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A polyhedral study of triplet formulation for single row facility layout problem

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ABSTRACT

The single row facility layout problem (SRFLP) is the problem of arranging n departments with given lengths on a straight line so as to minimize the total weighted distance between all department pairs. We present a polyhedral study of the triplet formulation of the SRFLP introduced by Amaral [A.R.S. Amaral, A new lower bound for the single row facility layout problem, Discrete Applied Mathematics 157 (1) (2009) 183–190]. For any number of departments n, we prove that the dimension of the triplet polytope is $n(n-1)(n-2)/3$ (this is also true for the projections of this polytope presented by Amaral). We then prove that several valid inequalities presented by Amaral for this polytope are facet-defining. These results provide theoretical support for the fact that the linear program solved over these valid inequalities gives the optimal solution for all instances studied by Amaral.

1. Introduction

In single row facility layout problem (SRFLP), the goal is to arrange n departments on a straight line. We are given the following data: an $n \times n$ symmetric matrix $C = [c_{ij}]$, where c_{ij} denotes the average daily traffic between two departments i and j, and the length l_i of each department $i \in N = \{1, \ldots, n\}$. The distance z_{ij} between two departments is considered to be the distance between their centroids. The objective is to find the permutation π that minimizes the total communication cost, i.e.

$$\min_{\pi} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} z_{ij}^{\pi}.$$

The SRFLP has several applications involving arranging rooms on a corridor, machines in a manufacturing system, and books on a shelf [9,15,16]. The *minimum linear arrangement problem (MLAP)* was proven to be NP-hard in [8]. The SRFLP is a generalization of MLAP and so is also NP-hard. Numerous heuristic solution approaches have been proposed for SRFLP (e.g. see [9,12,17,14]).

Several exact solution techniques have also been proposed including branch and bound algorithms [16], dynamic programming [15,11], nonlinear programming [10], and linear mixed integer programming [1,2,13]. Anjos et al. [5] and Anjos and Vanelli [6] provided lower bounds on the optimal cost of SRFLP using semidefinite programming (SDP) relaxations. Anjos and Yen [7] computed near optimal solutions for instances with up to 100 facilities using a new SDP relaxation. Amaral and Letchford [4] conducted a polyhedral study on the distance polytope formulation of SRFLP and developed several classes of valid inequalities. They achieved quick bounds for SRFLP using LP relaxations based on these valid inequalities. They are comparable to the bounds achieved in [5].

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Amaral [3] presented an alternate formulation of the SRFLP, herein referred to as the *triplet formulation*, and introduced a set of valid inequalities for it. It is shown in [3] that the linear program solved over these valid inequalities yields the optimal solution for several classical SRFLP instances of sizes $n = 5$ to $n = 30$. These problem instances are from [1,2,9,10,13,16]. The results in [3] are comparable to the results of [6] which are based on SDP relaxation with cutting planes added.

The fact that the LP relaxation over the valid inequalities of [3] gives the optimal solution to so many instances suggests that these valid inequalities are quite strong. In this paper, we conduct a polyhedral study of the *triplet polytope*, i.e. the convex hull of feasible integer points for the triplet formulation. We prove that almost all valid inequalities introduced in [3] are indeed facet-defining for the triplet polytope. More specifically, we first show that the three polytopes (triplet polytope and its two projections defined in [3]) are of dimension $n(n-1)(n-2)/3$. After establishing the dimension of these polytopes, we then prove the aforementioned facet-defining properties.

The paper is organized as follows: Section 2 briefly reviews the triplet polytope, its projections, and the valid inequalities developed for them in [3]. In Section 3 we prove that these polytopes are of dimension $n(n-1)(n-2)/3$. In Section 4 we prove the facet-defining properties of valid inequalities of [3], and we conclude in Section 5 with a few remarks.

2. Triplet polytope, its projections and valid inequalities

In the triplet formulation for the SRFLP [3], a binary vector $\zeta \in \{0, 1\}^{n(n-1)(n-2)}$ is used to represent a permutation of the departments in N. Each element of ζ is identified by a triplet subscript ijk, where $i, j, k \in N$ are distinct, and

$$\zeta_{ijk} = \begin{cases} 1 & \text{if department } k \text{ lies between departments } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the paper, all the department indices used in the subscript of a single variable, coefficient, or set are assumed to be distinct and we refrain from writing this in each case. We define

$$P = \left\{\zeta \in \{0, 1\}^{n(n-1)(n-2)} : \zeta \text{ represents a permutation of } 1, \dots, n\right\},$$

and refer to the convex hull of P, i.e. $conv(P)$, as the *triplet polytope*. Based on this formulation, the objective function of SRFLP can be written as

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} \left(\frac{1}{2}(l_i + l_j) + \sum_{k \neq i,\ k \neq j}^{n} l_k \zeta_{ijk} \right).$$

In [3] the following valid inequalities are presented for P:

$$0 \le \zeta_{ijk} \le 1 \quad i, j, k \in N \tag{1}$$

$$\zeta_{ijk} + \zeta_{ikj} + \zeta_{jki} = 1 \quad i, j, k \in N \tag{2}$$

$$\zeta_{ijd} + \zeta_{jkd} - \zeta_{ikd} \ge 0 \quad i, j, k, d \in N \tag{3}$$

$$\zeta_{ijd} + \zeta_{jkd} + \zeta_{ikd} \le 2 \quad i, j, k, d \in N. \tag{4}$$

Two projections of P are also introduced in [3]. We briefly review them here. It is clear that for any $\zeta \in P$

$$\zeta_{ijk} = \zeta_{jik} \quad 1 \le i < j \le n. \tag{5}$$

Using this identity, P can be projected onto the space $\{0, 1\}^{n'}$, where $n' = n(n-1)(n-2)/2$. We refer to this projection as P^1. The projection of a vector $\zeta \in P$ will be a vector $\lambda \in P^1 \subseteq \{0, 1\}^{n'}$ with elements λ_{ijk} such that $\lambda_{ijk} = \zeta_{ijk}$ for $i, j, k \in N$, $i < j$. So the valid inequalities (1)–(4) can also be projected yielding the following inequalities for P^1. Observe that (8)–(10) are obtained from projection of (3).

$$0 \le \lambda_{ijk} \le 1 \quad i, j, k \in N, i < j \tag{6}$$

$$\lambda_{ijk} + \lambda_{ikj} + \lambda_{jki} = 1 \quad i, j, k \in N,\ i < j < k \tag{7}$$

$$-\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \ge 0 \quad i, j, k, d \in N,\ i < j < k \tag{8}$$

$$\lambda_{ijd} + \lambda_{jkd} - \lambda_{ikd} \ge 0 \quad i, j, k, d \in N,\ i < j < k \tag{9}$$

$$\lambda_{ijd} - \lambda_{jkd} + \lambda_{ikd} \ge 0 \quad i, j, k, d \in N,\ i < j < k \tag{10}$$

$$\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} \le 2 \quad i, j, k, d \in N,\ i < j < k. \tag{11}$$

Amaral [3] also introduces a more complicated set of valid inequalities for $conv(P^1)$ as follows: for a positive even integer $\beta \le n$, consider the set of distinct indices $S = \{i_t : t = 1, \dots, \beta\} \subseteq \{1, \dots, n\}$ and $d \in S$. Let (S_1, S_2) be a partition of $S \setminus \{d\}$ such that $|S_1| = \beta 2$. Then, the inequality

$$\sum_{p,q \in S_1 : p<q} \lambda_{pqd} + \sum_{p,q \in S_2 : p<q} \lambda_{pqd} \le \sum_{p \in S_h,\ q \in S_{\{1,2\} \setminus h} : h=1,2,\ p<q} \lambda_{pqd} \tag{12}$$

is valid for $conv(P^1)$ [3]. Inequalities (8)–(10) are special cases of (12) for $\beta = 4$, as noted in [3].

P^1 can be further projected on a lower dimensional space using identity (7). Observe that based on (7), we have

$$\lambda_{ijk} = 1 - \lambda_{ikj} - \lambda_{jki} \quad i, j, k \in N,\ i < j < k.$$

Therefore, the number of variables can be reduced to $n'' = n' - \binom{n}{3} = n(n-1)(n-2)/3$. We refer to this projection as P^2. The projection of a vector $\lambda \in P^1$ will be a vector $\mu \in P^2 \subseteq \{0, 1\}^{n''}$ with elements μ_{ijk} such that $\mu_{ijk} = \lambda_{ijk}$ for $i, j, k \in N$, $i < j$, $k < j$. The set of valid inequalities (6)–(12) can also be projected yielding valid inequalities for P^2.

3. Dimension of convex hulls of P, P^1 and P^2

In this section, we prove that $conv(P^1)$ is of dimension n''. Based on the projection relationships between P, P^1 and P^2, we will then easily argue that the dimensions of $conv(P)$ and $conv(P^2)$ are n'' too. To prove that the dimension of P^1 is n'', we will show that any hyperplane passing through all points in P^1 can be expressed as a linear combination of the set of linearly independent equalities (7). Since the number of these inequalities is $\binom{n}{3}$, we will have $\dim(conv(P^1)) = n' - \binom{n}{3} = n''$.

We first define some notations that we will use throughout the paper. For any $N' \subseteq N$, we define $\Pi_{N'}$ as the set of all permutations of the departments in N'. Each $\lambda \in P^1$ corresponds to a member of Π_N. To denote the λ vector which corresponds to a given permutation $\pi \in \Pi_N$, we write λ^{π}. Similarly if for example $\pi^1 \in \Pi_{N\setminus\{x,y\}}$, then $\lambda^{x\pi^1 y}$ is the vector in P^1 corresponding to the permutation $x\pi^1 y$, i.e. the permutation in which x is the first department, y is the last one, and the rest are in the middle in the order π^1. Similar notations are also used for $\zeta \in P$ and $\mu \in P^2$ that correspond to a given permutation.

Based on the definition of P^1, λ_{ijk} is only defined when $i < j$. Therefore for any given three distinct departments i, j, and k, the variable representing whether k is between i and j or not, is λ_{ijk} if $i < j$, and is λ_{jik} if $i > j$. In many cases, just for the sake of notation simplicity, we would like to avoid differentiating between these two cases. In order to do so, wherever we have λ_{ijk}, where $i > j$, we mean λ_{jik}. We emphasize that this is just a notational substitute and does not mean that when $i > j$ the variable λ_{ijk} really exists. We also employ this practice for a_{ijk}, the coefficient of λ_{ijk} in a hyperplane, so the reader should be careful that when $i > j$, a_{ijk} is only a notational substitute for the real coefficient a_{jik}.

The following lemma is fundamental to the result in this section.

Lemma 1. *For some given departments $x, y, z \in N$ and permutations $\pi^1 \in \Pi_{N\setminus\{x,y\}}$, $\pi^2 \in \Pi_{N\setminus\{x,y,z\}}$, if $\lambda^{xy\pi^1}$, $\lambda^{yx\pi^1}$, $\lambda^{zxy\pi^2}$, and $\lambda^{zyx\pi^2}$ lie on the hyperplane*

$$\sum_{i,j,k \in N: i<j} a_{ijk}\lambda_{ijk} = b, \tag{13}$$

then $a_{yzx} = a_{xzy}$.

Proof. We substitute $\lambda^{xy\pi^1}$ and $\lambda^{yx\pi^1}$ in (13). The left-hand sides are both equal to b; therefore,

$$\sum_{i,j,k \in N: i<j} a_{ijk}\lambda_{ijk}^{xy\pi^1} = \sum_{i,j,k \in N: i<j} a_{ijk}\lambda_{ijk}^{yx\pi^1}. \tag{14}$$

Now observe that $\lambda_{ijk}^{xy\pi^1} = \lambda_{ijk}^{yx\pi^1}$ for any i, j, k such that $\{x, y\} \not\subset \{i, j, k\}$. Therefore, a_{ijk}'s for such terms cancel out from both sides. Also $\lambda_{xyh}^{xy\pi^1} = 0$, $\lambda_{yhx}^{xy\pi^1} = 0$, $\lambda_{xhy}^{xy\pi^1} = 1$, $\lambda_{xyh}^{yx\pi^1} = 0$, $\lambda_{xhy}^{yx\pi^1} = 0$, and $\lambda_{yhx}^{yx\pi^1} = 1$ for all $h \neq x, y$. Therefore, (14) reduces to

$$\sum_{h\neq x,y} a_{xhy} = \sum_{h\neq x,y} a_{yhx}. \tag{15}$$

Next we substitute the other two vectors $\lambda^{zxy\pi^2}$ and $\lambda^{zyx\pi^2}$ in (13) and equate the left-hand sides, we get

$$\sum_{i,j,k \in N: i<j} a_{ijk}\lambda_{ijk}^{zxy\pi^2} = \sum_{i,j,k \in N: i<j} a_{ijk}\lambda_{ijk}^{zyx\pi^2}. \tag{16}$$

Like above by substituting the variable values and canceling the common terms, it is easy to see that (16) reduces to

$$a_{yzx} + \sum_{h\neq x,y,z} a_{xhy} = a_{xzy} + \sum_{h\neq x,y,z} a_{yhx}. \tag{17}$$

Subtracting (17) from (15), we get $a_{xzy} - a_{yzx} = a_{yzx} - a_{xzy}$ or $a_{yzx} = a_{xzy}$, which concludes the proof. □

Amaral and Letchford [4] use a similar pairwise switching of departments to obtain the dimension of the distance polytope formulation they presented for SRFLP.

Theorem 2. *$conv(P^1)$ is of dimension n''.*

Proof. $conv(P^1) \subset \mathbb{R}^{n'}$ and any $\lambda \in P^1$ satisfies the set of $\binom{n}{3}$ equalities (7). Clearly these set of equalities are linearly independent because no variable appears in more than one equality. Hence, $\dim\left(conv(P^1)\right) \leq n' - \binom{n}{3} = n''$. To prove that the dimension is actually equal to n'', we just need to show that any other hyperplane like

$$\sum_{i,j,k\in N:i<j} a_{ijk}\lambda_{ijk} = b \tag{18}$$

satisfied by all $\lambda \in P^1$ will be a linear combination of the equalities (7). For this purpose observe that $\lambda^{\pi} \in P^1$ for any permutation $\pi \in \Pi_N$. Therefore, for any three distinct departments x, y, z, by choosing any two arbitrary permutations $\pi^1 \in \Pi_{N\setminus\{x,y\}}$ and $\pi^2 \in \Pi_{N\setminus\{x,y,z\}}$, the vectors $\lambda^{xy\pi^1}$, $\lambda^{yx\pi^1}$, $\lambda^{zxy\pi^2}$, $\lambda^{zyx\pi^2}$ are in P^1 and so lie on (18). Hence by Lemma 1, $a_{yzx} = a_{xzy}$. Also for any arbitrary $\pi^3 \in \Pi_{N\setminus\{y,z\}}$, the vectors $\lambda^{yz\pi^3}$, $\lambda^{zy\pi^3}$, $\lambda^{xyz\pi^2}$, $\lambda^{xzy\pi^2}$ are in P^1 and so lie on (18). Hence again by Lemma 1, $a_{xyz} = a_{xzy}$ (note that based on our notation the order of the first two departments in the subscript does not matter). Therefore in (18), for any three departments x, y, z we have

$$a_{xyz} = a_{xzy} = a_{yzx}. \tag{19}$$

Identity (19) along with equalities (7) shows that $b = \sum_{i,j,k\in N:i<j} a_{ijk}$ and (18) has to be a linear combination of equalities (7), which concludes the proof. □

Remember that P^1 is a projection P based on identities (5) and P^2 is a projection of P^1 based on identities (7). Therefore, dimensions of $conv(P)$ and $conv(P^2)$ can be derived as a corollary to Theorem 2. This corollary is based on the following simple Lemma, which we state first.

Lemma 3. *Let A be a $n_1 \times n_2$ matrix and b be a constant n_2-vector. If $S \subseteq \mathbb{R}^{n_1}$ and $T = \{(x, xA - b) \in \mathbb{R}^{n_1+n_2} : x \in S\}$, then* $\dim(S) = \dim(T)$.

Proof. The proof is the direct result of the fact that $x_1, \ldots, x_m \in S$ are affinely independent if and only if $(x_1, x_1A - b), \ldots, (x_m, x_mA - b) \in T$ are affinely independent. □

Observe that in Lemma 3, if we denote the elements of T by (x, y), then S is in fact the projection of T over $\mathbb{R}^{n_1}$ based on identity $y = xA - b$.

Corollary 4. *$conv(P)$ and $conv(P^2)$ are also of dimension n''.*

Proof. Based on the identities (5), $conv(P^1)$ and $conv(P)$ play the roles of S and T in Lemma 3, respectively (we would have $n_1 = n'$ and $n_1 + n_2 = 2n'$), so according to Lemma 3, $\dim(conv(P)) = \dim\left(conv(P^1)\right) = n''$.

Similarly, based on identities (7), $conv(P^2)$ and $conv(P^1)$ play the roles of S and T in Lemma 3, respectively (we would have $n_1 = n''$ and $n_1 + n_2 = n'$), so according to Lemma 3, $\dim\left(conv(P^2)\right) = \dim\left(conv(P^1)\right) = n''$. □

Therefore, $conv(P)$, $conv(P^1)$, and $conv(P^2)$ all have the same dimension n'' and $conv(P^2)$ is full dimensional.

4. Facet-defining properties of valid inequalities

In this section, we prove that inequalities (8)–(10) and (12) are facet-defining for $conv(P^1)$. Then as a result of Lemma 3, their corresponding inequalities for P and P^2 are also facet-defining for $conv(P)$ and $conv(P^2)$.

We note that trivial inequalities (6) as well as inequality (11) are not facet-defining in general. This can be easily seen by listing all $\lambda \in P^1$ that lie on the defining hyperplanes of these inequalities for $n = 3$ or $n = 4$ and checking their affine independence.

Theorem 5. *Inequalities (8)–(10) are facet-defining for $conv(P^1)$.*

Proof. Consider any chosen four departments i, j, k, d $(i < j < k)$. We prove the theorem for inequality (8). The proof for inequalities (9) and (10) is similar. By Theorem 2, we know $\dim\left(conv(P^1)\right) = n''$. Let P' be the face of $conv(P^1)$ defined by (8). Therefore, for every point in P', (8) is satisfied at equality, i.e.

$$-\lambda_{ijd} + \lambda_{jkd} + \lambda_{ikd} = 0. \tag{20}$$

To prove P' is a facet, we must show $\dim\left(P'\right) = n'' - 1$. To show this we prove any hyperplane like

$$\sum_{e,f,g\in N:e<f} a_{efg}\lambda_{efg} = b \tag{21}$$

that passes through P' has to be a linear combination of the hyperplanes (7) and the hyperplane (20). First we prove the following identity:

$$a_{efg} = a_{egf} = a_{fge} \quad \text{for any } \{e,f,g\} \neq \{i,j,d\},\{i,k,d\},\{j,k,d\}. \tag{22}$$

To show this observe that the following three cases are possible:

(i) $d \notin \{e,f,g\}$: Note that any for any $\pi \in \Pi_{N\setminus\{d\}}$, $\lambda^{\pi d}$ satisfies (20) and hence belongs to P'. Thus, it must satisfy (21). So in particular, for any arbitrary $\pi^1 \in \Pi_{N\setminus\{e,f,d\}}$, $\pi^2 \in \Pi_{N\setminus\{e,f,g,d\}}$, the vectors $\lambda^{ef\pi^1 d}$, $\lambda^{fe\pi^1 d}$, $\lambda^{gef\pi^2 d}$, and $\lambda^{gfe\pi^2 d}$ satisfy

(21). Therefore by Lemma 1, $a_{egf} = a_{fge}$. For the same reason, for any arbitrary $\pi^3 \in \Pi_{N\setminus\{f,g,d\}}$, $\lambda^{fg\pi^3 d}$, $\lambda^{gf\pi^3 d}$, $\lambda^{efg\pi^2 d}$, and $\lambda^{egf\pi^2 d}$ satisfy (21). So again by Lemma 1, $a_{efg} = a_{egf}$. Therefore (22) is true in this case.

(ii) $d \in \{e, f, g\}$ *and* $\{e, f, g\} \cap \{i, j, k\} = \emptyset$, $\{i\}$ *or* $\{j\}$: We assume $e = d$ (the arguments for the cases $f = d$ or $g = d$ are similar by symmetry). Now observe that for any arbitrary $\pi^1 \in \Pi_{N\setminus\{d,f\}}$, $\pi^2 \in \Pi_{N\setminus\{d,f,g\}}$, the vectors $\lambda^{df\pi^1}$, $\lambda^{fd\pi^1}$, $\lambda^{gdf\pi^2}$, and $\lambda^{gfd\pi^2}$ satisfy (20) and hence belongs to P' so they must satisfy (21) too. Therefore by Lemma 1, $a_{dgf} = a_{fgd}$. Also for the same reason, for any arbitrary $\pi^3 \in \Pi_{N\setminus\{d,g\}}$, $\lambda^{dg\pi^3}$, $\lambda^{gd\pi^3}$, $\lambda^{fdg\pi^2}$, and $\lambda^{fgd\pi^2}$ satisfy (21). So again by Lemma 1, $a_{dfg} = a_{fgd}$. Therefore, since $d = e$, identity (22) is true in this case too.

(iii) $d \in \{e, f, g\}$ and $\{e, f, g\} \cap \{i, j, k\} = \{k\}$: We assume $e = d$ and $f = k$ (the arguments for other possibilities are similar by symmetry). First observe that for any arbitrary $\pi^1 \in \Pi_{N\setminus\{g,k\}}$ and $\pi^2 \in \Pi_{N\setminus\{d,g,k\}}$, the vectors $\lambda^{gk\pi^1}$, $\lambda^{kg\pi^1}$, $\lambda^{dgk\pi^2}$, and $\lambda^{dkg\pi^2}$ satisfy (20) and hence belong to P'. So they satisfy (21). Therefore again by Lemma 1, $a_{dkg} = a_{dgk}$. Now to prove $a_{dkg} = a_{gkd}$ we cannot simply use Lemma 1 as before. The proof is as follows: note that for any arbitrary $\pi^3 \in \Pi_{N\setminus\{d,g\}}$, the vectors $\lambda^{dg\pi^3}$ and $\lambda^{gd\pi^3}$ satisfy (20) so they must satisfy (21) too. Similar to the proof of Lemma 1, by substituting these two vectors in the left-hand side of (21) and equating them, we get

$$\sum_{h\neq d,g} a_{dhg} = \sum_{h\neq d,g} a_{ghd}. \tag{23}$$

Moreover for any arbitrary $\pi^4 \in \Pi_{N\setminus\{d,g,i,k\}}$, the vectors $\lambda^{ikdg\pi^4}$ and $\lambda^{ikgd\pi^4}$ must satisfy (21) for the same reason. By substitution the two vectors in the left-hand side of (21) and equating, we get

$$a_{gkd} + a_{gid} + \sum_{h\neq d,g,i,k} a_{dhg} = a_{dkg} + a_{dig} + \sum_{h\neq d,g,i,k} a_{ghd}. \tag{24}$$

Subtracting (24) from (23) we get

$$a_{dkg} - a_{gkd} + a_{dig} - a_{gid} = a_{gkd} - a_{dkg} + a_{gid} - a_{dig}. \tag{25}$$

But $a_{gid} = a_{dig}$ according to case (ii). So (25) reduces to $a_{dkg} = a_{gkd}$. Therefore, identity (22) is true in this case too.

Moreover, for any arbitrary $\pi^1 \in \Pi_{N\setminus\{i,j,d\}}$, the vectors $\lambda^{ij\pi^1 d}$, $\lambda^{ji\pi^1 d}$, $\lambda^{dij\pi^1}$, $\lambda^{dji\pi^1}$ are in P' and hence satisfy (21). Therefore by Lemma 1,

$$a_{idj} = a_{jdi}. \tag{26}$$

By a similar argument, we also have

$$a_{idk} = a_{kdi}, \tag{27}$$

$$a_{jdk} = a_{kdj}. \tag{28}$$

Now observe that identities (22) imply that hyperplane (21) is a linear combination of equalities (7) for $\{e, f, g\} \neq \{i, j, d\}$,$\{i, k, d\}$,$\{j, k, d\}$ as well as equality (29) below (the coefficient of any particular equality (7) like $\lambda_{efg} + \lambda_{egf} + \lambda_{gfe} = 1$ in this linear combination is a_{efg} $(= a_{egf} = a_{fge})$ and we have $b_1 = b - \sum_{\{e,f,g:e<f\}\neq\{i,j,d\},\{i,k,d\},\{j,k,d\}} a_{efg}$):

$$a_{ijd}\lambda_{ijd} + a_{idj}\lambda_{idj} + a_{jdi}\lambda_{jdi} + a_{ikd}\lambda_{ikd} + a_{idk}\lambda_{idk} + a_{kdi}\lambda_{kdi} + a_{jkd}\lambda_{jkd} + a_{jdk}\lambda_{jdk} + a_{kdj}\lambda_{kdj} = b_1. \tag{29}$$

Furthermore having identities (26)–(28), equality (29) can be written as a linear combination of equalities (7) for $\{i, j, d\}$, $\{i, k, d\}$, and $\{j, k, d\}$ (with coefficients a_{idj}, a_{idk}, and a_{jdk}, respectively) as well as the equality

$$(a_{ijd} - a_{idj})\lambda_{ijd} + (a_{ikd} - a_{idk})\lambda_{ikd} + (a_{jkd} - a_{jdk})\lambda_{jkd} = b_2, \tag{30}$$

where $b_2 = b_1 - a_{idj} - a_{idk} - a_{jdk}$. This means any point in P' must satisfy (30) (because it satisfies (21) and equalities (7)). In particular for any arbitrary $\pi^1 \in \Pi_{N\setminus\{d,i\}}$, the vector $\lambda^{i\pi^1 d}$ is in P' and hence satisfies (30). If we substitute it in (30), we get $b_2 = 0$. The vector $\lambda^{id\pi^1}$ also belongs to P'. Substituting this vector in (30) gives

$$-(a_{ijd} - a_{idj}) = a_{ikd} - a_{idk}. \tag{31}$$

Also for any arbitrary $\pi^2 \in \Pi_{N\setminus\{d,i,k\}}$, the vector $\lambda^{ikd\pi^2}$ is in P' and hence satisfies (30). Substituting this vector in (30) gives

$$-(a_{ijd} - a_{idj}) = a_{jkd} - a_{jdk}. \tag{32}$$

Using identities (31) and (32) and the fact that $b_2 = 0$, equality (30) reduces to

$$(a_{ijd} - a_{idj})(-\lambda_{ijd} + \lambda_{ikd} + \lambda_{jkd}) = 0. \tag{33}$$

Therefore, (33) is equality (20) multiplied by $a_{ijd} - a_{idj}$. So we have shown that (21) is a linear combination of (20) and the hyperplanes (7). This concludes the proof. □

We mentioned that inequality (12) is a generalization of inequalities (8), (9), or (10). It turns out that this generalized inequality is also facet-defining. We prove this in Theorem 7 below; but first we prove the following lemma about a property of permutations that satisfy (12) at equality as we need it in proving Theorem 7.

Lemma 6. *Consider inequality (12) for given β, S, S_1, S_2 and d. Let $\pi \in \Pi_N$, and γ_1 and γ_2 be the number of departments in S_1 and S_2 which are to the left of d in π, respectively. Then, $\lambda^\pi \in P^1$ satisfies (12) at equality if and only if $\gamma_1 - \gamma_2 = 0$ or 1.*

Proof. Let $|S_1| = \alpha$. Hence, $|S_2| = \alpha - 1$. The number of departments in S_1 and S_2 to the left of d in π is γ_1 and γ_2, respectively; therefore, the number of departments in S_1 and S_2 to the right of d is $\alpha - \gamma_1$ and $\alpha - 1 - \gamma_2$, respectively. Now it is easy to see that on the left-hand side of (12), the first summation is equal to $\gamma_1(\alpha - \gamma_1)$ and the second summation is equal to $\gamma_2(\alpha - 1 - \gamma_2)$. Also the summation on the right-hand side of (12) is equal to $\gamma_1(\alpha - 1 - \gamma_2) + \gamma_2(\alpha - \gamma_1)$. So the validity of (12) is equivalent to the validity of

$$\gamma_1(\alpha - \gamma_1) + \gamma_2(\alpha - 1 - \gamma_2) \le \gamma_1(\alpha - 1 - \gamma_2) + \gamma_2(\alpha - \gamma_1).$$

This of course reduces to

$$(\gamma_1 - \gamma_2) \le (\gamma_1 - \gamma_2)^2, \tag{34}$$

which is trivial (and hence proves the validity of (12)). Now see that (34) is satisfied at equality if and only if $\gamma_1 - \gamma_2 = 0$ or 1, which means λ^π satisfies (12) at equality if and only if $\gamma_1 - \gamma_2 = 0$ or 1. □

Theorem 7. *Any of inequalities (12) is facet-defining for $conv(P^1)$.*

Proof. Consider inequality (12) for given β, S, S_1, S_2 and d. This proof is a generalization of the proof of Theorem 5 (in fact we had $S_1 = \{i, j\}$ and $S_2 = \{k\}$ in Theorem 5). Let P' be the face of $conv(P^1)$ defined by (12). Therefore, for every point in P', (12) is satisfied at equality, i.e.

$$\sum_{p,q \in S_1 : p<q} \lambda_{pqd} + \sum_{p,q \in S_2 : p<q} \lambda_{pqd} - \sum_{p \in S_1, q \in S_2} \lambda_{pqd} = 0. \tag{35}$$

Similar to Theorem 5, we need to show that any hyperplane like

$$\sum_{e,f,g \in N : e<f} a_{efg} \lambda_{efg} = b \tag{36}$$

that passes through P' is a linear combination of hyperplanes (7) and hyperplane (35). First notice that as a generalization of (22) we prove the following identity:

$$a_{efg} = a_{egf} = a_{fge} \quad \text{for any } e, f, g \text{ such that } d \notin \{e, f, g\} \text{ or } \{e, f, g\} \not\subset S. \tag{37}$$

To prove this see that the following cases are possible: (i) $d \notin \{e, f, g\}$; (ii) $d \in \{e, f, g\}$ and $(\{e, f, g\} \setminus d) \cap S = \emptyset$ or $\{i\}$, where $i \in S_1$; (iii) $d \in \{e, f, g\}$ and $(\{e, f, g\} \setminus d) \cap S = \{k\}$, where $k \in S_2$. The arguments for these three cases are very similar to the arguments for cases (i)–(iii) in the proof of Theorem 5, respectively. The λ vectors used are exactly the same and the reason why they satisfy (35) is Lemma 6 because in all given permutations $\gamma_1 - \gamma_2 = 0$ or 1. In case (iii), the i that is used in the proof of Theorem 5 represents any arbitrary member of S_1.

Moreover for any $p, q \in S_1$ and any arbitrary permutation $\pi^1 \in \Pi_{N \setminus \{p,q,d\}}$, the vectors $\lambda^{pq\pi^1 d}$, $\lambda^{qp\pi^1 d}$, $\lambda^{dpq\pi^1}$, and $\lambda^{dqp\pi^1}$ satisfy (35) by Lemma 6, so they must satisfy (36). Therefore by Lemma 1,

$$a_{pdq} = a_{qdp} \quad \text{for all } p, q \in S_1. \tag{38}$$

By a similar argument, we also have

$$a_{sdt} = a_{tds} \quad \text{for all } s, t \in S_2, \tag{39}$$

$$a_{pds} = a_{sdp} \quad \text{for all } p \in S_1, s \in S_2. \tag{40}$$

Now observe that having identities (38)–(40), hyperplane (36) can be written as a linear combination of equalities (7) as well as the equality:

$$\sum_{p,q \in S_1,\ p<q} (a_{pqd} - a_{pdq})\lambda_{pqd} + \sum_{s,t \in S_2,\ s<t} (a_{std} - a_{sdt})\lambda_{std} + \sum_{p \in S_1,\ s \in S_2} (a_{psd} - a_{pds})\lambda_{psd} = b_1. \tag{41}$$

Now for any arbitrary $\pi^1 \in \Pi_{N \setminus \{d\}}$, $\lambda^{\pi^1 d}$ is in P'. Substituting this vector in (41) gives $b_1 = 0$. Moreover, for any $p, q \in S_1$, $s \in S_2$ and arbitrary $\pi^2 \in \Pi_{N/\{d,p,q,s\}}$, the vector $\lambda^{pdqs\pi^2}$ belongs to P'. Substituting this vector in (41) gives

$$a_{pqd} - a_{pdq} = -(a_{psd} - a_{pds}) \quad \text{for all } p, q \in S_1, s \in S_2. \tag{42}$$

Also for any $p \in S_1$, $s, t \in S_2$ and arbitrary $\pi^3 \in \Pi_{N \setminus \{d,i,k\}}$, the vector $\lambda^{ptds\pi^3}$ is in P'. Substituting this vector in (41) gives

$$a_{std} - a_{sdt} = -(a_{psd} - a_{pds}) \quad \text{for all } p \in S_1, s, t \in S_2\,. \tag{43}$$

Identities (42) and (43) imply that all coefficients in equality (41) are equal. Let the constant K denote their common value. Therefore, (41) reduces to

$$K\left(\sum_{p,q\in S_1,\ p<q} \lambda_{pqd} + \sum_{s,t\in S_2,\ s<t} \lambda_{std} - \sum_{p\in S_1,\ s\in S_2} \lambda_{psd}\right) = 0. \tag{44}$$

Therefore, (44) is equality (35) multiplied by K. So we have shown that (36) is a linear combination of (35) and the hyperplanes (7). This concludes the proof. □

Corollary 8. *Inequalities (12), written for ζ instead of λ, and inequalities (3) are facet-defining for $conv(P)$. Also the projections of inequalities (8)–(10) and (12) for P^2 are facet-defining for $conv(P^2)$.*

Proof. The proof is a direct result of Theorems 5 and 7 and Lemma 3 applied to the faces defined by these inequalities. □

5. Conclusions

We proved that the convex hulls of the triplet formulation for SRFLP and its projections [3] are of dimension $n(n-1)(n-2)/3$, where n is the number of departments. We also showed that many valid inequalities presented in [3] for this polytope are facet-defining. Our result provides a theoretical support for the fact that the LP solution over these valid inequalities gives the optimal solution for all instances studied in [3]. A possible direction for future research is to develop new classes of valid inequalities and facets for the triplet polytope.

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