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On certain subclasses of meromorphic functions associated with certain differential operators

E.A. Elrifai, H.E. Darwish *, A.R. Ahmed

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt

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ABSTRACT

In this work, we study some subordination and convolution properties of certain subclasses of meromorphic functions which are defined by a previously mentioned differential operator.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the punctured unit disk $U^* := \{z : 0 < |z| < 1\} = U \setminus \{0\}$, with a simple pole at the origin.

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. If $g(z)$ is univalent in U, then the following equivalence relationship holds true:

$$f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For functions $f(z) \in \Sigma$ given by (1.1) and $g(z) \in \Sigma$ defined by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.3}$$

* Corresponding author.
E-mail addresses: Rifai@mans.edu.eg (E.A. Elrifai), Darwish333@yahoo.com (H.E. Darwish), Abdusalam5056@yahoo.com (A.R. Ahmed).

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In analogy with the operator defined by Frasin and Darus [1] on the normalized analytic functions, the differential operator I^n is now defined as follows:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = zf'(z) + \frac{2}{z},$$

$$I^2 f(z) = z(I^1 f(z))' + \frac{2}{z},$$

and for $n = 1, 2, 3, \ldots$,

$$I^n f(z) = z(I^{n-1} f(z))' + \frac{2}{z}$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} k^n a_k z^k. \quad (n \in N_0). \tag{1.4}$$

The operator I^n was introduced by Frasin and Darus [1] and studied by El-Ashwah and Aouf [2].

Now, we introduce the following subclasses of Σ associated with the differential operator $I^n f(z)$.

Definition 1. For fixed parameters A, B $(-1 \le B < A \le 1)$, a function $f(z) \in \Sigma$ is said to be in the class $\Sigma_n(\lambda, A, B)$ if

$$-z^2 \left\{(1+\lambda)(I^n f(z))' + \lambda(I^{n+1} f(z))'\right\} \prec 2\lambda + \frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.5}$$

where $n \in N$and $\lambda \ge 0$.

In this work, we derive some subordination results for the class $\Sigma_n(\lambda, A, B)$ and investigate several convolution properties of functions which have been defined here by means of the differential operator $I^n f(z)$.

2. Preliminaries

To prove our main result, we need the following lemmas.

Lemma 1 (*[3]; see also [4]*)**.** *Let $\phi(z)$ be analytic in U and $h(z)$ be analytic and convex (univalent) in U with $h(0) = \phi(0) = 1$. If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad (\mathrm{Re}(\gamma) \ge 0;\ \gamma \ne 0; z \in U), \tag{2.1}$$

then

$$\phi(z) \prec \Psi(z) = \gamma z^{-\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z) \quad (z \in U),$$

and $\Psi(z)$ is the best dominant of (2.1).

We denote by $P(\gamma)$ the class of functions $\phi(z)$ given by

$$\phi(z) = 1 + b_1 z + b_2 z^2 + \cdots, \tag{2.2}$$

which are analytic in U and satisfy the following inequality:

$$\mathrm{Re}(\phi(z)) > \gamma \quad (0 \le \gamma < 1, z \in U).$$

Lemma 2 (*[5]*)**.** *Let the function $\phi(z)$ given by (2.2) be in the class $P(0)$. Then*

$$\mathrm{Re}(\phi(z)) \ge \frac{1-|z|}{1+|z|} \quad (z \in U).$$

Lemma 3 (*[6]*)**.** *If $\phi_j \in P(\gamma_j)$ $(0 \le \gamma_j < 1; j = 1, 2)$, then $\varphi_1 * \varphi_2 \in P(\gamma_3)$, $\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)$.*
The result is best possible.

Lemma 4 (*[7]*)**.** *For real or complex numbers a, b and c $(c \ne 0, -1, -2, \ldots)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b, c; z) \quad (\mathrm{Re}(c) > \mathrm{Re}(b) > 0); \tag{2.3}$$

$${}_2F_1(a, b, c; z) = (1-z)^{-a} \quad {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{2.4}$$

and

$$ {}_2F_1\left(a, b; \frac{a+b+1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}. \tag{2.5} $$

3. The main results

Unless otherwise mentioned, we assume throughout this work that

$n \in N, \lambda > 0$ and $-1 \le B < A \le 1$.

Theorem 1. *Let $f(z)$ defined by (1.1) be in the class $\Sigma_n(\lambda, A, B)$; then*

$$ -z^2(I^n f(z))' \prec Q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{3.1} $$

where the function $Q(z)$ given by

$$ Q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1, 1, \frac{1}{\lambda}+1; \frac{Bz}{Bz+1}\right) & (B \neq 0) \\ 1 + \frac{1}{1+\lambda}Az & (B = 0), \end{cases} $$

is the best dominant of (3.1). Furthermore,

$$ \mathrm{Re}\left\{z^2(I^n f(z))'\right\} > \rho \quad (z \in U), \tag{3.2} $$

where

$$ \rho(\lambda, A, B) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1, 1, \frac{1}{\lambda}+1; \frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{1}{1+\lambda}A & (B = 0). \end{cases} $$

The result is best possible.

Proof. Set

$$ \phi(z) = -z^2(I^n f(z))' \quad (z \in U). \tag{3.3} $$

Then the function $\phi(z)$ is analytic in U with $\phi(0) = 1$. Differentiating (3.3) and with the aid of the identity (1.4), we get

$$ -z^2(I^{n+1} f(z))' = z\phi'(z) - \phi(z) + 2. $$

Now by (1.5), we get

$$ -z^2\left\{(1+\lambda)(I^n f(z))' + \lambda(I^{n+1} f(z))'\right\} - 2\lambda = \phi(z) + \lambda z\phi'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U). \tag{3.4} $$

Now, by using Lemma 1 for $\gamma = \frac{1}{\lambda}$, we deduce that

$$ \begin{aligned} \phi(z) \prec Q(z) &= \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1}\left(\frac{1+At}{1+Bt}\right) dt \\ &= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1, 1, \frac{1}{\lambda}+1; \frac{Bz}{Bz+1}\right) & (B \neq 0) \\ 1 + \frac{1}{1+\lambda}Az & (B = 0), \end{cases} \end{aligned} $$

by a change of variables followed by the use of identities (2.3) and (2.4) $\left(\text{with } a = 1, b = \frac{1}{\lambda}, c = b + 1\right)$.

This proves the assertion (3.1) of Theorem 1.

Next, to prove the assertion (3.2) of Theorem 1, it suffices to show that

$$ \inf_{|z|<1}\{\mathrm{Re}(Q(z))\} = Q(-1). \tag{3.5} $$

Indeed, for $|z| \le r < 1$,

$$ \mathrm{Re}\left(\frac{1+Az}{1+Bz}\right) \ge \frac{1-Ar}{1-Br}. $$

Setting

$$G(s,z)=\frac{1+Asz}{1+Bsz} \quad \text{and} \quad d\mu(s)=\frac{1}{\lambda}s^{\frac{1}{\lambda}-1}ds \quad (0\le s\le 1),$$

which is a positive measure on [0, 1], we get

$$Q(z)=\int_0^1 G(s,z)d\mu(s),$$

and so

$$\mathrm{Re}(Q(z))\ge\int_0^1\frac{1-Asr}{1-Bsr}d\mu(s)=Q(-r) \quad (|z|\le r<1).$$

Letting $r\longrightarrow 1^-$ in the above inequality, we obtain the assertion (3.5). The result in (3.2) is best possible as the function $Q(z)$ is the best dominant of (3.1). □

Remark 1. The left hand side of (1.5) is equivalent to

$$-z^2\left\{\lambda z(I^nf)''+(1+2\lambda)(I^nf)'\right\}$$

so (1.5) is equivalent to

$$-z^2\left\{\lambda z(I^nf)''+(1+2\lambda)(I^nf)'\right\}\prec\frac{1+Az}{1+Bz} \quad (z\in U). \tag{3.6}$$

Now putting $\lambda=\frac{\sigma}{1-2\sigma}$ $\left(0<\sigma<\frac{1}{2}\right)$ in Theorem 1, we get the following result:

Corollary 1. *If $f(z)\in\Sigma$ satisfies*

$$\frac{-z^2\left[(I^nf)'+\sigma z(I^nf)''\right]}{1-2\sigma}\prec\frac{1+Az}{1+Bz} \quad (z\in U), \tag{3.7}$$

then

$$-z^2(I^nf(z))'\prec Q(z)\prec\frac{1+Az}{1+Bz} \quad (z\in U), \tag{3.8}$$

where the function $Q(z)$ given by

$$Q(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1,\frac{1-\sigma}{\sigma},\frac{Bz}{Bz+1}\right) & (B\ne 0)\\ 1-\frac{1-2\sigma}{1-\sigma}Az \quad (B=0),\end{cases}$$

is the best dominant of (3.1). Furthermore,

$$\mathrm{Re}\left(-z^2\left(I^nf(z)\right)'\right)>\rho \quad (z\in U), \tag{3.9}$$

where

$$\rho(\lambda,A,B)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1,\frac{1-\sigma}{\sigma},\frac{B}{B-1}\right) & (B\ne 0),\\ 1-\frac{1-2\sigma}{\sigma}A \quad (B=0).\end{cases} \tag{3.10}$$

The result is best possible.

Taking $A=1-2\delta$ $(0\le\delta<1)$, $B=-1$ and $\lambda=1$ in Theorem 1, we have:

Corollary 2. *If $f(z)\in\Sigma$ satisfies the following inequality:*

$$\mathrm{Re}\left\{-z^2\left[3(I^nf(z))'+z(I^nf(z))''\right]\right\}>\delta \quad (0\le\delta<1;\ z\in U),$$

then

$$\operatorname{Re}\left\{-z^2(I^nf)'\right\} > 1+2(1-\delta)(\ln 2-1) \quad (z\in U). \tag{3.11}$$

The result is best possible.

Taking $A = 1-2\delta$ $(0\le\delta<1)$, $B=-1$ and $\lambda = 2$ in Theorem 1, and using (2.5) we have:

Corollary 3. *If $f(z)\in\Sigma$ satisfies the following inequality:*

$$\operatorname{Re}\left\{-z^2\left[5(I^nf(z))'+2z(I^nf(z))''\right]\right\} > -\frac{(\pi-2)}{4-\pi} \tag{3.12}$$

then

$$\operatorname{Re}\left\{-z^2(I^nf)'\right\} > 0. \tag{3.13}$$

The result is best possible.

Theorem 2. *If $f(z)\in\Sigma$ satisfies*

$$z\left\{(1+\lambda)I^nf(z)+\lambda I^{n+1}f(z)\right\}-2\lambda \prec \frac{1+Az}{1+Bz} \quad (z\in U), \tag{3.14}$$

then

$$zI^nf(z)\prec Q(z)\prec \frac{1+Az}{1+Bz} \quad (z\in U),$$

and

$$\operatorname{Re}(zI^nf(z)) > \rho \quad (z\in U),$$

where Q and ρ are given as in Theorem 1. The result is best possible.

Proof. Replace $\phi(z)$ by $zI^nf(z)$ in (3.3) and follow the lines of the proof of Theorem 1. □

Theorem 3. *Let $-1\le B_j<A_j\le 1$ $(j=1,2)$. If each of the functions $f_j(z)\in\Sigma$ satisfies the following subordination condition:*

$$z\left\{(1+\lambda)I^nf_j(z)+\lambda I^{n+1}f_j(z)\right\}-2\lambda \prec \frac{1+A_jz}{1+B_jz} \quad (j=1,2;\ z\in U) \tag{3.15}$$

and if $f(z)\in\Sigma$ is defined by

$$I^nf(z) = I^nf_1(z)*I^nf_2(z) \tag{3.16}$$

then

$$\operatorname{Re}\left\{(1+\lambda)zI^nf(z)+\lambda zI^{n+1}f(z)-2\lambda\right\} > \gamma, \tag{3.17}$$

where

$$\gamma = 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\,{}_2F_1\left(1,1,\frac{1}{\lambda}+1;\frac{1}{2}\right)\right].$$

The result is best possible when $B_1=B_2=-1$.

Proof. Suppose that each of the functions $f_j(z)\in\Sigma$ $(j=1,2)$ satisfies the condition (3.15). Then, on letting

$$\varphi_j(z) = (1+\lambda)zI^nf_j(z)+\lambda zI^{n+1}f_j(z)-2\lambda \quad (j=1,2) \tag{3.18}$$

we have

$$\varphi_j(z)\in P(\gamma_j)\left(\gamma_j=\frac{1-A_j}{1-B_j},\ j=1,2\right).$$

From (3.18), we have

$$I^nf_j(z) = \frac{1}{\lambda}z^{\frac{-1}{\lambda}}\int_0^z t^{\frac{1}{\lambda}-1}\varphi_j(t)dt \quad (j=1,2). \tag{3.19}$$

Now if $f(z) \in \Sigma$ is defined by (3.16), we find from (3.19) that

$$\begin{aligned} I^n f(z) &= I^n f_1(z) * I^n f_2(z) \\ &= \left(\frac{1}{\lambda} z^{\frac{-1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1} \varphi_1(t) dt\right) * \left(\frac{1}{\lambda} z^{\frac{-1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1} \varphi_2(t) dt\right) \\ &= \frac{1}{\lambda} z^{\frac{-1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1} \varphi_0(t) dt, \end{aligned} \tag{3.20}$$

where

$$\varphi_0(t) = \frac{1}{\lambda} z^{\frac{-1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1} (\varphi_1 * \varphi_2)(t) dt. \tag{3.21}$$

Since $\varphi_1(z) \in P(\gamma_1)$ and $\varphi_2(z) \in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(z) \in P(\gamma_3) \quad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)). \tag{3.22}$$

According to Lemma 2, we have

$$\mathrm{Re}(\varphi_1 * \varphi_2)(z) \geq \gamma_3 + (1-\gamma_3)\frac{1-|z|}{1+|z|} = (2\gamma_3 - 1) + \frac{2(1-\gamma_3)}{1+|z|}. \tag{3.23}$$

Now on using (3.23) in (3.21), we get

$$\begin{aligned} \mathrm{Re}\left\{(1+\lambda) z I^n f(z) + \lambda z I^{n+1} f(z) - 2\lambda\right\} &= \mathrm{Re}(\varphi_0(z)) \\ &= \frac{1}{\lambda} \int_0^1 u^{\frac{1}{\lambda}-1} \mathrm{Re}\left\{(\varphi_1 * \varphi_2)(uz)\right\} du \\ &\geq \frac{1}{\lambda} \int_0^1 u^{\frac{1}{\lambda}-1} \left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|}\right) du \\ &> \frac{1}{\lambda} \int_0^1 u^{\frac{1}{\lambda}-1} \left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right) du \\ &= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left[1 - \frac{1}{\lambda} \int_0^1 u^{\frac{1}{\lambda}-1} (1+u)^{-1} du\right] \\ &= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left[1 - \frac{1}{2}\, {}_2F_1\left(1, 1; \frac{1}{\lambda}+1, \frac{1}{2}\right)\right] \\ &= \gamma \quad (z \in U), \end{aligned}$$

where $2\gamma_3 - 1 = 1 - 4\left(\frac{A_1-B_1}{1-B_1}\right)\left(\frac{A_2-B_2}{1-B_2}\right)$ and $2(1-\gamma_3) = 4\left(\frac{A_1-B_1}{1-B_1}\right)\left(\frac{A_2-B_2}{1-B_2}\right)$ which completes the proof of assertion (3.17).

For $B_1 = B_2 = -1$, we consider the functions $f_j(z) \in \Sigma$ $(j = 1, 2)$ defined by

$$I^n f_j(z) = \frac{1}{\lambda} z^{\frac{-1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1} \left(\frac{1+A_j t}{1-t}\right) dt \quad (j = 1, 2)$$

for which we have

$$\varphi_j(z) = (1+\lambda) z I^n f_j(z) + \lambda z I^{n+1} f_j(z) - 2\lambda = \frac{1+A_j z}{1-z} \quad (j = 1, 2)$$

and

$$\begin{aligned} (\varphi_1 * \varphi_2)(z) &= \left(\frac{1+A_1 z}{1-z}\right) * \left(\frac{1+A_2 z}{1-z}\right) \\ &= 1 + \frac{(1+A_1)(1+A_2) z}{1-z}. \end{aligned}$$

Hence, for $f(z) \in \Sigma$ given by (3.16), we obtain

$$\begin{aligned} &\left\{(1+\lambda) z I^n f(z) + \lambda z I^{n+1} f(z) - 2\lambda\right\} = \varphi_0(z) \\ &\quad = \frac{1}{\lambda} \int_0^1 u^{\frac{1}{\lambda}-1} \left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right) du \end{aligned}$$

$$= 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1}\,{}_2F_1\left(1, 1, \frac{1}{\lambda} + 1, \frac{z}{z-1}\right)$$

$$\to 1 - (1 + A_1)(1 + A_2) + \frac{1}{2}(1 + A_1)(1 + A_2)\,{}_2F_1\left(1, 1, \frac{1}{\lambda} + 1; \frac{1}{2}\right)$$

as $z \to -1$

which evidently completes the proof of Theorem 3. □

Letting $A_j = 1 - \eta_j, B_j = -1\ (j = 1, 2)$ and $\tau = \lambda$ in Theorem 3, we get the following result.

Corollary 4. *If* $f(z) \in \Sigma$ *satisfies*

$$z\left\{(1 + \tau)(I^n f_j(z)) + \tau z(I^n f_j(z))'\right\} > \eta_j \quad (j = 1, 2, z \in U),$$

then

$$\text{Re}\left\{(1 + \tau)z(I^n f_1(z) * I^n f_2(z)) + \tau z(I^n f_1(z) * I^n f_2(z))'\right\} > \gamma, \tag{3.24}$$

where

$$\gamma = 1 - 4(1 - \eta_1)(1 - \eta_2)\left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1, \frac{1}{\tau} + 1, \frac{1}{2}\right)\right].$$

Remark 2. The technique used in this work is similar to that used by Lashin [8].

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