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On the nonexistence of three-dimensional tiling in the Lee metric II

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Abstract

We prove that there does not exist a tiling of R^3 with Lee spheres of radius greater than 0 such that the radius of at least one of them is greater than one.

1. Introduction

First, let us recall notations and definitions given in [3].

Let $(0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ be an orthogonal basis of n-dimensional space R^3, and let $X = (x_1, \ldots, x_n)$ be a point of R^n. The *n-cube* centered on X is defined by the set:

$$C(X) = \left\{ Y = (y_1, \ldots, y_n) \,|\, \forall i, y_i = \alpha_i + x_i \text{ with } \frac{-1}{2} \leqslant \alpha_i \leqslant \frac{1}{2} \right\}.$$

From the definition, it is clear that $C(X)$ is a convex closed set. Let $\operatorname{Int}(C(X))$ denotes the set of inner points of the n-cube $C(X)$.

The *Lee distance* between two points $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ of Z^n, is defined by

$$d(X, Y) = \sum_{i=1}^{n} |x_i - y_i|.$$

Let r be a nonnegative integer. The *r-Lee sphere* in R^n centered on 0 of *major axes* $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, is the set of n-cubes $C(Y)$ where $d(0, Y) \leqslant r$ and Y has integer coordinates. The *border* of an r-Lee sphere, is the set of n-cubes $C(Y)$ where $d(0, Y) = r$ and Y has integer coordinates. More generally, an r-Lee sphere in R^n centered on X of major

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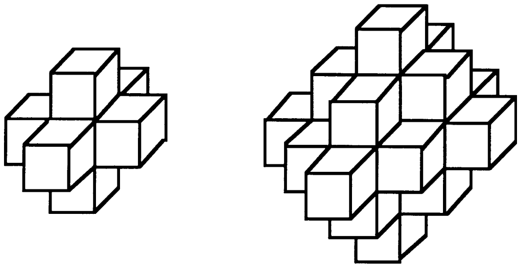

Fig. 1. A 1-Lee sphere and a 2-Lee sphere in R^3.

axes $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n$ (see Fig. 1), denoted by $B^n_r(X)$, is a moving (translation and rotation) ϕ of the r-Lee sphere centered on 0 such that $\phi(0)=X$ and $\phi(\boldsymbol{e}_i)=\boldsymbol{u}_i$ for every $i=1,\ldots,n$.

The distance between two Lee spheres is the distance between their centers. A tiling is called *regular* if neighboring spheres meet along entire $(n-1)$-dimensional faces of the original cubes. Thus, the center of each sphere of a regular tiling with Lee spheres belongs to an integer grid.

In the preceeding paper [3], we proved:

Theorem 1. *There does not exist a tiling of three-dimensional space with Lee spheres of radii at least* 2 (*even with different radii*).

This theorem confirms, for the three-dimensional space, a conjecture due to Gollomb and Welch [1,2]. Moreover, a Corollary of the proof of Theorem 1 asserts that:

Corollary 1. *There does not exist a non-regular tiling of three-dimensional space with Lee spheres of radii at least* 1, *even with different radii.*

The previous corollary and theorem suggested the following conjecture.

Conjecture 1. There does not exist a tiling of n-dimensional $(n>2)$ space with Lee spheres of radii greater than 0 such that the radius of at least one of them is greater than 1.

In the submitted version of our paper [3], we proposed Conjecture 1 for three-dimensional space. Finally, at printing time, we announced, without proof, this result:

Theorem 2. *Conjecture* 1 *holds for* $n=3$.

The aim of the present paper is to give a proof of Theorem 2.

2. Notations and preliminaries

Let $\mathscr{T}$ be a tiling of R^3 with three-dimensional Lee spheres. As observed in [3], it is straightforward to prove that all spheres of $\mathscr{T}$ have same major axes. Let u be one of these axes. For some integer i, the intersection of $\mathscr{T}$ with the plane orthogonal to u, denoted $\mathscr{T}_u^i = \{X = (x_1, x_2, x_3) \mid x_u = i\}$, defines a tiling of this plane, denoted $\mathscr{T}_u^i$, by two-dimensional Lee spheres of radius $0, 1, \ldots$.

If we move along the axis u in a fixed direction then the intersections of an r-Lee sphere $B_r^3(X)$ with a plane orthogonal to u are successively Lee spheres $B_s^2(X_u)$ of the two-dimensional space of radii $s = 0, \ldots, r, r-1, \ldots, 0$, where X_u^i is the projection of X on this plane. The first r intersections Lee spheres of radii $0, \ldots, r-1$ will be denoted by the letter **L** for *low* positions in the Lee sphere. The last r intersections Lee spheres of radii $r-1, \ldots, 0$ will be denoted by the letter **H** for *high* positions in the Lee sphere. And the $(r+1)$th intersection (Lee sphere of radius r) will be denoted by the letter **M**.

We mark a 2-cube $C(X_u^i)$ belonging to the border of a Lee sphere of $\mathscr{T}_u^i$ by $+$ (respectively $-$) if $B_0^2(X_u^{i+1}) \in \mathscr{T}_u^{i+1}$ (resp. $B_0^2(X_u^{i-1}) \in \mathscr{T}_u^{i-1}$). Observe that

> Let $X_u^i = (a, b)$ be the center of a 2-cube belonging to the border of a Lee sphere of $\mathscr{T}_u^i$. If $(a, b-1)$, $(a-1, b)$, (a, b), $(a, b+1)$ and $(a+1, b)$ belong to Lee spheres of types **L** (resp. type **H**) or **M** then $C(X_u^i)$ is marked $+$ (resp. $-$). (1)

If we have two Lee spheres of type **L** (or two of type **H**) then we can move along u in one of the directions such that both radii increase. Using this observation, we obtain the following lemma.

Lemma 1. *The distance between two* 2*-cubes in the border of two Lee spheres belonging to* $\mathscr{T}_u^i$, *is at least* 3 *if these spheres are both* **L** *or both* **H**.

By the previous lemma, we have

> If $C(X_u^i)$ and $C(Y_u^i)$ are two 2-cubes marked both $+$ or both $-$ then $d(X_u^i, Y_u^i) \geqslant 3$. (2)

3. Proof of Theorem 2

Assume that $\mathscr{T}$ is a tiling of R^3 with Lee spheres of radii at least one such that there exists a Lee sphere $B_r^3(X) \in \mathscr{T}$ with $r \geqslant 2$. Let u be one of the major axes of $B_r^3(X)$. We consider the orthogonal plane of u which cuts $B_r^3(X)$ in a way to obtain the 1-Lee sphere $B_1^2(X_u^0)$ of type **L** (see Fig. 2).

By Corollary 1, $\mathscr{T}$ must be regular.

Fig. 2. $B_1^2(X_u^0)$.

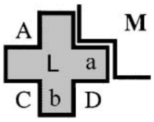

Fig. 3. An **M** Lee sphere in the neighbourhood of $B_1^2(X_u^0)$.

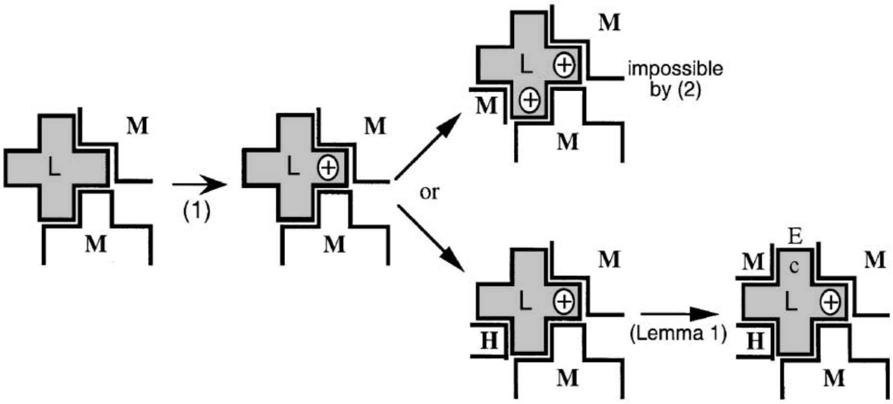

Fig. 4. D belongs to a Lee sphere of type **M**.

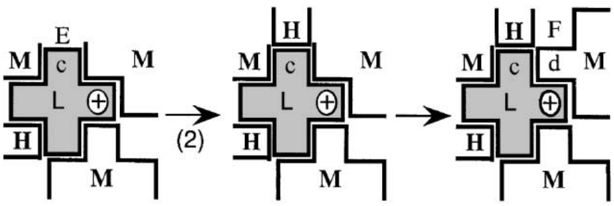

Fig. 5. E belongs to a Lee sphere of type **H**.

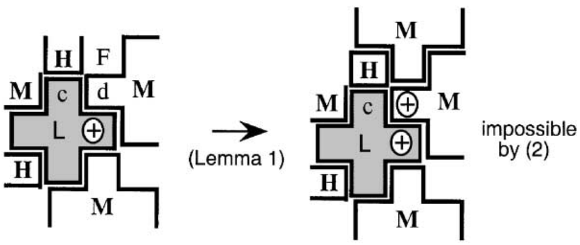

Fig. 6. A contradiction.

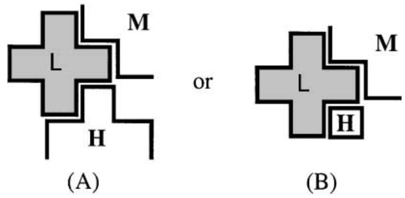

Fig. 7. D belongs to a Lee sphere of type **H**.

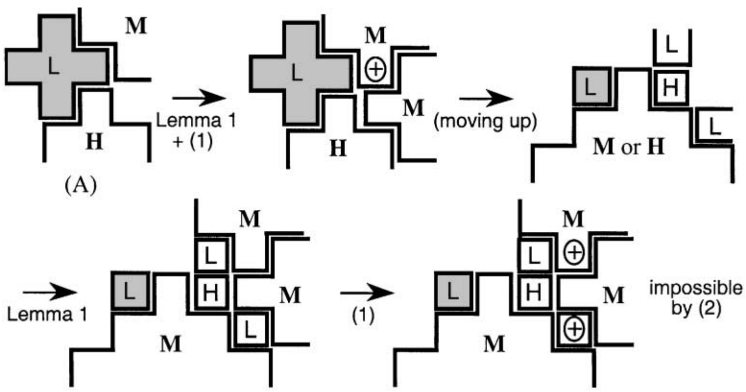

Fig. 8. Case (A).

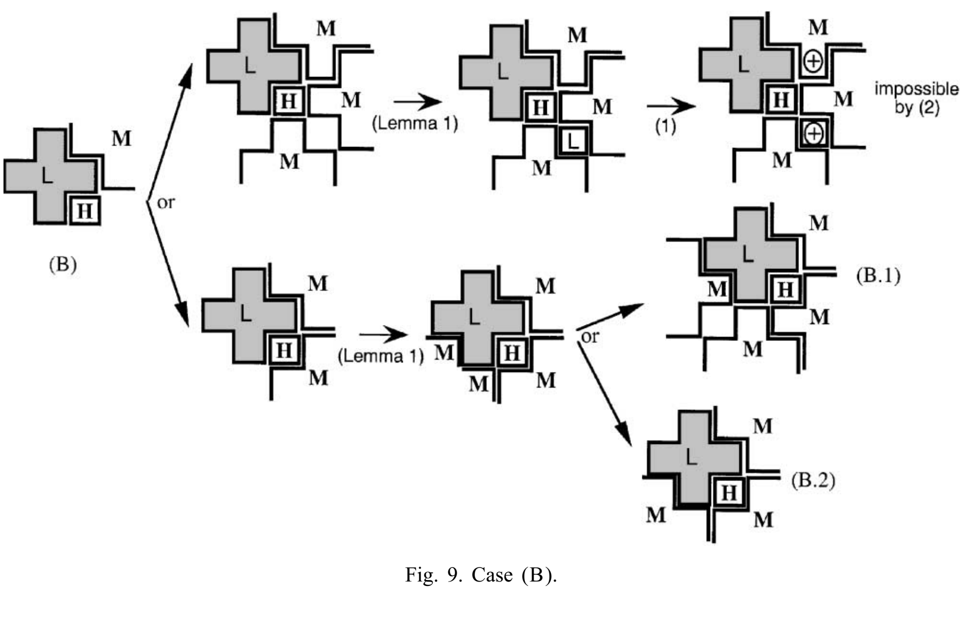

Fig. 9. Case (B).

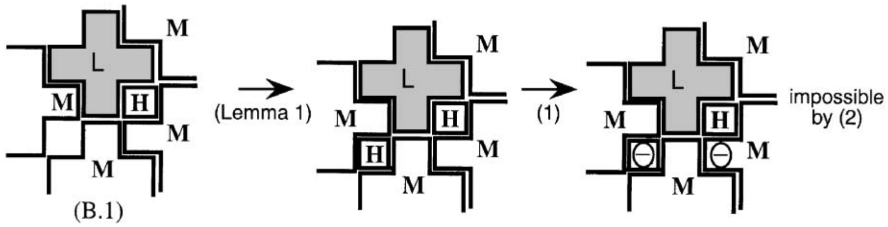

Fig. 10. Subcase (B.1).

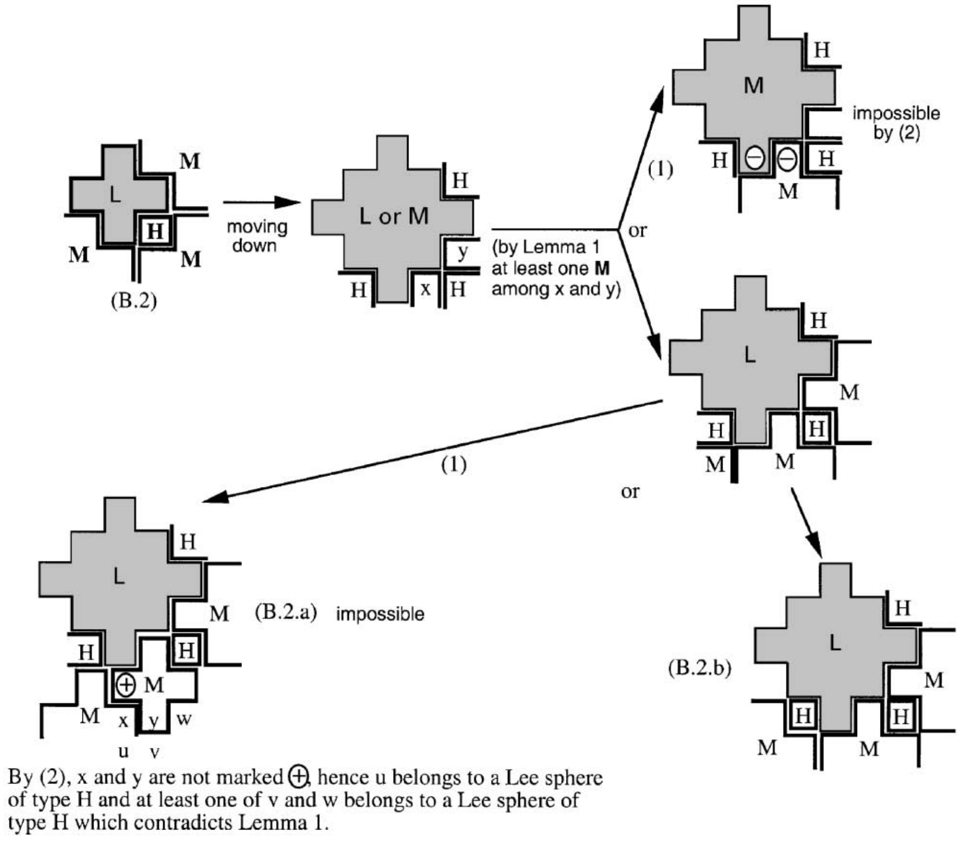

Fig. 11. Subcase (B.2).

By Lemma 1, at least one of the 2-cubes A, B, C or D must belong to a Lee sphere of type **M**. Since the radius of each sphere of $\mathscr{T}$ is greater than 0, we obtain a situation similar to one describe in Fig. 3.

By Lemma 1, D belongs to a Lee sphere of type **M** or **H**.

First, assume that D belongs to a Lee sphere of type **M**. By (1), we have that the 2-cube 'a' is marked $+$. If C belongs to a Lee sphere of type **M** then the 2-cube 'b' will be marked $+$, which is impossible by (2). So, we may assume that C belongs to

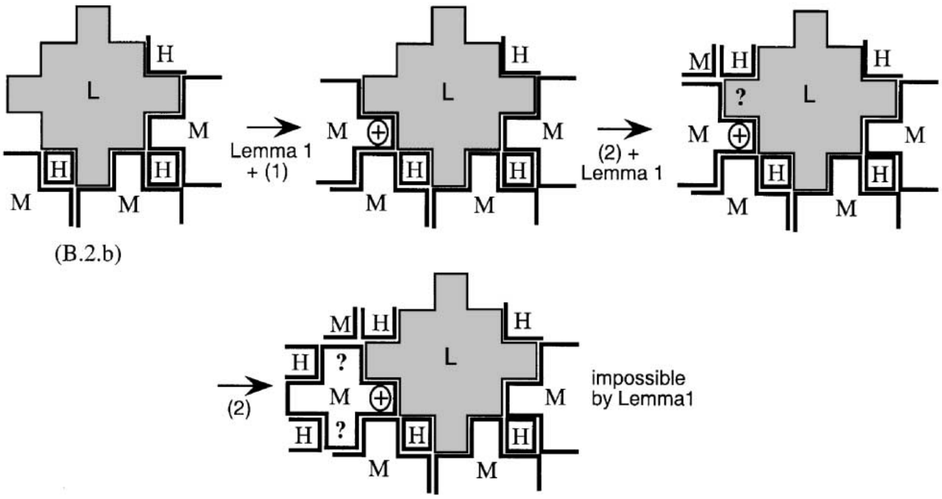

Fig. 12. Subcase (B.2.b).

a Lee sphere of type **H** which implies by Lemma 1 that A belongs to a Lee sphere of type **M**. This analysis is summarized in Fig. 4.

By (2), the 2-cube ‘c’ cannot be marked +. Hence, E belongs to a Lee sphere of type **H** (see Fig. 5).

By Lemma 1, F belongs to a Lee sphere of type **M**. But, now by (1), the 2-cube ‘d’ should be marked +, which contradicts (2) (see Fig. 6).

Now, we may assume that D belongs to a Lee sphere of type **H**. We will examine the tiling $\mathscr{T}_u^0$ in the neighborhood of $B_1^2(X_u^0)$ and similarly to the previous case, we will obtain a contradiction. Our proof is given by Figs. 7–12. In some cases, we will be led to move according to the axis u (denoted by ‘moving up’ or ‘moving down’) to look at the neighbourhood of $B_0^2(X_u^1)$ or $B_2^2(X_u^{-1})$ in the tiling on the ‘next’ or ‘previous’ plane $\mathscr{T}_u^1$ or $\mathscr{T}_u^{-1}$, respectively.

We have the two cases describe in Fig. 7.

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