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Improvement of Newman inequality ☆

Tingfan Xie [a], Xinlong Zhou [b,*]

[a] *Department of Mathematics, China Jiliang University, 31000 Hangzhou, China*
[b] *Department of Mathematics, University of Duisburg-Essen, 47048 Duisburg, Germany*

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Abstract

Let $a = \exp(-1/\sqrt{n})$. Newman inequality is

$$\prod_{k=1}^{n-1} \frac{1-a^k}{1+a^k} < e^{-\sqrt{n}}, \quad \forall n \geqslant 5.$$

We prove in this paper that

$$\prod_{k=1}^{s-1} \frac{1-a^k}{1+a^k} = n^{\frac{1}{4}} e^{-\frac{\pi^2}{4}\sqrt{n}+\mathcal{O}(1)}, \quad \forall s \geqslant n,$$

which will be applied to improve the estimate concerning the approximation of $|x|$ by using Newman's construction.

Keywords: Newman inequality; Rational approximation

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* Corresponding author.
E-mail addresses: xietf@cjlu.edu.cn (T. Xie), zhou@math.uni-duisburg.de (X. Zhou).

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1. Introduction

Let $a = \exp(-1/\sqrt{n})$. In 1964 Newman established the following well-known inequality:

$$\prod_{k=1}^{n-1} \frac{1-a^k}{1+a^k} < e^{-\sqrt{n}}, \quad \forall n \geqslant 5. \tag{1.1}$$

Using this inequality Newman proved

$$\max_{|x|\leqslant 1} \big| |x| - N_n(x) \big| \leqslant 3e^{-\sqrt{n}}, \quad \forall n \geqslant 5, \tag{1.2}$$

where the rational functions $N_n(x)$ are given by

$$N_n(x) = x\frac{P(x) - P(-x)}{P(x) + P(-x)} \quad \text{and} \quad P(x) = \prod_{k=1}^{n-1} \big(a^k + x\big).$$

Because of the simple construction of N_n Newman's approach was used widely to construct interesting rational functions in approximation theory (see [1–4,6] and the papers cited there).

Recently we find out that the right-hand side of (1.1) can be replaced by a smaller number, thus (see [7]), there holds

$$\prod_{k=1}^{n-1} \frac{1-a^k}{1+a^k} < C'e^{-1.3\sqrt{n}}, \tag{1.3}$$

where $C' > 0$ is an absolute constant. Using this inequality, we obtain the asymptotic express of $\max_{|x|\leqslant 1} ||x| - N_n(x)|$ (see [7]):

$$\max_{|x|\leqslant 1} \big| |x| - N_n(x) \big| = \frac{A}{\sqrt{n}} e^{-\sqrt{n}} + \mathcal{O}\left(\frac{1}{n} e^{-\sqrt{n}}\right), \tag{1.4}$$

where $A = \max_{0\leqslant t\leqslant\infty} t/(1+e^t)$. Therefore, the exact approximation rate for $|x|$ by $N_n(x)$ is $e^{-\sqrt{n}}/\sqrt{n}$.

Comparing (1.1) and (1.3) with Newman's approach it is natural to ask what is the exact order in (1.3) and whether can we modify this term to improve the approximation rate of $|x|$. The aim of this paper is to answer these questions. For this goal let us denote

$$\Delta_s = \prod_{k=1}^{s-1} \frac{1-a^k}{1+a^k}, \quad \forall s = n, n+1, \ldots.$$

Let further

$$N_{n,s}(x) = x\frac{P_s(x) - P_s(-x)}{P_s(x) + P_s(-x)} \quad \text{and} \quad P_s(x) = \prod_{k=1}^{s-1} \big(a^k + x\big).$$

The main result of this paper is

Theorem 1.1. *For all $s \geqslant n \geqslant 1$ there holds*

$$\Delta_s = n^{\frac{1}{4}} \exp\left(-\frac{\pi^2}{4}\sqrt{n} + \mathcal{O}(1)\right), \tag{1.5}$$

where $\mathcal{O}(1)$ does not depends on s and n. Consequently,

$$\max_{|x|\leqslant 1} \big||x| - N_{n,s}(x)\big| \sim \frac{1}{\sqrt{n}} e^{-\frac{s}{\sqrt{n}}} + n^{\frac{1}{4}} e^{-\frac{\pi^2\sqrt{n}}{4}}, \tag{1.6}$$

where $B_1 \sim B_2$ means that there is an absolute positive constant C such that $C^{-1}B_2 \leqslant B_1 \leqslant CB_2$.

Therefore, (1.3) can be improved by

$$\prod_{k=1}^{n-1} \frac{1-a^k}{1+a^k} = n^{\frac{1}{4}} \exp\left(-\frac{\pi^2}{4}\sqrt{n} + \mathcal{O}(1)\right).$$

Moreover, we can modify Newman's function $N_n(x)$ to get better approximation rate. Indeed, for suitable s the estimate (1.6) is better than (1.4). In particular, for $s = vn$ we have

Corollary 1.2. *There holds*

$$\max_{-1\leqslant x\leqslant 1} \big||x| - N_{n,2n}(x)\big| = \mathcal{O}(1)e^{-2\sqrt{n}}.$$

Moreover,

$$\max_{-1\leqslant x\leqslant 1} \big||x| - N_{n,vn}(x)\big| = \mathcal{O}(1)n^{\frac{1}{4}} e^{-\frac{\pi^2}{4}\sqrt{n}}, \quad \forall v = 3, 4, \ldots.$$

We notice that if we use $N_{vn}(x)$ to approximate $|x|$ then (1.4) tells us that the exact approximation rate is $\exp(-\sqrt{vn})/\sqrt{vn}$. Therefore, for $2 \leqslant v < \pi^4/16$ (i.e., $v = 2, 3, \ldots, 6$) the rational function $N_{n,vn}(x)$ is better than $N_{vn}(x)$ for the approximation of $|x|$. We notice also that the degree for both $N_{n,vn}$ and N_{vn} as well as the construction are the same. However, this modification is "saturated." The best approximation rate by this approach is $n^{1/4}\exp(-\pi^2\sqrt{n}/4)$, which can be reached if

$$\frac{1}{\sqrt{n}} e^{-\frac{s}{\sqrt{n}}} = \mathcal{O}\left(n^{\frac{1}{4}} e^{-\frac{\pi^2\sqrt{n}}{4}}\right).$$

In other words, for those s there holds

$$\max_{|x|\leqslant 1} \big||x| - N_{n,s}(x)\big| \sim n^{\frac{1}{4}} e^{-\frac{\pi^2\sqrt{n}}{4}}.$$

We will prove Theorem 1.1 in the next section.

2. Proof of Theorem 1.1

We need the following lemma.

Lemma 2.1. *Let $a = \exp(-1\sqrt{n})$, $n = 1, 2, \ldots,$ and $s \geqslant n$. Then there holds*

$$\Delta_s = \exp\left(-2\sum_{m=1}^{\infty}\frac{a^{2m+1}}{(2m+1)(1-a^{2m+1})} + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

where $\mathcal{O}$ does not depend on s.

Proof. It is known that for $0 < t < 1$ one has

$$\ln\frac{1-t}{1+t} = -2t - \frac{2}{3}t^3 - \frac{2}{5}t^5 - \cdots.$$

Hence,

$$\frac{1-t}{1+t} = \exp\left(-2\sum_{m=0}^{\infty}\frac{1}{2m+1}t^{2m+1}\right),$$

which in turn implies

$$\Delta_s = \prod_{k=1}^{s-1}\frac{1-a^k}{1+a^k} = \exp\left(-2\sum_{m=0}^{\infty}\frac{1}{2m+1}\sum_{k=1}^{s-1}a^{k(2m+1)}\right). \tag{2.1}$$

It is clear that

$$\sum_{k=1}^{s-1}a^{k(2m+1)} = \frac{a^{2m+1} - a^{s(2m+1)}}{1-a^{2m+1}}. \tag{2.2}$$

Thus, replacing a by $\exp(-1/\sqrt{n})$, we conclude

$$\frac{a^{s(2m+1)}}{1-a^{2m+1}} \leqslant \frac{\sqrt{n}}{2m+1}e^{-(s-1)\frac{2m+1}{\sqrt{n}}}.$$

Now as $s \geqslant n$ we obtain

$$e^{-(s-1)\frac{2m+1}{\sqrt{n}}} = \mathcal{O}\big(n^{-2}(2m+1)^{-4}\big).$$

In particular, one has

$$\frac{a^{s(2m+1)}}{1-a^{2m+1}} = \mathcal{O}\left(\frac{1}{n(2m+1)}\right). \tag{2.3}$$

The assertion follows from (2.1)–(2.3). □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We first verify (1.5). According to Lemma 2.1, we need only to calculate

$$\sum_{m=0}^{\infty}\frac{a^{2m+1}}{(2m+1)(1-a^{2m+1})}.$$

To this end we observe first the terms with $m \leqslant \sqrt{n}$. By Taylor's formula, we conclude

$$\frac{1}{e^{\frac{2m+1}{\sqrt{n}}}-1}=\frac{\sqrt{n}}{2m+1}\times\frac{1}{1+\frac{2m+1}{2\sqrt{n}}+\frac{1}{6}\left(\frac{2m+1}{\sqrt{n}}\right)^2+\cdots}$$
$$=\frac{\sqrt{n}}{2m+1}\left\{1-\frac{2m+1}{2\sqrt{n}}+\mathcal{O}\left(\left(\frac{2m+1}{\sqrt{n}}\right)^2\right)\right\}.$$

Thus,

$$\sum_{m=0}^{[\sqrt{n}]}\frac{a^{2m+1}}{(2m+1)(1-a^{2m+1})}=\sum_{m=0}^{[\sqrt{n}]}\frac{\sqrt{n}}{(2m+1)^2}-\frac{1}{2}\sum_{m=0}^{[\sqrt{n}]}\frac{1}{2m+1}+\mathcal{O}(1).$$

On the other hand, it is easy to see that

$$\frac{1}{2}\sum_{m=0}^{[\sqrt{n}]}\frac{1}{2m+1}=\frac{1}{8}\ln n+\mathcal{O}(1).$$

Therefore, we get

$$\sum_{m=0}^{[\sqrt{n}]}\frac{a^{2m+1}}{(2m+1)(1-a^{2m+1})}=\sum_{m=0}^{[\sqrt{n}]}\frac{\sqrt{n}}{(2m+1)^2}-\frac{1}{8}\ln n+\mathcal{O}(1)$$
$$=\sum_{m=0}^{\infty}\frac{\sqrt{n}}{(2m+1)^2}-\frac{1}{8}\ln n+\mathcal{O}(1)$$
$$=\frac{\pi^2}{8}\sqrt{n}-\frac{1}{8}\ln n+\mathcal{O}(1). \tag{2.4}$$

Next, we consider the terms with $m>\sqrt{n}$. Clearly, in this case one has always

$$\frac{1}{e^{\frac{2m+1}{\sqrt{n}}}-1}=\mathcal{O}\left(\frac{\sqrt{n}}{2m+1}\right).$$

Hence,

$$\sum_{m=[\sqrt{n}]+1}^{\infty}\frac{a^{2m+1}}{(2m+1)(1-a^{2m+1})}=\mathcal{O}(1). \tag{2.5}$$

We conclude from (2.4) and (2.5) that

$$\sum_{m=0}^{\infty}\frac{a^{2m+1}}{(2m+1)(1-a^{2m+1})}=\frac{\pi^2}{8}\sqrt{n}-\frac{1}{8}\ln n+\mathcal{O}(1).$$

The desired assertion follows from Lemma 2.1 and the last estimate.

To show (1.6), we use Newman's approach (see [5]) to obtain

$$\max_{a^s\leqslant x\leqslant 1}\left|\prod_{k=1}^{s-1}\frac{a^k-x}{a^k+x}\right|=\prod_{k=1}^{s-1}\frac{1-a^k}{1+a^k}.$$

Hence,

$$\max_{a^s\leqslant x\leqslant 1}\left|x\prod_{k=1}^{s-1}\frac{a^k-x}{a^k+x}\right|=\prod_{k=1}^{s-1}\frac{1-a^k}{1+a^k}. \tag{2.6}$$

But $N_{n,s}(x)$ is an even function, so there is

$$\max_{a^s\leqslant |x|\leqslant 1}\big||x|-N_{n,s}(x)\big|=\max_{a^s\leqslant x\leqslant 1}\big||x|-N_{n,s}(x)\big|=\max_{a^s\leqslant x\leqslant 1}\frac{2x|P_s(-x)|}{|P_s(x)+P_s(-x)|}$$

$$=\max_{a^s\leqslant x\leqslant 1}\frac{2x\left|\prod_{k=1}^{s-1}\frac{a^k-x}{a^k+x}\right|}{\left|1+\prod_{k=1}^{s-1}\frac{a^k-x}{a^k+x}\right|}.$$

It follows from (2.6) that

$$\frac{2\prod_{k=1}^{s-1}\frac{1-a^k}{1+a^k}}{1+\prod_{k=1}^{s-1}\frac{1-a^k}{1+a^k}}\leqslant \max_{a^s\leqslant |x|\leqslant 1}\big||x|-N_{n,s}(x)\big|\leqslant \frac{2\prod_{k=1}^{s-1}\frac{1-a^k}{1+a^k}}{1-\prod_{k=1}^{s-1}\frac{1-a^k}{1+a^k}}.$$

From (1.5) we conclude

$$\max_{a^s\leqslant |x|\leqslant 1}\big||x|-N_{n,s}(x)\big|=n^{\frac{1}{4}}e^{-\frac{\pi^2}{4}\sqrt{n}+\mathcal{O}(1)}. \tag{2.7}$$

Next we estimate $\max_{|x|\leqslant a^s}||x|-N_{n,s}(x)|$. Thus, for $0\leqslant x\leqslant a^s$ we have

$$\big||x|-N_{n,s}(x)\big|=\frac{2x}{1+\prod_{k=1}^{s-1}\frac{a^k+x}{a^k-x}}. \tag{2.8}$$

On the other hand, using

$$\ln\frac{1+t}{1-t}=2t+\frac{2}{3}t^3+\frac{2}{5}t^5+\cdots,$$

we obtain

$$\prod_{k=1}^{s-1}\frac{a^k+x}{a^k-x}=\exp\left(2\sum_{m=0}^{\infty}\frac{1}{2m+1}\sum_{k=1}^{s-1}a^{-k(2m+1)}x^{2m+1}\right). \tag{2.9}$$

Denote

$$y=2\sum_{m=0}^{\infty}\frac{1}{2m+1}\sum_{k=1}^{s-1}a^{-k(2m+1)}x^{2m+1}.$$

Obviously,

$$y\geqslant 2x\sum_{k=1}^{s-1}a^{-k}.$$

Thus, it follows from (2.8) and (2.9) that for $0\leqslant x\leqslant a^s$,

$$\big||x| - N_{n,s}(x)\big| \leqslant \frac{\frac{y}{1+e^y}}{\sum_{k=1}^{s-1} a^{-k}} = \frac{y}{1+e^y} \times \frac{e^{\frac{1}{\sqrt{n}}} - 1}{e^{\frac{s}{\sqrt{n}}} - e^{\frac{1}{\sqrt{n}}}}.$$

Clearly,

$$e^{\frac{1}{\sqrt{n}}} - 1 = \frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right) \quad \text{and} \quad \max_{0\leqslant y\leqslant\infty} \frac{y}{1+e^y} = \frac{y_0}{1+e^{y_0}},$$

where $y_0 = 1.2784\ldots$. Hence, for $A = y_0/(1+e^{y_0})$ we get

$$\max_{0\leqslant x\leqslant a^s} \big||x| - N_{n,s}(x)\big| \leqslant \frac{A}{\sqrt{n}} e^{-\frac{s}{\sqrt{n}}} + \mathcal{O}\left(\frac{1}{n} e^{-\frac{s}{\sqrt{n}}}\right). \tag{2.10}$$

Moreover, if x_0 satisfies

$$2\left(x_0 \sum_{k=1}^{s-1} a^{-k} + x_0^3 \sum_{k=1}^{s-1} a^{-3k}\right) = y_0,$$

then $x_0 = \mathcal{O}(1)\exp(-\frac{s}{\sqrt{n}})/\sqrt{n}$. Hence, there exists n_0 such that for all $n > n_0$,

$$x_0 e^{\frac{s}{\sqrt{n}}} < \frac{1}{2}. \tag{2.11}$$

But $(\ln(1-t)/(1+t)) < 2(t+t^2)$ for $0 < t < 1/2$. Therefore, we obtain for $n > n_0$,

$$\prod_{k=1}^{s-1} \frac{a^k + x_0}{a^k - x_0} \leqslant \exp\left(2x_0 \sum_{k=1}^{s-1} a^{-k} + 2x_0^3 \sum_{k=1}^{s-1} a^{-3k}\right).$$

Combining the last inequality with (2.8), we obtain

$$\big||x_0| - N_{n,s}(x_0)\big| \geqslant \frac{A}{\sum_{k=1}^{s-1} a^{-k} + x_0^2 \sum_{k=1}^{s-1} a^{-3k}},$$

while (2.11) implies

$$\sum_{k=1}^{s-1} a^{-k} + x_0^2 \sum_{k=1}^{s-1} a^{-3k} \leqslant \frac{e^{\frac{s}{\sqrt{n}}} + x_0^2 e^{\frac{3s}{\sqrt{n}}}}{e^{\frac{1}{\sqrt{n}}} - 1}.$$

Therefore,

$$\big||x_0| - N_{n,s}(x_0)\big| \geqslant A \frac{e^{\frac{1}{\sqrt{n}}} - 1}{e^{\frac{s}{\sqrt{n}}} + x_0^2 e^{\frac{3s}{\sqrt{n}}}} = \frac{A}{\sqrt{n}} e^{-\frac{s}{\sqrt{n}}} + \mathcal{O}(1)\frac{1}{n} e^{-\frac{s}{\sqrt{n}}}.$$

It follows from this inequality and (2.10) that

$$\max_{0\leqslant x\leqslant a^s} \big||x| - N_{n,s}(x)\big| = \frac{A}{\sqrt{n}} e^{-\frac{s}{\sqrt{n}}} + \mathcal{O}(1)\frac{1}{n} e^{-\frac{s}{\sqrt{n}}}.$$

The desired assertion follows from (2.7) and the last inequality. □

From the above proof one can easily obtain an analogue of (1.4).

Corollary 2.2. *Let $n \leqslant s < \pi^2 n/4 - 5/4\sqrt{n}\ln n$. Then*

$$\max_{|x|\leqslant 1}\big||x| - N_{n,s}(x)\big| = \frac{A}{\sqrt{n}}e^{-\frac{s}{\sqrt{n}}} + \mathcal{O}(1)\frac{1}{n}e^{-\frac{s}{\sqrt{n}}}.$$

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References

[1] L. Bruteman, E. Passow, On rational interpolation to $|x|$, Constr. Approx. 13 (1997) 381–391.
[2] X. Han, On the order of approximation for the rational interpolation to $|x|$, Approx. Theory Appl. 18 (2002) 58–64.
[3] G.L. Ilive, U. Opitz, Comonotone approximation of $|x|$ by rational functions, Serdica Bulgar. Math. Publ. 10 (1984) 88–105.
[4] A.L. Levin, V.V. Maimeskul, E.B. Saff, Rational approximation with geometric rate, J. Approx. Theory 92 (1998) 308–330.
[5] D. Newman, Rational approximation to $|x|$, Michigan Math. J. 11 (1964) 11–14.
[6] T.F. Xie, S.P. Zhou, Approximation Theory of Real Functions, Hangzhou Univ. Press, Hangzhou, 1998 (in Chinese).
[7] T.F. Xie, S.P. Zhou, The asymptotic property of approximation to $|x|$ by Newman's rational operators, Acta Math. Hungar. 103 (2004) 313–319.