau \in \mathcal{E}$, $\chi \in X$. This action of $\mathcal{E}$ on $\Psi$ is essentially determined by $\Delta$, $\Delta_0$ and $[\sigma]$. Following (Tits, 1966) the quadruple $(X, \Delta, \Delta_0, [\sigma])$ is called an index of $\mathcal{E}$ or an $\mathcal{E}$-index. We will also use the name $\mathcal{E}$-diagram, following the notation in Satake (1971, 2.4).

4.7. As in Tits (1966) we make a diagrammatic representation of the index of $\mathcal{E}$ by coloring black those vertices of the ordinary Dynkin diagram of $\Phi$, which represent roots in $\Delta_0(\mathcal{E})$ and indicating the action of $[\sigma]$ on $\Delta$ by arrows.

In the cases of $\mathcal{E} = \Theta$ and $\mathcal{E} = \Gamma$ we get the well known $\theta$-index and $\Gamma$-index, which are essential in the respective classifications. For $k$-involutions we do not use the $\Gamma_\theta$-index, but we combine it with the $\Gamma$-index and $\theta$-index to add additional information. These indices are defined as follows.

4.8. $(\Gamma, \theta)$-order

Assume $A_0 \subset A$ is a maximal $(\theta, k)$-split torus of $G$, $S$ a maximal $\theta$-split torus of $G$ such that $A_0 \subset S \subset T$. By Helminck and Wang (1993, 4.5) tori $A_0$, $A$, $S$ and $T$...
exist. A linear order on $X = X^*(\Gamma)$ which is simultaneously a $\Gamma$, $\theta$- and $\Gamma_0$-order is called a $(\Gamma, \theta)$-order. A fundamental system of $\Phi$ with respect to a $(\Gamma, \theta)$-order is called a $(\Gamma, \theta)$-fundamental system of $\Phi$.

An $\Gamma_0$-order on $(X; \Phi)$ is a $(\Gamma, \theta)$-order if and only of the following condition is satisfied:

If $\Phi_1 \subset \Phi_0(\Gamma_0)$ is an irreducible component then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$.

**Remarks 4.9.** (1) A $(\Gamma, \theta)$-order, as above, is completely determined by the sextuple $(X, \Delta, \Delta_0(\Gamma), \Delta_0(\theta), [\sigma], \theta^*)$.

We will call this sextuple an index of $(\Gamma, \theta)$ or an $(\Gamma, \theta)$-index. This terminology again follows (Tits, 1966). We will also use the name $(\Gamma, \theta)$-diagram, following the notation in Satake (1971, 2.4).

(2) We can make a diagrammatic representation of the $(\Gamma, \theta)$-index by coloring black those vertices of the ordinary Dynkin diagram of $\Phi$, which represent roots in $\Delta_0(\Gamma, \theta)$ and giving the vertices of $\Delta_0(\Gamma) \cup \Delta_0(\theta)$ which are not in $\Delta_0(\Gamma) \cap \Delta_0(\theta)$ a label $k$ or $\theta$ if $\alpha \in \Delta_0(\Gamma) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ or $\alpha \in \Delta_0(\theta) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ respectively. The actions of $[\sigma]$ and $\theta^*$ are indicated by arrows.

(3) The above index of $(\Gamma, \theta)$ determines the indices of both $\Gamma$ and $\theta$ and vice versa.

We conclude this section with some additional results for the action of the involution on $(X, \Phi)$.

**4.10. $\theta$-orders on quasi-$k$-split tori**

For $\theta$-stable maximal quasi-$k$-split tori we will need both the action of $\theta$ and $-\theta$. The action of $\theta$ is as defined in Definition 4.4. The action of $-\theta$ can be defined in a similar manner. We will discuss this action in the following. We will use the same notation as in Subsection 4.2. In particular let $A$ be a $\theta$-stable maximal quasi-$k$-split torus of $G$ and write $X = X^*(A)$ and $\Phi = \Phi(A)$. Similarly to as in Subsection 4.2 let

$$X_0(\theta) = \{ \chi \in X \mid \theta(\chi) = \chi \}, \quad \Phi_0(\theta) = \Phi \cap X_0(\theta)$$

$$X_0(-\theta) = \{ \chi \in X \mid \theta(\chi) = -\chi \}, \quad \Phi_0(-\theta) = \Phi \cap X_0(-\theta).$$

Let $\pi$ be the natural projection from $X$ to $X/X_0(\theta)$ and $\pi^-$ the natural projection from $X$ to $X/X_0(-\theta)$. As in Helminck (1991) we define a $\theta$-order (resp. $-\theta$-order) on $\Phi$ by choosing orders on $X_0(\theta)$ and $X/X_0(\theta)$ (resp $X_0(-\theta)$ and $X/X_0(-\theta)$). To be more precise, we give the following definition.

**Definition 4.11.** Let $\succ$ be a linear order on $X$. The order $\succ$ is called a $\theta^+$-order if it has the following property:

$$\text{if } \chi \in X, \quad \chi \succ 0 \quad \text{and} \quad \chi \notin X_0(\theta), \quad \text{then } \theta(\chi) \prec 0. \quad (9)$$

The order $\succ$ is called a $\theta^-$-order if it has the following property:

$$\text{if } \chi \in X, \quad \chi \succ 0 \quad \text{and} \quad \chi \notin X_0(\theta), \quad \text{then } \theta(\chi) \succ 0. \quad (10)$$

Similarly to as in Helminck (1988), a $\theta^+$-order on $X$ will also be called a $\theta$-order on $X$. Note that a $\theta^-$-order on $X$ is a $\theta$-order on $X$ for the involution $-\theta$ of $(X, \Phi)$. 
A basis $\Delta$ of $\Phi$ with respect to a $\theta^+$-order (resp. $\theta^-$-order) on $X$ will be called a $\theta^+$-basis (resp. $\theta^-$-basis) of $\Phi$. If $\Delta$ is a basis of $\Phi$ with respect to a $\theta^+$-order on $X$, then we write $\Delta_0(\theta) = \Delta \cap \Phi_0(\theta)$ and $\Delta_\varphi = \pi(\Delta - \Delta_0(\theta))$. Similarly, if $\Delta$ is a basis of $\Phi$ with respect to a $\theta^-$-order on $X$, then we write $\Delta_0(-\theta) = \Delta \cap \Phi_0(-\theta)$ and $\Delta_{-\theta} = \pi(\Delta - \Delta_0(-\theta))$. Clearly $\Delta_0(\theta)$ (resp. $\Delta_0(-\theta)$) is a basis of $\Phi_0(\theta)$ (resp. $\Phi_0(-\theta)$). A similar property holds for $\Delta_\varphi$ and $\Delta_{-\theta}$ (see Helminck, 1988, 2.4).

4.12. A CHARACTERIZATION OF $\theta$ ON A $\theta$-BASIS OF $\Phi$

Let $\Delta_1$ be a $\theta$-basis of $\Phi$. As in Helminck (1988, 2.8) we can write $\theta = -\text{id} \theta_1^* w_0(\theta)$, where $w_0(\theta) \in W_0(\theta)$ is the longest element of $W_0(\theta)$ with respect to $\Delta_0(\theta)$, and $\theta_1^* \in \text{Aut}(X, \Phi, \Delta_1, \Delta_0(\theta)) = \{ \phi \in \text{Aut}(X, \Phi) \mid \phi(\Delta_1) = \Delta_1 \text{ and } \phi(\Delta_0(\theta)) = \Delta_0(\theta) \}$ with $(\theta_1^*)^2 = \text{id}$. For more details see Helminck (1988, Section 2). This is called a characterization of $\theta$ on its $(+1)$-eigenspace (because $W_0(\theta)$ is the Weyl group of $\Phi_0(\theta)$).

Similarly we get a characterization of $\theta$ on a $\theta^-$-basis of $\Phi$ as follows. Let $\Delta_2$ be a $\theta^-$-basis of $\Phi$. Then $\theta = \theta_2^* : w_0(-\theta)$, where $w_0(-\theta) \in W_0(-\theta)$ is the longest element of $W_0(-\theta)$ with respect to $\Delta_0(-\theta)$, and $\theta_2^* \in \text{Aut}(X, \Phi, \Delta_2, \Delta_0(-\theta)) = \{ \phi \in \text{Aut}(X, \Phi) \mid \phi(\Delta_2) = \Delta_2 \text{ and } \phi(\Delta_0(-\theta)) = \Delta_0(-\theta) \}$ with $(\theta_2^*)^2 = \text{id}$. This is called a characterization of $\theta$ on its $(-1)$-eigenspace.

5. $(\theta, k)$-singular Involutions

In this section we introduce two classes of involutions in $W$ which will be used in the characterization of the $W$-orbits in $\varphi(V)$ and $\varphi_k(V_k)$.

5.1. We use the notation of Section 2. In particular, let $G$ be a connected reductive $k$-group, $\theta$ an involution of $G$ defined over $k$ and $H$ a $k$-open subgroup of $G_\theta$.

For quasi-$k$-split tori we defined singular roots with respect to the involution (see 3.8). For $k$-split tori we have to combine this with the $k$-structure of the group itself. Before we can define this all we need a bit more notation. Let $A$ be a $\theta$-stable maximal $k$-split torus. For $\alpha \in \Phi(A)$ let $A_\alpha = \{ a \in A_1 \mid s_\alpha(a) = a \}^0$, $G_\alpha = Z_G(A_\alpha)$ and $\overline{G_\alpha} = \{ G_\alpha, G_\alpha \}$. If $\alpha$ is either real or imaginary, then $G_\alpha$ is $\theta$-stable. We first define $\theta$-singular roots:

**Definition 5.2.** Let $A$ be as above. A root $\alpha \in \Phi(A)$ with $\theta(\alpha) = \pm \alpha$ is called $\theta$-singular (resp. $\theta$-compact) if $\overline{G_\alpha} \not\subset H$ (resp. $\overline{G_\alpha} \subset H$).

The $\theta$-singular roots can be divided in those which are singular with respect to the $k$-structure and those which are not. These are now defined as follows.

**Definition 5.3.** A real $\theta$-singular root $\alpha \in \Phi(A)$ is called $(\theta, k)$-singular (resp. $\theta$-singular anisotropic) if $\overline{G_\alpha} \cap H$ is isotropic (resp. $\overline{G_\alpha} \cap H$ is anisotropic). Similarly a imaginary $\theta$-singular root is called $(\theta, k)$-singular (resp. $\theta$-singular anisotropic) if $\overline{G_\alpha}$ has a non-trivial $(\theta, k)$-split torus (resp. $\overline{G_\alpha}$ has no non-trivial $(\theta, k)$-split tori).

Similar as for $\theta$-stable maximal tori we have the following result (see Helminck, 1997).

**Proposition 5.4.** Let $A$ be a $\theta$-stable maximal $k$-split torus of $G$. Then we have the following:
Remark 5.5. In Helminck (1997, 3.8) it was shown that every real root of a \( G \)-split torus of \( H \) if and only if \( \Phi(A) \) has no \((\theta,k)\)-singular real roots.

(2) \( A^- \) is a maximal \((\theta,k)\)-split torus of \( G \) if and only if \( \Phi(A) \) has no \((\theta,k)\)-singular imaginary roots.

Proof. We will prove (1). The proof of (2) is similar.

Assume first that \( A^+ \) is a maximal \( k \)-split torus of \( H \). If \( \alpha \in \Phi(A) \) is a \((\theta,k)\)-singular real root, then \( G_{\alpha} \cap H \) contains a \( k \)-split torus, say \( S \). Since \( \alpha \) is real, \( A^+ \subset \ker(\alpha) \subset Z(G_{\alpha}) \). So \( S \cdot A^+ \) is a \( k \)-split torus of \( G_{\alpha} \cap H \), what contradicts the maximality of \( A^+ \).

Conversely assume \( \Phi(A) \) has no \((\theta,k)\)-singular real roots. By passing to \( Z_G(A^+) \), we can assume that \( A^+ = \{ e \} \). Since \( \Phi(A) \) has no \((\theta,k)\)-singular real roots, it follows that \( G_{\alpha} \cap H \) is anisotropic for all \( \alpha \in \Phi(A) \). However, since \( A \) is maximal \( k \)-split, we get that \( H \) is anisotropic and hence does not contain any \( k \)-split tori. \( \square \)

Remark 5.5. In Helminck (1997, 3.8) it was shown that every real root of a \( \theta \)-stable maximal \( k \)-split torus \( A \) is \( \theta \)-singular. If \( A^-_\theta \) is maximal \((\theta,k)\)-split then all \( \theta \)-singular are real and the set of \( \theta \)-singular roots in \( \Phi(A) \) is completely determined by the \((\Gamma,\theta)\)-index of the \( k \)-involution \( \theta \).

Unfortunately not all real roots are \((\theta,k)\)-singular. Whether a \( \theta \)-singular root is \((\theta,k)\)-singular or not depends on other properties of the \( k \)-involution, like the quadratic elements. We illustrate this in the following example.

Example 5.6. Let \( k = \mathbb{R} \). Assume \( G_{\mathbb{R}} = SU^*(4,\mathbb{R}) \) and \( \theta \) is of type \( AII \). Then the \((\Gamma,\theta)\)-index of \((G,\theta)\) is

\[
\begin{array}{c}
\bullet \\
1 \\
\circ \\
\end{array}
\]

There are two \( k \)-involutions related to this \((\Gamma,\theta)\)-index. Let \( \theta_1 \) be the standard involution of type \( A_2^1(II) \) and \( \theta_2 = \theta_1 \text{Int}(e_1) \) the involution of type \( A_2^1(II)(e_1) \). We use here the same notation as in Helminck (1988). See also Helminck (2000a).

In both these cases \( G \) has a \( \theta \)-split maximal \( k \)-split torus \( A \) and \( \Phi(A) = \Phi(A^-) \) is of type \( A_1 \). In the first case the Lie algebra of \((G_{\theta_1})_k \) is \( su(2) \) and \((G_{\theta_1})_k^0 \) is compact. So all roots of \( \Phi(A) \) are \( \theta \)-singular, but not \((\theta,k)\)-singular. In the second case the Lie algebra of \((G_{\theta_2})_k \) is \( su(1,1) \) and \((G_{\theta_2})_k^0 \) is isotropic. So all roots of \( \Phi(A) \) are \((\theta,k)\)-singular.

Remarks 5.7. (1) From Helminck (1997, 3.8) it follows that every real root is \( \theta \)-singular.

(2) In general the root system \( \Phi(A) \) of a \( \theta \)-stable maximal quasi-\( k \)-split torus \( A \) is reduced. It would be quite natural to expect that if \( \lambda, 2\lambda \in \Phi(A) \) and \( \lambda \) is \( \theta \)-singular, then also \( 2\lambda \) is \( \theta \)-singular. This is true for real roots, however for imaginary \( \theta \)-singular roots this is in general not true, as can be seen from the following example.

Example 5.8. Assume the \((\Gamma,\theta)\)-index of \((G,\theta)\) is

\[
\begin{array}{c}
1 \\
\circ \\
\bullet \\
\end{array}
\]

\[
\theta^* \& \Gamma
\]

Then \( G \) has a maximal \( k \)-split torus \( A \), which is also \( \theta \)-split. Let \( T \supset A \) be a \( \theta \)-stable
Let $\imath$ coincide with those defined in Definitions 5.2 and 5.3. Then we set $
abla_\imath$ if there exists a $\theta$-stable maximal quasi-$k$-split torus $A_1$ and the long root $\beta \in \Phi(A_1) \cap \Phi(T_1)$ is $\theta$-singular. On the other hand, since $\Phi(T_1)$ contains $\theta$-compact roots, there exists $w \in W(T_1)$ such that $w(\beta)$ is $\theta$-compact and consequently $w(\beta)$ is not $\theta$-singular. However, $\frac{1}{2}w(\beta)$ is still $\theta$-singular, since $G_{\frac{1}{2}w(\beta)} = G$. Note that $\mathcal{T}_{w(\beta)}$ is of type $A_1$. In this example $G$ has a no $\theta$-stable minimal parabolic $k$-subgroup, but $G$ has a $\theta$-stable minimal quasi parabolic $k$-subgroup.

5.9. Now we have defined $\theta$-singular and $(\theta, k)$-singular roots we can generalize these definitions to include involutions as well. For the remainder of this section let $A_0$ be a $\theta$-stable maximal $k$-split torus with $A_0^\theta$ a maximal $(\theta, k)$-split torus of $G$. Let $A$ be a $\theta$-stable maximal quasi-$k$-split torus with $A^- \subset A_0^\theta$ and $A^+ \supset A_0^\theta$. For $\alpha \in \Phi(A)$ let $A_\alpha = \{ a \in A_1 \mid s_\alpha(a) = a \}^0$, $G_\alpha = Z_G(A_\alpha)$ and $G_\alpha = [G_\alpha, G_\alpha]$. If $\alpha$ is either real or imaginary, then $G_\alpha$ is $\theta$-stable. Similarly if $w \in W(A)$ satisfies $w^2 = e$ and $w\theta = \theta w$, then we set $G_w = Z_G(A_w^\theta)$ and $\mathcal{T}_w = [G_w, G_w]$. Let $n$ be a preimage of $w$ in $N_G(A)$. Then $n \in Z_G(A_w^\theta)$ and $A_w^\theta \cap Z(G_w)$ is finite. As a consequence, $A_w^\theta$ is a $\theta$-stable maximal $k$-split torus of $[G_w, G_w]$.

Recall that a $k$-involution $\sigma$ of a connected reductive $k$-group $M$ is called $(\sigma, k)$-split if there exists a $\tau$-split maximal $k$-split torus of $M$.

In Definitions 5.2 and 5.3 we defined $\theta$-singular and $(\theta, k)$-singular roots. We can lift these definitions to involutions in the Weyl group by defining the following.

**Definition 5.10.** Let $A$ be a $\theta$-stable maximal quasi-$k$-split torus with $A^- \subset A_0^\theta$ and $A^+ \supset A_0^\theta$ and $w \in W(A)$. Then $w$ is called $\theta$-singular if

1. $w^2 = e$.
2. $\theta w = w\theta$.
3. The involution $\theta \mid [G_w, G_w]$ is $(\theta, k)$-split.
4. $[G_w, G_w] \cap H$ contains a maximal quasi-$k$-split torus of $[G_w, G_w]$.

An involution $w \in W(A)$ is called $(\theta, k)$-singular if it satisfies (1), (2), (3) and if

5. $[G_w, G_w] \cap H$ contains a maximal $k$-split torus of $[G_w, G_w]$.

A root $\alpha \in \Phi(A)$ is called $\theta$-singular (resp. $(\theta, k)$-singular) if the corresponding reflection $s_\alpha \in W(A)$ is $\theta$-singular (resp. $(\theta, k)$-singular).

**Remark 5.11.** Note that these definitions of $\theta$-singular (resp. $(\theta, k)$-singular) roots coincide with those defined in Definitions 5.2 and 5.3.

**Lemma 5.12.** Let $A$ be a $\theta$-stable maximal $k$-split torus with $A^-$ a maximal $(\theta, k)$-split torus of $G$ and $w \in W(A)$ a $(\theta, k)$-singular (resp. $\theta$-singular) involution. Then we have the following conditions:

1. $A^-_w = (A^-_w)^\sim$ is a $\theta$-split maximal $k$-split torus of $[G_w, G_w]$.
2. $A^+_w = A^+(A^-_w)^\sim$.
Hence (A) Theorem 5.13. Let \( \theta \) split torus of \( G \) case that \( w \). 

Corollary 5.14. Let the \( (\theta,k) \) classes of \( \theta \), \( \theta,k \). We show the assertion for the case that \( w \). 

Proof. Let \( \theta \), \( \theta \), \( \theta,k \). This result is immediate from Theorem 5.13. Hence \( (A^-)^w = A^- \).

(ii) is immediate from (i) and the observation that \( A^+_w = (A^+_w)^+(A^+_w)^- \).

The \( \theta \)-singular involutions in \( W(A_0) \) can be characterized as follows.

Theorem 5.13. Let \( A_0 \) be a \( \theta \)-stable maximal \( k \)-split torus of \( G \) with \( A^-_0 \) a maximal \( (\theta,k) \)-split torus of \( G \) and \( w \in W(A_0) \), \( w^2 = e \). Then the following are equivalent:

(i) \( w \) is \( \theta \)-singular
(ii) \( (A_0)_w^\theta \subset A_0^\theta \)

Proof. (ii) \( \Rightarrow \) (i). From \( (A_0)_w^\theta \subset A_0^\theta \) it follows that \( w \) and \( \theta \) commute. Since \( (A_0)_w^\theta \) is a \( \theta \)-split maximal \( k \)-split torus of \( [G_w,G_w] \) it suffices to show that \( [G_w,G_w] \cap H \) contains a maximal quasi-\( k \)-split torus of \( [G_w,G_w] \). However, this follows from Helminck (1997, 8.13).

(i) \( \Rightarrow \) (ii) follows from Lemma 5.12.

Corollary 5.14. Let \( A_0 \) be a \( \theta \)-stable maximal \( k \)-split torus of \( G \) with \( A^-_0 \) a maximal \( (\theta,k) \)-split torus of \( G \). Then we have the following:

(i) Every involution in \( W(A_0) \cap W(A^-_0) \) is \( \theta \)-singular.
(ii) \( \alpha \in \Phi(A_0) \) is \( \theta \)-singular if and only if \( \alpha \) is a real root.

This result is immediate from Theorem 5.13.

Remark 5.15. Let \( A \) be a \( \theta \)-stable maximal \( k \)-split torus of \( G \). Using a simple induction one can show that an involution \( w \in W(A) \) is \( \theta \)-singular if and only if there exists a set of strongly orthogonal \( \theta \)-singular roots \( \{\alpha_1, \ldots, \alpha_r\} \) of \( \Phi(A) \), such that \( w = s_{\alpha_1} \cdots s_{\alpha_r} \).

Similarly \( w \in W(A) \) is \( (\theta,k) \)-singular if and only if there exists a set of strongly orthogonal \( (\theta,k) \)-singular roots \( \{\alpha_1, \ldots, \alpha_r\} \) of \( \Phi(A) \), such that \( w = s_{\alpha_1} \cdots s_{\alpha_r} \).

6. Twisted \( W \)-orbits in \( I \)

In this section we establish the correspondence between \( W \)-twisted isomorphy classes of the involutions \( w_0^1 \in I \) and the \( W \)-conjugacy classes of the involutions \( w_0^1 \in I \).

6.1. Let \( A \) be a fixed \( \theta \)-stable maximal \( k \)-split torus and \( W = W(A) \) the Weyl group of \( A \) with respect to \( G \). The map \( \phi^{-1} : A^\theta/H \rightarrow \varphi(V)/W \subset I/W \), where \( \gamma \) is as in Proposition 2.9 and \( \phi \) as in Subsection 3.4, enables us to use results proved for conjugacy classes of \( \theta \)-stable quasi-\( k \)-split tori in Helminck (1997, 1999) to characterize \( \varphi(V) \). In the following we use this correspondence to prove that every \( W \)-orbit in \( \varphi(V) \) contains a \( \theta \)-singular involution.
6.2. We give first another description of the involutions $w_0^0$ in the characterization of the twisted involutions as in Proposition 3.13.

Let $A_0$ be a $\theta$-stable maximal $k$-split torus of $G$ with $A_0^-$ a maximal $(\theta,k)$-split torus and $A_1$ a $\theta$-stable maximal quasi-$k$-split torus of $G$. There exists $g \in G$ such that $A_1 = gA_0g^{-1}$. Let $w = \tau(g)Z_G(A_0) \in W(A_0)$. Fix a basis $\Delta_0$ of $\Phi_0 = \Phi_0(-\theta) = \{\alpha \in \Phi(A_0) | \theta(\alpha) = -\alpha\}$. We can extend $\Delta_0$ to a $\theta$-basis $\Delta$ of $\Phi(A_0)$. Let $w_0$ be the opposite involution of $W(\Phi_0)$ with respect to $\Delta_0$. Then by Subsection 4.12 it follows that $\theta = \theta^*w_0$. So $\theta' = \theta w_0 = \theta^*$ and hence $\theta'(\Delta) = \Delta$. Write $w' = w w_0 = s_{\alpha_1} \cdots s_{\alpha_k} w_0^0 \theta'(s_{\alpha_1}) \cdots \theta'(s_{\alpha_k})$ as in Proposition 3.13. Here $\Pi$ is a subset of $\Delta$ and $\Pi$ as in Proposition 6.3. We can extend $\alpha_1, \ldots, \alpha_k \in N_G(A_0)_{\tau}$ with images $s_1, \ldots, s_k$ in $W(A_0)$ respectively. Set $u = g n_1 \ldots n_k$, $m = \tau(u) = u^{-1} \theta(u)$, $w_1 = \varphi(u) \in W(A_0)$. Then $w_1 w_0 = w_0^0$. So assume $w' = w w_0 = w_0^0$.

**Proposition 6.3.** Let $g$, $w$, $w'$ and $w_0$ be as above. Then there exists $w_1 \in W(A_0)$ such that $w_1 w_0^0 \theta'(w_1)^{-1} = w_0^0 \Pi_1$, with $\Pi_1 \subset \Delta_0$ and $w_0^0 \Pi_1$ the longest involution of $\Phi_{\Pi_1}$ with respect to $\Pi_1$.

**Proof.** From the paper Helminck (1997, 5.8) there exists $h \in H^0$ such that $h A_1^\tau h^{-1} \subset A_0^\tau$ and $h A_1^\tau h^{-1} \subset A_0^\tau$. Let $A_2 = h A_1 h^{-1}$ and let $g_1 \in Z_G(A_2, A_0^\tau)$ such that $g_1 A_0 g_1^{-1} = A_2$. Then $h^{-1} g_1$ and $g$ both map $A_0$ to $A_1$, namely $h^{-1} g_1 A_0 g_1^{-1} = h A_0 g^{-1} = A_1$. So there exists $n \in N_G(A_0)$ such that $h^{-1} g_1 = g n$. Let $w_1 \in W(A_0)$ be the Weyl group element corresponding to $n$ and let $w_2 = \tau(h^{-1} g_1) Z_G(A_0) = \tau(g_1) Z_G(A_0) \in W(A_0)$. Then $w_2 = w_1^{-1} \theta(\varphi(w_1))$ and $(A_0)^{-1} w_2 \subset A_0^\tau$. Since $\theta' = \theta_0^{-1} \theta = \theta w_0$ and $w' = w_0^0$, we get

$$w_2 = w_1^{-1} \theta(\varphi(w_1)) = w_1^{-1} w w_0 \theta'(w_1 \theta = w_1^{-1} w' \theta' w_1 \theta w_0^{-1} = w_1^{-1} w_0^0 \theta' w_0^{-1}. \quad (11)$$

From the paper by Helminck (1991, 2.14) there exists $w_3 \in W(\Phi_0)$ such that $w_3 w_2 w_3^{-1} \theta w_0 = w_0^0 \Pi_1$ with $\Pi_1 \subset \Delta_0$. Together with (11) this gives:

$$w_0^0 \Pi_1 = w_3 w_2 w_3^{-1} \theta w_0 = w_3 w_1^{-1} w_0^0 \theta' w_1 \theta w_0^{-1} w_3^{-1} \theta w_0 = w_3 w_1^{-1} w_0^0 \theta' w_1 w_3^{-1} \theta w_0 = w_3 w_1^{-1} w_0^0 \theta' (w_1 w_3^{-1}).$$

This proves the result. \( \square \)

**Corollary 6.4.** Let $A_0$ be as above. Then we have the following:

1. Every $W(A_0)$-orbit in $\varphi(V)$ contains a $\theta$-singular involution.
2. Every $W(A_0)$-orbit in $\varphi(V)$ has a representative $w_0^0 \in W(A_0) \cap W(A_0^-)$.

**Proof.** These results are immediate from Proposition 6.3, using the isomorphism between $I_\theta/W(A_0)$ and $I_{\theta^*}/W(A_0)$ as in Proposition 3.26. \( \square \)

For $\varphi_k(V_k)$ we will prove in Theorem 8.5 a similar result using the $(\theta,k)$-singular involutions.

6.5. Since $w_0^0 \in W(A_0) \cap W(A_0^-)$ we can characterize $\varphi(V)/W(A_0)$ by looking at $W(A_0^-)$-conjugacy classes of the involution $w_0^0$. For this it will be useful to have another
characterization of $W(A_0)$ by linking it to the group $W(A_0, H)$. Let $X = X^*(A_0)$, $X_0(\theta)$ and $\Phi_0(\theta)$ be as in Subsection 4.2. Write

$$W_1(\theta) = \{ w \in W(A_0) \mid w(X_0(\theta)) \subseteq X_0(\theta) \},$$

and $W_0(\theta) = W_0(A_0, \theta) = W(\Phi_0(\theta))$. Then by Satake (1971, 2.1.3) we have

$$W(A_0) \simeq W_1(\theta)/W_0(\theta).$$

The group $W(A_0, H)$ corresponds with $W_1(\theta)$ due to the following result.

**Proposition 6.6.** Let $A_0 \in A_0^\theta$ be a $\theta$-stable maximal $k$-split torus of $G$ with $A_0^-$ a maximal $(\theta, k)$-split torus of $G$. Then we have the following:

(i) Any $w \in W(A_0^-)$ has a representative in $(H^0 Z_G(A_0))_k \cap N_G(A_0^-)$.

(ii) $N_G(A_0^-) = N_{H^0}(A_0^-) Z_G(A_0^-)$.

**Proof.** (i) Let $n \in N_G(A_0^-)$ and $P$ a minimal $\theta$-split parabolic $k$-subgroup of $G$. Then $P_1 = n P n^{-1}$ is also a minimal $\theta$-split parabolic $k$-subgroup of $G$ containing $A_0$. By Helminck and Wang (1993, 4.9) there exists $x \in (H^0 P)_k$ such that $x P x^{-1} = P_1$. Let $P_0$ be a minimal parabolic $k$-subgroup of $P$ containing $A_0$. By Helminck and Wang (1993, 4.8) $H^0 P_0 = H^0 P$. On the other hand $(H^0 P)_k = (H^0 Z_G(A_0))_k U_k$, where $U = R_u(P_0)$ (see Helminck and Wang, 1993, 10.2). It follows that $x = hzu$ with $h \in H^0$, $z \in Z_G(A_0)$ and $u \in U_k$. If we take $g = h z \in (H^0 Z_G(A_0))_k$, then $g P g^{-1} = P_1$ and $g A_0^- g^{-1}$ is $(\theta, k)$-split. Moreover $g A_0^- g^{-1} \subset P_1 \cap \theta(P_1) = Z_G(A_0^-)$, so $g A_0^- g^{-1} A_0^-$ is a $(\theta, k)$-split torus of $G$. Since $A_0$ is maximal $(\theta, k)$-split it follows that $g A_0^- g^{-1} = A_0^-$, which proves (i). Finally (ii) is immediate from (i). □

**Remark 6.7.** Ideally one would like $W(A_0^-)$ to have representatives in $H_k$, but as the proof of this result indicates, this will unfortunately not always be true. However, in many cases, including the standard pairs for $k = \mathbb{R}$ (for a definition see Helminck, 1988), one can show that $W(A_0^-)$ has representatives in $H_k$. It is an interesting open question to give necessary and sufficient conditions such that $W(A_0^-)$ has representatives in $H_k$.

**Corollary 6.8.** Let $A_0 \in A_0^\theta$ be a $\theta$-stable maximal $k$-split torus of $G$ with $A_0^-$ a maximal $(\theta, k)$-split torus of $G$. Then $W_0(A_0, H) = W_1(\theta)$.

**Proof.** Clearly $W_0(A_0, H) \subset W_1(\theta)$. As for the other inclusion, it suffices by Proposition 6.6 to show that $W_0(\theta) = W_0(A_0, \theta) \subset W_0(A_0, H)$. However, this follows from the fact that every root $\alpha \in \Phi_0(A_0, \theta)$ is compact and therefore $G_\alpha \subset H$. □

To prove the correspondence between $W_1(\theta)$-conjugacy classes of involutions in $W(A_0) \cap W(A_0^-)$ and the $W(A_0^-)$-conjugacy classes of involutions we need the following result from Helminck (1996). Write $E = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ and for $\sigma \in \text{Aut}(\Phi)$ denote the eigenspace of $\sigma$ for the eigenvalue $\xi$, by $E(\sigma, \xi)$. Let $\Phi = \Phi(A_0)$.

**Proposition 6.9.** (Helminck, 1996, 9.7) Let $\theta$ be an involution of $\Phi$ such that $\Phi_\theta$ is a root system with Weyl group $W_\theta$. If $w_1, w_2 \in W(A_0)$ are involutions with $E(w_i, -1) \subset E(\theta, -1)$ ($i = 1, 2$), then the following are equivalent:
(i) $w_1$ and $w_2$ are conjugate under $W(A_0)$.
(ii) $w_1$ and $w_2$ are conjugate under $W_1(\theta)$.
(iii) $w_1\theta$ and $w_2\theta$ are conjugate under $W(A_0)$.
(iv) $w_1\theta$ and $w_2\theta$ are conjugate under $W_1(\theta)$.

**Remark 6.10.** Note that the involutions $w^n_1$ satisfy the condition $E(w^n_1, -1) \subset E(\theta, -1)$. So this result reduces the characterization of the $W_1(\theta)$-conjugacy classes of these involutions to $W(A_0)$-conjugacy classes. A characterization of the latter can be found in Helminck (1991).

For the $\theta$-singular involutions in a twisted $W(A_0)$-orbit we can show now the following proposition.

**Proposition 6.11.** Let $w_1, w_2 \in \varphi(V)$ be $\theta$-singular involutions. Then $w_1$ and $w_2$ are in the same twisted $W(A_0)$-orbit if and only if $w_1$ and $w_2$ are $W(A_0)$-conjugate.

**Proof.** Assume first that $w_1$ and $w_2$ are $W(A_0)$-conjugate. Since $A_0^+$ is a maximal $(\theta, k)$-split torus, we have $(A_0)_{w_1} \subset A_0^+$ ($i = 1, 2$) and hence $w_1, w_2 \in W(A_0) \cap W(A_0^+)$. By Proposition 6.9 $w_1$ and $w_2$ are conjugate under $W_1(A_0)$. Let $w \in W_1(A_0)$ such that $ww_1w^{-1} = w_2$. Since $W_1(A_0) = W(A_0, H)$ (see Corollary 6.8) it follows that $\theta(w) = w$ and hence $ww_1\theta(w^{-1}) = w_1w^{-1} = w_2$.

For the converse statement assume $w \in W(A_0)$ such that $ww_1\theta(w^{-1}) = w_2$. Then by Lemma 3.28 we have $w((A_0^-)_{w_1}^+) = (A_0^-)_{w_2}$ and with a similar argument we also have $w((A_0^-)_{w_1}^-A_0^+) = (A_0^-)_{w_2}^-A_0^+$. Consequently $ww_1\theta w^{-1} = w_2\theta$. Then by Proposition 6.9 $w_1$ and $w_2$ are $W(A_0)$-conjugate, what proves the result. □

7. Computing $\varphi(V)$ and $V$

In this section we give an algorithm to compute $\varphi(V)$ and $V$. This algorithm is essentially a modification of the one used for computing orbits or Borel subgroups acting on symmetric varieties as in Helminck (1996).

7.1. Characterization of $w_0$

To compute $\varphi(V)$ or $\varphi'(V')$ we first need a characterization of the involution $w_0$ as in Subsection 3.10. For this we give a characterization of the open orbit which will lead to a description of $w_0$. We use the same notation as in Sections 2 and 3. In particular let $A$ be a $\theta$-stable maximal $k$-split torus of $G$, $P \supset A$ a minimal parabolic $k$-subgroup, $\Phi = \Phi(A)$, $\Phi^+ = \Phi(P, A)$ the set of positive roots of $\Phi$ related to $P$, $\Delta$ the corresponding basis of $\Phi$, $w_0$ the element in the Weyl group $W = W(A) = N_G(A)/Z_G(A)$ with $w_0(\Phi^+) = \theta(\Phi^+)$ and $\theta' = \theta w_0$. If $\Pi \subset \Delta$, then we write $\Psi_{\Pi}$ for the subsystem of $\Phi(G, A)$ consisting of integral combinations of $\Pi$ and we write $P_{\Pi}$ for the standard parabolic $k$-subgroup of $G$ containing $P$ with $\Phi(P_{\Pi}, A) = \Phi_{\Pi} \cup \Phi^+$. The following result from Helminck and Wang (1993, Proposition 9.2) characterizes the open orbit.

**Proposition 7.2.** Let $v \in V$, $n = x(v)\theta(x(v))^{-1}$, $w$ the image of $n$ in $W$ and $w' = w w_0$. Let $\zeta$ be the involution of $G$ corresponding to $w'\theta' = w\theta$ (i.e. $\zeta$ is given by $\zeta(x) = n\theta(x)n^{-1}$, $x \in G$). The following conditions are equivalent:
(i) $P \ast n$ is open in $Q = \{x\theta(x)^{-1} \mid x \in G\}$.
(ii) Let $\Pi = I(w) \cap \Delta$. Then $C''(w) \cap \Delta = \emptyset$ and $\zeta$ is trivial on $G_{\Phi_n}$.
(iii) $w' = w_1^0g_\Delta^0$ and $\zeta$ is trivial on $G_{\Phi_n}$.
(iv) $x(v)^{-1}P_1x(v)$ is a minimal $\theta$-split parabolic $k$-subgroup of $G$.
(v) There exists a minimal $\theta$-split parabolic $k$-subgroup of $G$ containing $x(v)^{-1}P_x(v)$.

Combining this result with Corollary 3.22 we get the following characterization of $w_0$ and $\theta'$.

**Corollary 7.3.** Let $A$ be a $\theta$-stable maximal $k$-split torus of $G$ such that $A^{-}$ is a maximal $(\theta,k)$-split torus and $P \supset A$ a minimal parabolic $k$-subgroup of $G$ such that $PH \subset G$ is open. Let $w_0 \in W$ satisfy $\theta(\Phi^+) = w_0(\Phi^+)$, and take $v_0 \in V$ such that if $x_0 = x(v_0)$ and $n_0 = x_0\theta(x_0)^{-1} \in N_G(A)$ then $n_0$ induces $w_0$ in $W$; let $\zeta$ be the involution of $G$ given by $\zeta(x) = n_0^{-1}\theta(x)n_0$ for $x \in G$. Then we have the following.

(1) $w_0 = w_0^0\Delta_0(\theta)w_\Delta^0$, where $\Delta_0(\theta) = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha\}$ is as in Subsection 4.10.
(2) $\zeta|\Phi = w_0^{-1}\theta = \theta w_0 = \theta'$.

**Remark 7.4.** The element $w_0^0\Delta_0(\theta) \in W$ follows from the $(\Gamma,\theta)$-index of the $k$-involution $\theta$. For $k$ the real numbers, $p$-adic numbers, finite field or numbers field the $(\Gamma,\theta)$-indices of $k$-involutions are given in Helminck (2000a) (see also Helminck, 1994). For $k$ algebraically closed the $(\Gamma,\theta)$-indices are given in Helminck (1998). The element $w_0^0 \in W$ follows from the classification of involutions in Helminck (1991). Combining these two classifications and we get a list of the elements $w_0 \in W$. From the classification of $(\Gamma,\theta)$-indices in Helminck (2000a) we also get a list of the involutions $\theta'$ for each of the irreducible $(\Gamma,\theta)$-indices. Note that in each case $\theta' = id$ or a diagram automorphism.

### 7.5. Image and Fibers of $\varphi$

In a similar way as in Helminck (1996) the classification of the image and fibers of $\varphi : V \to \mathcal{I}$ (or $\varphi' : V' \to \mathcal{I}'$) can be reduced to a problem related to the involutions $w_0^0$ as in the characterization of the twisted involutions in Proposition 3.13. The results about these involutions as proved in Helminck (1996, Section 9) carry over to the present situation. Let $\Delta$ be a basis of $\Phi$, $w_0 \in W = W(A)$ be such that $\theta(\Phi^+) = w_0(\Phi^+)$, $\theta' = \theta w_0 = w_0^{-1}\theta$ and let $\mathcal{I}_\theta$ be as in Subsection 3.23. Write

$$
\Lambda_\Delta = \{\Pi \subset \Delta \mid \theta'(\Pi) = \Pi \quad \text{and} \quad w_0^0\theta'(\alpha) = -\alpha, \quad \forall \alpha \in \Phi_\Pi\},
$$

$$
\mathcal{I}_\Delta = \{w_0^0 \mid \Pi \in \Lambda_\Delta\}. \quad (14)
$$

The set $\mathcal{I}_\Delta$ contains a set of representatives of $\mathcal{I}_\theta/W$ and also of $\varphi'(V'/W)$. Since by Proposition 3.26 $\varphi'(V'/W) \simeq \varphi(V)/W$ we have the following result.

**Lemma 7.6.** Let $W$ act on $\mathcal{I}$, $\mathcal{I}_\theta$, $V$ and $V'$ as in Subsections 3.23 and 3.25. Let $\Delta$ be a basis of $\Phi$ and let $\mathcal{I}_\Delta$ be as in (14). Then we have the following:

(i) Each orbit in $\mathcal{I}_\theta/W$ and $\varphi'(V'/W)$ has a representative in $\mathcal{I}_{\Delta}w_0^0^{-1}$.
(ii) Each orbit in $\mathcal{I}/W$ and $\varphi(V)/W$ has a representative in $\mathcal{I}_{\Delta}w_0^0^{-1}$. 

7.7. Classification of $\varphi(V)/W$ and $\varphi'(V')/W$

We need to classify the involutions in $I_{\Delta}$ which represent the different classes in $I_{\varphi}/W$ resp. $\varphi'(V')/W$. By Propositions 6.9 and 6.11 it suffices to look at the $W$-conjugacy classes of the involutions $w_{\Pi}^0$. A classification of conjugacy classes of involutions in the Weyl group is given in Helminck (1991). Similar as in Helminck (1996) let

$$I_{\Delta}^0 = \{w_{\Pi_1}^0, \ldots, w_{\Pi_k}^0\} \subset I_{\Delta},$$

be a set of representatives of $I_{\varphi}/W$ and let

$$\Lambda_{\Delta}^0 = \{\Pi_1, \ldots, \Pi_k\} \subset \Lambda_{\Delta},$$

be the corresponding subset of $\Lambda_{\Delta}$. Similarly let

$$I_{\Delta}(V') = I_{\Delta}^0 \cap \varphi'(V'),$$

and

$$I_{\Delta}(V) = I_{\Delta}^0 \cdot w_{0}^{-1} \cap \varphi(V).$$

Then $I_{\Delta}(V')$ is a set of representatives of $\varphi'(V')/W$ and similarly $I_{\Delta}(V)$ is a set of representatives of $\varphi(V)/W$. The sets $I_{\Delta}(V')$ and $I_{\Delta}(V')$ are related as follows:

$$I_{\Delta}(V') = I_{\Delta}(V) \cdot w_0^{-1}.$$

7.8. Next we show that the orbits in $\varphi'(V')$ (resp. $\varphi(V)$) under the twisted action of $W$ correspond with the conjugation classes of the involutions $w_{\Pi}^0 \in I_{\Delta}(V')$ (resp. $I_{\Delta}(V)$). For this we need to first choose a suitable maximal torus to characterize the sets $I_{\Delta}(V')$ and $I_{\Delta}(V)$. In the following let $A$ be a $\theta$-stable maximal $k$-split torus with $A^-$ a maximal $(\theta, k)$-split torus of $G$. Let $\Theta = \Theta_{\theta}$ be a basis of $\Phi_{\theta}$ and extend this to a $\theta^*$-basis $\Delta$ of $\Phi$. Then by Subsection 4.12 we have $\theta^* = \theta \Theta_{\theta}$. As in Helminck (1991, 7.4) we call an involution $w_{\Pi}^0$ a $\Delta_0$-standard involution if $\Pi \subset \Delta_0$. We can choose now $\Delta_0$-standard involutions as representatives for both $I_{\Delta}(V)$ and $I_{\Delta}(V')$ (see also Helminck, 1991, Section 7). Using Proposition 6.3 and a similar argument as in Helminck (1996) we get the following proposition.

Proposition 7.9. Let $W$, $V$, etc. be as above. Then

(i) Every $W$-orbit in $\varphi(V)$ contains a $\Delta_0$-standard involution.

(ii) Every $W$-orbit in $\varphi'(V') = \varphi(V) \cdot w_{\Pi}^0$ contains a $\Delta_0$-standard involution.

Remarks 7.10. (1) It follows from the above results that in the case that $A^-$ is a maximal $(\theta, k)$-split torus of $G$, we can represent the elements of $I_{\Delta}(V)$ and $I_{\Delta}(V')$ as involutions $w_{\Pi_1}^0$ or $w_{\Pi_2}^0$. For $I_{\Delta}(V')$ we will only use the first characterization, but for $I_{\Delta}(V)$ we will use both characterizations depending on whether $\theta^* = id$ or a diagram automorphism.

(2) If $I_{\Delta}(V') = \{w_{\Pi_1}^0, \ldots, w_{\Pi_k}^0\}$, where $w_{\Pi_1}^0, \ldots, w_{\Pi_k}^0$ are $\Delta_0$-standard involutions, then $I_{\Delta}(V) = \{w_{\Pi_1}^0 w_0, \ldots, w_{\Pi_k}^0 w_0\}$ and $w_{\Pi_1}^0 w_0, \ldots, w_{\Pi_k}^0 w_0$ are involutions. In most cases we will use this characterization of $I_{\Delta}(V')$ and $I_{\Delta}(V)$.

(3) Using the above results, the classification of $(\Gamma, \theta)$-indices of $k$-involutions in Helminck (2000a) and the classification of involutions in the Weyl group in Helminck
(1991) we get a list of the possible subsets $\Pi \in \Lambda_\Delta^0$, which correspond to involutions which are not $W$-conjugate.

With only very few exceptions, the isomorphism class of an involution $w_\Pi^0 \in \mathcal{I}_\Delta(V)$ is determined by the type of the root system $\Phi_\Pi$ spanned by $\Pi$. Only in the case that $(G, \theta)$ is $(\theta, k)$-split (i.e., $G$ contains a $\theta$-split maximal $k$-split torus) can it happen that two involutions $w_\Pi^0_{\Pi_1}$ and $w_{\Pi_2}^0$ are not $W$-conjugate, while the root systems $\Phi_\Pi$, and $\Phi_{\Pi_1}$ $(\Pi_1, \Pi_2 \subset \Delta)$ are of the same type. In these cases it is easy to find two subsets $\Pi_1, \Pi_2 \subset \Delta$, such that the root systems $\Phi_{\Pi_1}$ and $\Phi_{\Pi_2}$ are of the same type and $w_{\Pi_1}^0$ and $w_{\Pi_2}^0$ are not $W$-conjugate. For more details, see Helminck (1991, Section 7).

7.11. IMAGE OF $\varphi$

To compute $\varphi'(V')$ (or $\varphi(V)$) from the subsets $\Pi \in \Lambda_\Delta^0$ as above we can use the same method as in Helminck (1988). If $w_\Pi^0 \in \mathcal{I}_\Delta$, then we also write $W(\Pi)$ for the stabilizer subgroup $W_{w_\Pi^0} = \{w \in W \mid w * w_\Pi^0 = w_\Pi^0\}$. The orbits $W * w_\Pi^0 \subset \mathcal{I}_{\theta'}$ and $W * w_\Pi^0 w_0^{-1} \subset \mathcal{I}$ can be characterized as follows.

**Lemma 7.12.** (Helminck, 1996, 9.12) Let $\theta, \theta', \mathcal{I}$ and $\mathcal{I}_{\theta'}$ be as above. If $w_\Pi^0 \in \mathcal{I}_\Delta$, $W(\Pi)$ as above and $w_1, \ldots, w_n$ minimal coset representatives of $W/W(\Pi)$, then we have the following:

(i) $W * w_\Pi^0 = \{w_1 * w_\Pi^0, \ldots, w_n * w_\Pi^0\}$.

(ii) $W * w_\Pi^0 w_0^{-1} = \{w_1 * w_\Pi^0 w_0^{-1}, \ldots, w_n * w_\Pi^0 w_0^{-1}\}$.

7.13. It remains to determine the subsets $\Pi \in \Lambda_\Delta^0$ and compute the corresponding subgroups $W(\Pi)$ for $\Pi \in \Lambda_\Delta^0$. We first recall the following. A pair $(G, \theta)$ is called $(\theta, k)$-split if there exists a $\theta$-split maximal $k$-split torus of $G$. The pair $(G, \theta)$ is called quasi-$(\theta, k)$-split if there exists a $\theta$-split minimal parabolic $k$-subgroup of $G$. Note that if $(G, \theta)$ is $(\theta, k)$-split and $A$ is a $\theta$-stable maximal $k$-split torus with $A_{\theta}$ a maximal $(\theta, k)$-split torus of $G$, then $\theta|\Phi(A) = -\text{id}$. This means that if $s \in \mathcal{I}_\theta$ and $w \in W$, then $w * s * w = w * \theta(w^{-1}) = w * w^{-1}$. So $\mathcal{I}_\theta$ consists of the set of involutions in $W(A)$ and the $W(A)$-orbits in $\mathcal{I}_\theta$ are precisely the $W(A)$-conjugacy classes of involutions in $W(A)$. We summarize this in the following result.

**Lemma 7.14.** Let $(G, \theta)$ be $(\theta, k)$-split, $P$ a $\theta$-split minimal parabolic $k$-subgroup of $G$, $A \subset P$ a $\theta$-split maximal $k$-split torus of $G$ and $\mathcal{I}$ the set of twisted involutions in $W(A)$. Then $\mathcal{I}$ consists of the set of involutions in $W(A)$.

Similar as in Helminck (1988) we can now show that to compute the subgroups $W(\Pi)$ of $W$ as above, it suffices to consider $\mathcal{I}_{\text{id}}$ or $\mathcal{I}_{-\text{id}}$. The $W$-orbits in all other cases can be identified with orbits for these two cases. This can be seen as follows.

7.15. COMPUTING THE SUBGROUPS $W(\Pi)$

Let $A$ be a maximal $k$-split torus of $G$, $\Phi = \Phi(A)$, $W = W(A)$, $\theta \in \text{Aut}(G, A)$ an involution with $A_{\theta}$ a maximal $(\theta, k)$-split torus and $\Delta$ a $\theta^{-}$-basis of $\Phi$. Write $\theta = \theta^0 w_{\Delta_{\theta}}$ as in Subsection 4.12 and let $\mathcal{T}_\theta \subset W$ denote the set of involutions in $W$. Assume $\Phi$ is irreducible. Then we have three cases.
Lemma 7.17. Let $A$ be a $\theta$-stable maximal $k$-split torus, $W = W(A)$, $\Phi = \Phi(A)$ and assume $\Phi$ is irreducible. Let $\Delta_1$ be a $\theta$-basis of $\Phi$, $\Delta_2$ a $\theta^*$-basis of $\Phi$, $w_0(\theta)$ the longest element of $W_0(\theta)$ with respect to $\Delta_1$ and $w_0(-\theta)$ the longest element of $W_0(-\theta)$ with respect to $\Delta_0(-\theta) \subset \Delta_2$. If $I_1 = I_0 \cdot w_0(\theta) \subset W$ and $I_2 = I_0 \cdot w_0(-\theta) \subset W$, then $I_1$ or $I_2$ consists of the set of involutions in $W$.

Remarks 7.18. (1) The same argument as in Helminck (1996) can be used compute the fibers of $\varphi$. Again $\varphi$ is injective if and only if there is a unique closed orbit of $P$ on $G/H$ and $\varphi$ is surjective if and only if $(G, \theta)$ is quasi-($\theta, k$)-split.

(2) The computation of the orbits in $V$ can be reduced again to the case that $G$ is semisimple and simply connected. For more details see Helminck (1996, 9.21).

7.19. An Algorithm to Classify $V$

Combining the results in this paper we can now modify the algorithm in Helminck (1996) for orbits of Borel subgroups on symmetric varieties and obtain an algorithm to classify the orbits of minimal parabolic $k$-subgroups acting on symmetric $k$-varieties, i.e. the orbits in $V$. Each step of this algorithm can be implemented in LiE or a number of other programs, like GAP4, Magma or Maple. For a further discussion of this, see Subsection 7.20. First we describe the algorithm in the following. Assume that $G$ is semisimple and simply connected and assume that the pair $(G, \theta)$ is irreducible. In the following let $A$ be a $\theta$-stable maximal $k$-split torus such that $A^\perp$ is a maximal $(\theta, k)$-split torus of $G$, $W = W(A)$, $\Phi = \Phi(A)$ and $\Delta$ a $\theta^*$-basis of $\Phi$. In a similar was to Helminck (1996) we need to determine first $I_{id} = I_{-id}$. The algorithm for this is the following.
STEP 1: Obtain a list of the involutions $w_0^0_{II}$ in $T_0^0$ and write them as the standard product of reflections in the Weyl group. A complete list of these involutions can be found in Helminck (1991, Table II). Note that in almost all cases these Weyl group elements are determined by the type of $\Phi_{II}$, so these are easily expressed as a product of reflections.

STEP 2: For each involution $w_0^0_{II} \in T_0^0$ compute the group $W(II)$, which is in these cases precisely the commutator subgroup of $w_0^0_{II}$. For a discussion on this, see also Subsection 7.15.

STEP 3: Compute a minimal set of coset representatives of $W/W(II)$. This classifies the $W$-orbit of $w_0^0_{II}$ in $I_{\nu'}$, using Lemma 7.12.

These three steps classify $I_{id} = I_{-id}$. The next step is to determine $\theta'$ and $w_0$ and to classify $\varphi(V)$ or $\varphi'(V')$ by identifying the corresponding $W$-orbits in $I_{id}$ or $I_{-id}$. Recall that by Corollary 7.3 the involution $w_0$ is completely determined by the subset $\Delta_0(\theta) = \{ \alpha \in \Delta \mid \theta(\alpha) = \alpha \}$.

STEP 4: Get a list of the pairs $(\Delta_0(\theta), \theta')$ from the classification of $(\Gamma, \theta)$-indices of $k$-involutions of $G$ in Helminck (2000a) (see also Helminck, 1994) and the classification of involutions of $W$ in Helminck (1991, Section 7).

In the following, if $\theta' = id$, then compute $\varphi'(V')$ and if $\theta' \neq id$ (i.e. a diagram automorphism), then compute $\varphi(V)$ by identifying $\varphi(V)$ as a subset of $I_{-id}w_0^0_{\Delta_0(-\theta)}w_0^0_{\Delta'}$.

STEP 5: Find $I_\Delta(V') \subset T_0^0$ or $I_\Delta(V) \subset T_0^0w_0^0_{\Delta_0(-\theta)}w_0^0_{\Delta}$. For $I_\Delta(V')$ this easily follows from Proposition 7.9 and the classification in Helminck (1991, Table IV). For $I_\Delta(V)$ note that we can choose $\Delta_0(-\theta)$-standard involutions as representatives for the $W$-orbits in $\varphi(V)$. Since these involutions commute with $w_0^0_{\Delta_0(-\theta)}w_0^0_{\Delta}$ a list of these also follows from Proposition 7.9 and the classification of involutions of $W$ in Helminck (1991, Table IV).

STEP 6: Determine $\varphi'(V') \subset I_{\nu'}$ or $\varphi(V) \subset I_{-id}w_0^0_{\Delta_0(-\theta)}w_0^0_{\Delta}$, by finding the $W$-orbits of the involutions in $I_\Delta(V')$ or $I_\Delta(V)$. For this we use **STEP 1** and **STEP 2**.

After this one can finally find $V$ as follows.

STEP 7: Determine $|\varphi^{-1}(w_0^0_{II})|$ for each involution $w_0^0_{II}$ in $I_\Delta(V')$ or $I_\Delta(V)$. Then, using the same argument as in Helminck (1996, 9.17) we find the $W$-orbits in $V$ and $V'$.

**Remark.** The results about the Bruhat order on the set of twisted involutions in Helminck (1996) and Richardson and Springer (1990) carry over to this situation and can also be used to characterize the Bruhat order on $V$. For $k$ a p-adic or real field one can also look at the Bruhat order on $V_k$. This situation is however much more complicated and will need a lot of additional work. The main complication is the fact that there is no longer a unique open orbit. We intend to deal with this in a future paper.

7.20. Computational Considerations

To implement this algorithm one can use most of the code already written for the computation of the orbits of Borel subgroups acting on symmetric varieties, as discussed
in Helminck (1996). Similar as in that case the above algorithm can be implemented on a computer for each symmetric $k$-variety $G_k/H_k$, as in Helminck (2000a). The computation of the images $\varphi(V)$ only depends on the $(\Gamma, \theta)$-indices of the involutions and is relatively straightforward using the above algorithm. Nevertheless there will be a lot of additional work, mainly due to the fact that there are many more $(\Gamma, \theta)$-indices than in the case of orbits of a Borel subgroup acting on a symmetric variety as in Helminck (1996). There are 137 different types of absolutely irreducible $(\Gamma, \theta)$-indices, while there are only 24 absolutely irreducible $\theta$-indices related to symmetric varieties.

The eventual goal is to be able to compute the orbits of all finite cases and the infinite families up to a certain dimension. We should note that some of the finite cases are extremely large and it will be hard to compute these. The highest dimension of the infinite cases that can be handled will depend on several factors. First of all it depends, of course, on the processors used, but more importantly, it depends on the efficiency of both the algorithm and its implementation. To optimize the algorithm one will have to use some specific properties of each of the symmetric $k$-varieties $G_k/H_k$.

The amount of work needed to implement the algorithm can be reduced considerably by using one of the available symbolic manipulation programs for which a lot of Weyl group algorithms already have been implemented. For example, one could use the excellent package of Stembridge (see Stembridge, 1992), who has implemented several Weyl group algorithms in Maple. The algorithm can also be implemented in GAP4 or Magma. Probably most suited is the package LiE written by CAN (see Leeuwen et al., 1992). Using this package the calculation of the subgroup $W(\Pi)$ and the calculation of the minimal coset representatives in $W/W(\Pi)$ are easily implemented. Another reason that LiE is a good choice is that the source code (written in C) is available, so one can also optimize its algorithms to suit the above calculation of $I_\theta$ and $V$. An implementation in LiE should be able to handle all the finite-dimensional cases.

Given the large number of absolutely irreducible symmetric $k$-varieties, it will take a considerable amount of work to implement the algorithm in all these cases. So far we have implemented the algorithm for a number of the infinite families of type $A_n$ and $B_n$. In all these cases the above algorithm does a good job in computing the orbits, however, in its present form the algorithm is not yet very efficient in computing the orbit closures. The present algorithm uses an inductive procedure to compute a reduced expression for a twisted involution. This procedure is similar to the one typically used for computing reduced expressions for elements of the Weyl group (see Snow, 1990; Humphreys, 1990). To compute this reduced expression for a twisted involution $w$ involves an amount of work, which is approximately proportional to the length of $w$ times the rank of the root system. The big disadvantage of this method is that it carries out the same or similar calculations several times in a row. For the Weyl group Casselman (see Casselman, 1994) recently described a much more efficient method to compute these reduced expressions (and the underlying Bruhat order) by using automata. He expends on the ideas laid out in Brink and Howlett (1993). It is likely that the ideas in Casselman (1994) can be generalized to the setting of twisted involutions, but for this further research will be needed. This has the potential to drastically reduce the amount of computational work involved. We hope to address this problem in a forthcoming paper.
8. Computing \( \varphi_k(V_k) \)

In this section we give an algorithm to compute \( \varphi_k(V_k) \). This algorithm necessarily depends on the classification of \( k \)-involutions. To get a classification of these one needs to classify both the \( (\Gamma, \theta) \)-index and the quadratic elements (see Helminck, 2000a, for more on this). The \( (\Gamma, \theta) \)-indices have been classified for \( k \) the real numbers, \( p \)-adic numbers, finite field, numbers field or algebraically closed. However, for most of these fields a classification of the quadratic elements is still lacking. For \( k = \mathbb{R} \) and \( k = \mathbb{F}_q \) a classification of the \( k \)-involutions can be found in Helminck (1988). In these cases the algorithm in this section can compute \( \varphi_k(V_k) \).

8.1. We will use the same notation as in the previous sections. In particular let \( A \) be a maximal \( k \)-split torus of \( G \), \( \Phi = \Phi(A) \), \( W = W(A) \), \( \theta \in \text{Aut}(G,A) \) an involution with \( A^- \) a maximal \( (\theta,k) \)-split torus. Let \( \Delta_0 = \Delta_0(\theta) \) be a basis of \( \Phi_0(\theta) \) and extend this to a \( \theta^- \)-basis \( \Delta \) of \( \Phi \). Then by Subsection 4.12 we have \( \theta' = \theta w_0^{-1} \).

8.2. For the characterization of \( \varphi(V) \) we used the map \( \phi_{\gamma}^{-1} : \mathcal{A}^\theta/H \rightarrow \varphi(V)/W \subset \mathcal{I}/W \). To compute \( \varphi_k(V_k) \) we need to look at the map \( \phi_k \gamma_k^{-1} : \mathcal{A}_k^\theta/H_k \rightarrow \varphi_k(V_k)/W \), where \( \gamma_k \) is as in Proposition 2.9 and \( \phi_k \) as in Subsection 3.4.

Recall from Subsection 2.10 that we have a natural embedding \( \zeta : \mathcal{A}_k^\theta/H_k \rightarrow \mathcal{A}^\theta/H \) sending the \( H_k \)-conjugacy class of a \( \theta \)-stable maximal \( k \)-split torus onto its \( H \)-conjugacy class and we have that \( \varphi_k(V_k) \subset \varphi(V) \subset \mathcal{I} \) is \( W \)-stable. From Helminck (1997, 4.13) it now follows that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{A}_k^\theta/H_k & \xrightarrow{\phi_k \gamma_k^{-1}} & \varphi_k(V_k)/W \\
\downarrow{\zeta} & & \downarrow{\text{id}} \\
\mathcal{A}_0^\theta/H & \xrightarrow{\phi \gamma_0^{-1}} & \varphi(V)/W \\
\end{array}
\]

This means that in order to find representatives for the \( W \)-orbits in \( \varphi_k(V_k) \) we can look at \( H \)-conjugacy classes of \( \theta \)-stable maximal \( k \)-split tori instead of \( H_k \)-conjugacy classes. This will enable us to characterize the \( \Delta_0 \)-standard involutions in \( \varphi_k(V_k)/W \) by using the classification of \( \theta \)-stable maximal \( k \)-split tori in Helminck (1991, 1997, 1999).

8.3. Let \( A \) be a \( \theta \)-stable maximal \( k \)-split torus of \( G \) with \( A^- \) a maximal \( (\theta,k) \)-split torus and let \( S \) be a \( \theta \)-stable maximal \( k \)-split torus of \( G \) such that \( S^+ \) a maximal \( k \)-split torus of \( H \), \( A^+ \subset S^+ \) and \( A^- \subset S^- \). By Helminck (1997, 4.10) such a pair \( (A,S) \) always exists. Let \( g_1 \in Z_G(A^+ S^-) \) such that \( g_1 S g_1^{-1} = A \), \( n_1 = g_1 \theta g_1^{-1} \in N_G(A) \) and \( w_1 \in W(A) \) corresponding Weyl group element. One easily checks that \( w_1 \) is an involution and by Definition 5.3 \( w_1 \) is \( (\theta,k) \)-singular. It is easy to see that we can choose \( w_1 \) to be \( \Delta_0 \)-standard. Let \( \Pi_1 \subset \Delta_0(-\theta) \) such that \( w_1 = w_{\Pi_1}^0 \). This is called the maximal \( (\theta,k) \)-singular involution. Since all \( \theta \)-stable maximal \( k \)-split tori \( G \) containing a maximal \( k \)-split torus of \( H \) are conjugate under \( H \) we get the following result.

**Lemma 8.4.** All maximal \( (\theta,k) \)-singular involutions are \( W \)-conjugate.

Combining the results in Helminck (1997, Section 4) with the results of Section 7 we get the following result.
THEOREM 8.5. Let $A$, $S$, $w^0_{\Pi}$ be as above and let $A_1$ be a $\theta$-stable maximal $k$-split torus of $G$. Then we have the following:

1. There exists $h \in H$ such that $A_2 = hA_1h^{-1}$ satisfies $A^+ \subset A^+_1 \subset S^+$ and $A^- \supset A^-_1 \supset S^-$.
2. There exists $g \in Z_G(A^+A^-)$ such that $gSg^{-1} = A$.
3. Let $n = g\theta(g)^{-1} \in N_G(A)$ and $w \in W(A)$ corresponding Weyl group element. Then $w$ is a $(\theta,k)$-singular involution.
4. If $w^0_{\Pi} \in \varphi_k(V_k)$ is a $\Delta_0$-standard involution, then $w^0_{\Pi}$ is $(\theta,k)$-singular and there exists $s \in W$ such that $s(\Pi) \subset \Pi_1$.
5. Every $W$-orbit in $\varphi_k(V_k)$ has a representative $w^0_{\Pi}$ with $\Pi \subset \Pi_1$.

REMARK 8.6. It follows from the above result that in order to compute representatives for $\varphi_k(V_k)/W$ it suffices to determine the maximal $(\theta,k)$-singular involutions, which we can choose $\Delta_0$-standard and representatives for all other $W$-orbits in $\varphi_k(V_k)$ are determined by the subsets of $\Pi_1$ which correspond to involutions in $\Lambda^0_{\Delta}$. So to compute $\varphi_k(V_k)$ we need to extend the algorithm for $\varphi(V)$ in Subsection 7.19 with the following additional step:

**STEP 8:** For each $k$-involution determine the subset $\Pi \subset \Delta_0$ which represents the maximal $(\theta,k)$-singular involution. The involutions $w^0_{\Pi} \in \Lambda^0_{\Delta} \cap \varphi(V)$ with $\Pi \subset \Pi_1$ are representatives for the $W$-orbits in $\varphi_k(V_k)$.

REMARKS 8.7. (1). The maximal $(\theta,k)$-singular involutions depend on the $k$-involution and not on the $(\Gamma,\theta)$-index. In order to get a list of these for a specific field $k$ one needs first a classification of the $k$-involutions for that field. So far that classification is only known for $k = \bar{k}$ and $k = \mathbb{R}$ (see Helminck, 1988).
(2). For $k = \mathbb{R}$ a list of the maximal $(\theta,k)$-singular involutions is given in Helminck (1999). So combined with the algorithm in Subsection 7.19 we can compute $\varphi_k(V_k)$ for $k = \mathbb{R}$.
(3). The computation of the orbit closures in $V_k$ is much more complicated than for $V$.
For a further discussion of this see Helminck (2000b).

8.8. In Helminck (1999) we showed that for $k = \mathbb{R}$ there exists a one-to-one correspondence between $(\theta,k)$-singular involutions in $A$ and $H_k$-conjugacy classes of $\theta$-stable maximal $k$-split tori. This leads to the following result.

THEOREM 8.9. Assume $k = \mathbb{R}$ and let $A$, $V_k$, $\varphi_k$, etc. be as above. The map $\varphi_k \gamma_k^{-1} : A^0_k/H_k \to \varphi_k(V_k)/W \subset I/W$ is a bijection.

REMARK 8.10. It follows from this result that similar as in the case of $k = \bar{k}$ in Helminck (1996) the fibers of $\varphi_k : V_k \to I$ for $k = \mathbb{R}$ differ a Weyl group element. So these can be computed in a similar way as in Helminck (1996) for $k = \bar{k}$. We note, however, that a more efficient description of these is still needed. We hope to address this question in a future paper.

We conclude with giving an example, which illustrates the above algorithm.
Table 1. Calculation of $\varphi(V) = I$.

<table>
<thead>
<tr>
<th>$w^0_\Pi$</th>
<th>$W/W(\Pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2$</td>
<td>$\text{id}$</td>
</tr>
<tr>
<td>$w_0 = s_1$</td>
<td>${\text{id}}$</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>$\text{id}$</td>
</tr>
<tr>
<td>$w_0 = s_2s_1s_2$</td>
<td>${\text{id}}$</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>$\text{id}$</td>
</tr>
<tr>
<td>$w_0 = s_1s_2s_1s_2$</td>
<td>${\text{id}}$</td>
</tr>
</tbody>
</table>

Example 8.11. Let $G = \text{SL}_{2n}(k)$ and assume $k = \mathbb{R}$. Let $\theta(g) = L(g^{-1})L^{-1}$, where $L = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then $H \simeq \text{SP}_{2n}$. Using the notation in Helminck (2000a) (see also Helminck, 1988, or Helminck, 1994) the pair $(G, \theta)$ is of type $A_{2n-1}^n$. The corresponding $(\Gamma, \theta)$-index is

$$
\begin{array}{ccccccc}
1 & 2 & \cdots & n-1 & n \\
\theta^* & I & I & I & I & I & I \\
\end{array}
$$

Let $B$ be the Borel subgroup of upper triangular matrices, $T$ the group of diagonal matrices and $A = \{\text{diag}(a_1, \ldots, a_n, a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in k^*\}$. Clearly $A \subset T$, $T$ is maximal $k$-split and $A$ is both maximal $\theta$-split and $(\theta, k)$-split. The orbit $BH$ is open in $G$. Let $\Phi = \Phi(T)$ be the root system of $T$ with respect to $\Phi^+$ the set of positive roots of $\Phi$ related to $B$, $\Delta$ the corresponding basis of $\Phi$, $W = W(T)$ the Weyl group of $T$ and $X = X^+(T)$ the group of characters of $T$. Let $w_0 \in W$ such that $\theta(\Phi^+) = w_0(\Phi^+)$, $\theta' = \theta w_0$, $v_0 \in V$ and $n_0 = x(v_0)\theta(x(v_0))^{-1} \in N_G(T)$ such that $n_0$ induces $w_0$ in $W$ and let $\zeta$ be the involution of $G$ given by $\zeta(x) = n_0^{-1}\theta(x)n_0$, $x \in G$. By Corollary 7.3 $w_0 = w^0_{\Delta_0(\theta)} w^0_{\Delta}$, where $\Delta_0(\theta) = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha\}$. Since $(G, \theta)$ is quasi-$(\theta, k)$-split, we have $\Delta_0(\theta) = \emptyset$ and $w_0 = w^0_{\Delta}$. The set $I$ consists of the set of involutions in $W$. Let $I^0 = I \cdot w_0$ as in Lemma 3.24.

From Helminck (1991, Table II) it follows that the involutions $w^0_{\Pi}$ are all of type $A_1 + \cdots + A_1$, $r$ times and there is only one conjugacy class for each type.

Since $(G, \theta)$ is quasi-$(\theta, k)$-split it follows that the map $\varphi : V \rightarrow I$ is surjective. So in particular $\varphi(V) = I$. However from Helminck (1999, Table 2) it follows that the longest $(\theta, k)$-split involution is the id, so $\varphi_k(V_k) = \{\text{id}\}$. The computation of $I$ is as in Helminck (1996).

An implementation of the above algorithm in LiE is illustrated in Table 1. We use there the following notation. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and write $s_i = s_{\alpha_i}$ for the reflection defined by $\alpha_i \in \Delta$. In the table we list the minimal coset representatives of $W/W(\Pi)$ for each of the involutions $w^0_{\Pi} \in I \Delta \simeq I \Delta(V)$.

References


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