Some Geometric Properties of Compressible Fluid Flows and a Class of Three-Dimensional Flows Obtained by Intrinsic Methods

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INTRODUCTION

There are no known general methods for solving the system of nonlinear partial differential equations which characterizes steady nonviscous nonheat-conducting compressible fluid flow in the absence of external forces. Some classes of solutions have been obtained, however, upon making simplifying assumptions. Certain of these assumptions are that the flow is incompressible and irrotational [1], the flows are supersonic plane flows [2], or that pressure is constant on streamlines [3].

One objective of this paper is to obtain, under the assumption that pressure is not constant on streamlines, a class of threedimensional flows in which the velocity vector \( \mathbf{v} \) is such that \( \mathbf{B} \times (\mathbf{V} \times \mathbf{v}) \) is a normal congruence. Prior to doing this, however, we make some observations concerning the geometry of the type of flows mentioned in the first paragraph with the hope that an understanding of the geometry will give some insight into the behavior of such flows. In particular, we find a necessary and sufficient condition for the \( \mathbf{B} \times \nabla \times \mathbf{v} \) congruence of curves to be normal and then investigate the geometry of flows having such a normal congruence.

THE SYSTEM OF EQUATIONS

We now introduce the system of partial differential equations which characterizes steady nonviscous nonheat-conducting flow of a compressible fluid, assuming a separable equation of state of the form

\[
\rho = P(p) S(\eta),
\]

where \( \rho \) is density, \( p \) is pressure, \( \eta \) is entropy, and \( P(p) \) and \( S(\eta) \) are given functions of \( p \) and \( \eta \), respectively. Letting \( \mathbf{v} \) and \( \nabla \) denote velocity and...
covariant differentiation, respectively, this system of equations may be written:

\[
\nabla_i \rho v^i = 0, \quad (2)
\]

\[
v^i \nabla_i v_j = -\frac{1}{\rho} \nabla_j \rho, \quad (A)
\]

\[
v^i \nabla_i \eta = 0. \quad (4)
\]

Since \( \rho \) may be eliminated from system (A) through the equation of state, it may be considered as a system of five equations in five unknowns, viz. the three velocity components, \( p \), and \( \eta \).

**The Geometry of Compressible Flows**

R. C. Prim [4, p. 436], in connection with the substitution principle, has mentioned flows, in which the strong Bernoulli law holds, which have the special property of possessing a family of surfaces with unit normals \( \vec{N} \) for which \( \vec{N} \times (\vec{v} \times \nabla \times \vec{v}) = 0 \). In general, however, given a vector field \( \vec{v} \), the problem of finding a family of surfaces with unit normals \( \vec{N} \) such that \( \vec{N} \times (\vec{v} \times \nabla \times \vec{v}) = 0 \) is not solvable [5, p. 202]. When such a family of surfaces does exist, the integral curves of \( \vec{v} \times \nabla \times \vec{v} \) are called a normal congruence of curves. Letting \( \vec{\omega} = \nabla \times \vec{v} \), our first theorem sheds light on conditions under which the \( \vec{v} \times \vec{\omega} \) congruence is normal in a compressible flow.

**Theorem 1.** For a compressible fluid flow characterized by Eqs. (1)-(4), the \( \vec{v} \times \vec{\omega} \) congruence is normal iff at least one of the following holds:

(a) Entropy is constant on the vortex lines,

(b) Pressure is constant on the streamlines.

*Proof.* We write (3) in the form

\[
(\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla \rho
\]

and apply the vector identity

\[
(\vec{v} \cdot \nabla) v = \frac{1}{2} (\nabla (\vec{v} \cdot \vec{v}) - \vec{v} \times \nabla \times \vec{v}),
\]

with \( \vec{v} = q \hat{t} \) where \( |\hat{t}| = 1 \). We obtain

\[
\nabla \rho = \rho (v \times \omega) - \rho (v \omega^2 / 2). \quad (5)
\]
Forming the curl results in
\[ \overline{0} = \nabla \rho \times (\vec{v} \times \vec{\omega}) + \rho \nabla \times (\vec{v} \times \vec{\omega}) - \left[ \nabla \rho \times \nabla \left( \frac{\rho^2}{2} \right) \right]. \]

Next, we take the scalar product with \( \vec{v} \times \vec{\omega} \), transpose, and make use of the equation of state, which yields
\[ (\vec{v} \times \vec{\omega}) \cdot \nabla \times (\vec{v} \times \vec{\omega}) = \frac{1}{\rho} \left[ (\vec{v} \times \vec{\omega}) \cdot \left( \frac{\partial \rho}{\partial \rho} \nabla \rho + \frac{\partial \rho}{\partial \eta} \nabla \eta \right) \times \nabla \left( \frac{\rho^2}{2} \right) \right]. \]

From Eq. (5) we see that \( \nabla \rho \), \( \vec{v} \times \vec{\omega} \), and \( \nabla (\rho^2/2) \) are coplanar. Hence, the last equation may be written
\[ (\vec{v} \times \vec{\omega}) \cdot \nabla \times (\vec{v} \times \vec{\omega}) = \frac{1}{\rho} \frac{\partial \rho}{\partial \eta} \left[ (\vec{v} \times \vec{\omega}) \cdot \nabla \eta \times \nabla \left( \frac{\rho^2}{2} \right) \right]. \]

Applying Lagrange's identity and making use of Eq. (4) allows us to write this equation in the form
\[ (\vec{v} \times \vec{\omega}) \cdot \nabla \times (\vec{v} \times \vec{\omega}) = -\frac{1}{\rho} \frac{\partial \rho}{\partial \eta} \left[ \vec{v} \cdot \nabla \left( \frac{\rho^2}{2} \right) \right] [\vec{\omega} \cdot \nabla \eta]. \]

Since \( \partial \rho/\partial \eta \neq 0 \), \( \vec{v} \times \vec{\omega} \cdot \nabla \times (\vec{v} \times \vec{\omega}) = 0 \) iff
\[ \vec{v} \cdot \nabla \left( \frac{\rho^2}{2} \right) = 0 \quad \text{or} \quad \vec{\omega} \cdot \nabla \eta = 0. \]

From (5), \( \vec{v} \cdot \nabla (\rho^2/2) = 0 \) iff \( \vec{v} \cdot \nabla \rho = 0 \), and the theorem is proved.

Thus, we have shown that entropy constant on vortex lines or pressure constant on streamlines guarantees the existence of a family of surfaces orthogonal to the \( \vec{v} \times \vec{\omega} \) congruence. In keeping with the terminology of Truesdell [6, p. 133], we call these surfaces Lamb surfaces.

**Corollary 1.1.** If entropy is constant on the vortex lines, then entropy is constant on each Lamb surface of the flow.

**Proof.** This follows from (4) and the identity
\[ \nabla \eta \times (\vec{v} \times \vec{\omega}) = (\vec{\omega} \cdot \nabla \eta) \vec{v} - (\vec{v} \cdot \nabla \eta) \vec{\omega}. \]

**Corollary 1.2.** If entropy is constant on the vortex lines, the Bernoulli constant \( B \) is the same for each streamline in a Lamb surface. That is, \( B \) is a constant on each Lamb surface.
Proof. The Crocco–Vazsonyi equation [7, p. 186] states that
\[ \bar{v} \times \bar{\omega} = \nabla B - T \nabla \eta, \]
where \( T \) represents absolute temperature. By the previous corollary, \( \bar{v} \times \bar{\omega} = \alpha \nabla \eta \) for some scalar \( \alpha \). Therefore,
\[ \nabla B = (\alpha + T) \nabla \eta, \]
and we see the level surfaces of \( B \) and \( \eta \) coincide.

Using the term Bernoulli surface, for a level surface of \( B \), we observe that if pressure is not constant on streamlines, then the existence of Lamb surfaces implies the existence of Bernoulli surfaces and constant entropy surfaces, and these families of surfaces are the same.

Since an article by P. Smith [8] as well as that already mentioned by R. C. Prim motivated this investigation, in the remainder of this paper we shall use the term *Prim–Smith flow* to designate a flow with the following two properties.

(a) It possesses Lamb surfaces.
(b) Its constant pressure surfaces are not stream surfaces.

A Special Congruence of Curves Lying on the Lamb Surfaces

**Theorem 2.** At each point of a Prim–Smith flow or a flow in which constant pressure surfaces are stream surfaces, the vectors \( \nabla \eta, \nabla \rho, \nabla p, \) and \( \nabla q \) are coplanar.

**Proof. Case 1.** Prim–Smith flow: The equation of motion in the form
\[ \nabla (q^2/2) - \bar{v} \times \bar{\omega} = (-1/\rho) \nabla \rho \]
implies that \( \nabla q, \bar{v} \times \bar{\omega}, \) and \( \nabla \rho \) are coplanar. In a Prim–Smith flow, however, \( \bar{v} \times \bar{\omega} \) and \( \nabla \eta \) are collinear, so \( \nabla q \) lies in the plane of \( \nabla \eta \) and \( \nabla \rho \). Using the equation of state, we see that the four gradients \( \nabla \eta, \nabla \rho, \nabla p, \) and \( \nabla q \) are coplanar.

**Case II.** Constant pressure surfaces are stream surfaces: In this case, \( \bar{t} \cdot \nabla \rho = 0 \). This, together with Eq. (6), implies that \( \bar{t} \cdot \nabla q = 0 \). Since \( \bar{t} \cdot \nabla \eta = 0 \) in the flows under consideration, the equation of state implies that \( \bar{t} \cdot \nabla \rho = 0 \). So at each point of the flow, \( \nabla \eta, \nabla \rho, \nabla p, \) and \( \nabla q \) lie in a plane normal to the streamline through the point.

We observe in Case II that \( \eta, \rho, p, \) and \( q \) are constant on the streamlines and extend this property in the following corollary.
COROLLARY 2.1. In a flow with constant pressure surfaces and Lamb surfaces that are distinct, \( \eta, \rho, p, \) and \( q \) are constant on the unit vector field given by

\[
\vec{x} = \frac{[i \cdot \nabla(q^2/2)] \vec{w} - [\vec{w} \cdot \nabla(q^2/2)] i}{\left( [i \cdot \nabla(q^2/2)]^2 + [\vec{w} \cdot \nabla(q^2/2)]^2 - 2[i \cdot \nabla(q^2/2)] [\vec{w} \cdot \nabla(q^2/2)] \vec{w} \cdot i \right)^{1/2}} \tag{7}
\]

where \( \vec{\omega} = \omega \vec{w} \) with \( |\vec{w}| = 1 \).

Proof. Case I. Prim-Smith flow. Clearly, from (7), \( \vec{x} \cdot \nabla q = 0 \). For a Prim-Smith flow, the energy equation implies \( \vec{x} \cdot \nabla \eta = 0 \). From the theorem, \( \nabla p \) and \( \nabla \rho \) lie in the plane spanned by \( \nabla q \) and \( \nabla \eta \). Therefore

\[ \vec{x} \cdot \nabla p = \vec{x} \cdot \nabla \rho = 0, \]

and the corollary holds for a Prim-Smith flow.

Case II. Constant pressure surfaces are stream surfaces. As Eq. (6) indicates, \( t \cdot \nabla p = 0 \) iff \( t \cdot \nabla(q^2/2) = 0 \). From the hypothesis of this corollary, it follows that \( \vec{w} \cdot \nabla(q^2/2) \neq 0 \), and hence \( \vec{x} = \pm t \). As observed immediately preceding this corollary, \( \eta, \rho, p, \) and \( q \) are constant on the streamlines in this case. Thus, the corollary is proved.

STREAMLINES AS GEODESICS

Since the equation of motion implies that \( \vec{b} \cdot \nabla \rho = 0 \), pressure constant on streamlines immediately implies that streamlines are geodesics on constant pressure surfaces. Conversely, if streamlines are geodesics on constant pressure surfaces, as previously shown, \( \eta, \rho, p, \) and \( q \) are constant on the streamlines.

Consequently, the functions \( \eta, \rho, p, \) and \( q \) are constant on the streamlines of a flow iff the streamlines are geodesics on the surfaces of constant pressure.

We notice that in the case of constant pressure surfaces, the streamlines need merely to lie on the surfaces in order to be geodesics. In general, of course, this is not the case, and we now obtain a necessary and sufficient condition for a streamline to be a geodesic on a Lamb surface. To do this we make use of the following expression for the vorticity derived by N. Coburn [9, p. 118]:

\[
\omega_j = q \left( b_k \frac{\partial t_k}{\partial n} - n_k \frac{\partial t_k}{\partial b} \right) t_j + \frac{\partial q}{\partial b} n_j + \left( q_k - \frac{\partial q}{\partial n} \right) b_j. \tag{8}
\]

Forming the cross product with \( \vec{\omega} \), we obtain

\[
\vec{\omega} \times \vec{w} = q \left( \frac{\partial q}{\partial n} - q_k \right) \vec{n} + q \frac{\partial q}{\partial b} \vec{b}. \tag{9}
\]
If \( \vec{v} \times \vec{w} \neq 0 \), then (9) implies that \( \partial q/\partial b = b \cdot \nabla q = 0 \) iff \( \vec{v} \times \vec{w} \) and \( \vec{n} \), the principal normal of the streamline, are collinear. Since, at each point of a Lamb surface, \( \vec{v} \times \vec{w} \) is collinear with a surface normal, we see that the streamlines are geodesics on Lamb surfaces iff \( b \cdot \nabla q = 0 \), i.e., \( q \) is constant on each curve of the \( b \) congruence. Theorem 2 and the fact that \( b \cdot \nabla p = 0 \) imply that \( \eta, \rho, \rho, \) and \( q \) are constant on each curve of the \( b \) congruence. We have thus proved the following:

**Theorem 3.** A necessary and sufficient condition for streamlines to be geodesics on Lamb surfaces is that \( \eta, \rho, \rho, \) and \( q \) are constant on the \( b \) congruence.

Theorems 2 and 3 relate the dynamic properties of a flow to the geometric properties since they say that \( t \cdot \nabla q = 0 \) is a necessary and sufficient condition for streamlines to be geodesics on constant pressure surfaces and \( b \cdot \nabla q = 0 \) is a necessary and sufficient condition for streamlines to be geodesics on Lamb surfaces when such surfaces exist in a flow. Thus, it follows that \( t \cdot \nabla q = b \cdot \nabla q = 0 \) iff constant pressure surfaces and Lamb surfaces coincide. In the next section we shall pursue further same relationships between the dynamics and the geometry of a compressible flow.

**Relations Between \( \vec{v} \times \vec{w} \) and the Magnitude of the Velocity Vector**

It has been proved by M. H. Martin [10, p. 470] that for plane flow, a necessary and sufficient condition for an irrotational flow is that \( q \), the magnitude of the velocity vector, depends on pressure only. We shall extend these remarks by showing relationships between \( q \) and \( \vec{v} \times \vec{w} \) in three-dimensional flows.

We assume throughout this section that surfaces of constant pressure are not stream surfaces, and we denote by \( \psi = \text{const} \) and \( \phi = \text{const} \), where \( \psi \) and \( \phi \) are scalar point functions two distinct families of stream surfaces. Thus, we consider \( q \) to be a function of \( p, \psi, \) and \( \phi \).

Upon expanding the term \( \nabla(q^2/2) \) in the equation of motion, we obtain

\[
\vec{v} \times \vec{w} = (qq_p + 1/\rho) \nabla p + qq_\phi \nabla \phi + qq_\psi \nabla \psi.
\]

In view of the above assumptions, Eq. (10) implies that

\[
qq_p + 1/\rho = 0,
\]

so

\[
\vec{v} \times \vec{w} = q(q_\phi \nabla \phi + q_\psi \nabla \psi).
\]

(11)
If $\vec{v} \times \vec{\omega} = 0$, then

$$\nabla \phi = -\frac{q_{\phi}}{q} \nabla \psi$$

or

$$q_{\phi} = q_{\psi} = 0.$$  

Case (a) contradicts the known independence of $\phi$ and $\psi$, so $q_{\phi} = q_{\psi} = 0$ and $q$ is a function of pressure only. Conversely, it is clear from Eq. (11) that $\vec{v} \times \vec{\omega} = 0$ if $q$ depends on pressure only. Thus we have

**Theorem 4.** If constant pressure surfaces are not stream surfaces of a flow, the flow is Beltrami iff the magnitude of the velocity vector depends only on the pressure.

Proceeding to flows in which streamlines and vortex lines do not coincide but form Lamb surfaces, we have the following theorem.

**Theorem 5.** If constant pressure surfaces are not stream surfaces, a compressible fluid flow contains Lamb surfaces iff the velocity magnitude is a function of pressure and entropy only.

**Proof.** Since $\eta$ is constant on Lamb surfaces, Eq. (11) may be written

$$\vec{v} \times \vec{\omega} = q(q_{\eta} \nabla \eta + q_{\psi} \nabla \psi). \tag{12}$$

Forming the dot product of (12) with $\vec{\omega}$, our hypotheses imply that

$$0 = q_{\eta} \vec{\omega} \cdot \nabla \psi.$$

Thus $q_{\eta} = 0$ or $\vec{\omega} \cdot \nabla \psi = 0$. The latter equation cannot hold because of the independence of $\psi$ and $\eta$. Therefore $q_{\eta} = 0$ and $q = q(p, \eta)$. Conversely, let us consider a flow in which $q = q(p, \eta)$. Then

$$\vec{v} \times \vec{\omega} = q q_{\eta} \nabla \eta,$$

and by Theorem 1, the $\vec{v} \times \vec{\omega}$ congruence is normal, or the flow is of Prim-Smith type.

We summarize these results for flows in which $t \cdot \nabla p \neq 0$ in the following table.

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A Class of Three-Dimensional Flows

We seek a three-dimensional flow in which streamlines are geodesics on Lamb surfaces. Continuing under the assumption that a separable equation of state holds, we may rewrite the dynamical equations replacing $\eta$ with $S$ in the last equation obtaining

\begin{align*}
\nabla_i \nu^i &= 0, \\
v^i \nabla_i \nu &= -\frac{1}{\rho} \nabla_i p, \\
v^i \nabla_i S &= 0.
\end{align*}

(B)

As in the previous section, we continue under the assumption that surfaces of constant pressure are not stream surfaces, and $\psi = \text{const}$ and $\phi = \text{const}$ denote two distinct families of stream surfaces. Thus, we have coordinate curves consisting of streamlines and two distinct families of curves on which pressure is constant. We denote the variables $p, \psi,$ and $\phi$ by $X^1, X^2,$ and $X^3,$ respectively.

It has been shown [11] that for our coordinate system, the set of Equations (B) is equivalent to the equations.

\begin{align*}
\frac{\partial}{\partial p} \left( \frac{P \mu_g}{g_{11}} \right) &= 0, \\
2 + P \frac{\partial u}{\partial p} &= 0, \\
\frac{\partial g_{11}}{\partial X^a} &= 2 \left[ (g_{11})^{1/2} \frac{\partial}{\partial p} \left( \frac{g_{12}}{(g_{11})^{1/2}} \right) - \frac{g_{12}}{Pu} \right], \\
\frac{\partial g_{11}}{\partial X^a} &= 2 \left[ (g_{11})^{1/2} \frac{\partial}{\partial p} \left( \frac{g_{13}}{(g_{11})^{1/2}} \right) - \frac{g_{13}}{Pu} \right],
\end{align*}

(C)

where $u = Sq^a, g_{ij}$ are elements of the metric tensor and $g$ is the determinant of that tensor.

Since $S$ is constant on the Lamb surface of a Prim-Smith flow, we may take $X^3 = S.$ Also, the assumption that streamlines are geodesics on Lamb surfaces implies that $g_{12} = 0.$ This allows us to omit Eq. (15) from system (C).

To solve system (C), we assume there exists a particular relationship among the rectangular Cartesian coordinates $x, y, z$ and the variables $p, \psi,$ and $S,$ viz. one of the form

\begin{align*}
x &= x(p, \psi), \\
y &= y(p, S), \\
z &= z(p, \psi).
\end{align*}

(18)
in which the Jacobian $J$, is not zero. The metric tensor is then of the form

$$g_{ij} = \begin{pmatrix} x_p^2 + y_p^2 + z_p^2 & 0 & y_p y_z \\ 0 & x_o^2 + z_o^2 & 0 \\ y_o y_z & 0 & y_s^2 \end{pmatrix}.$$ 

The requirement that $g_{12} = 0$ forces

$$x_p x_o = z_p z_o = 0.$$  \hfill (19)

Seeking a solution of system (C) for which $g_{13} \neq 0$ allows us to replace Eq. (16) with the equation

$$\frac{\partial g_{11}}{\partial x^3} = g_{13}(\partial/\partial p) \ln\left(\frac{uy_p y_s^2}{x_p^2 + y_p^2 + z_p^2}\right).$$ \hfill (20)

Substituting from the metric tensor into this equation yields

$$\frac{\partial}{\partial S} \left(x_p^2 + y_p^2 + z_p^2\right) = y_p y_s \frac{\partial}{\partial p} \ln\left(\frac{uy_p y_s^2}{x_p^2 + y_p^2 + z_p^2}\right).$$

Performing the indicated differentiation with respect to $S$, dividing by $y_p y_s$, integrating with respect to $p$, and solving for $u$, we obtain

$$u = \frac{F^2(\psi, S) \left(x_p^2 + y_p^2 + z_p^2\right)}{y_p^2},$$ \hfill (21)

where $F^2(\psi, S)$ is an arbitrary positive valued function.

After a bit of simplification, substitution for $u$ and the metric coefficients in (13) yields

$$\frac{\partial}{\partial p} \left[P^2(p) y_s \left(x_p^2 + z_p^2\right) \left(x_p^2 + z_p^2\right)\right] = 0.$$ \hfill (22)

Substituting for $u$ in Eq. (14) we obtain

$$P(p) \frac{\partial}{\partial p} \left(\frac{x_p^2 + z_p^2}{y_p^2}\right) = \frac{-2}{F^2(\psi, S)}.$$ \hfill (23)

Thus, to find a three-dimensional flow of the form indicated by Eqs. (18) with the property that streamlines are geodesics on Lamb surfaces is reduced to finding $x$, $y$, $z$ and $u$ in terms of $p$, $\psi$, and $S$ such that (19) and (21)–(23) are satisfied.

In accordance with a standard separation of variable technique, we assume there is a solution to the equations just mentioned of the form

$$x = \alpha_1(p) \beta_1(\psi),$$

$$y = \alpha_2(p) \gamma(S),$$

$$z = \alpha_3(p) \beta_2(\psi).$$ \hfill (24)
Substitution into (19) results in
\[
\frac{\alpha_1(p) \alpha_1'(p)}{2} \frac{d}{dp} \left[ \beta_1^2(\psi) + \beta_2^2(\psi) \right] = 0.
\]

Therefore, we must have (for \( \alpha_1(p) \) not constant)
\[
\beta_1^2(\psi) + \beta_2^2(\psi) = A,
\]
where \( A \) is an arbitrary positive constant.

Substituting from (24) into (22), we find that
\[
\frac{d}{dp} \left[ P(p) \frac{\alpha_2(p) \alpha_1'(p)}{\alpha_2'(p)} \right] = 0.
\] (26)
provided the various functions appearing in (24) are not constant.

Finally, we substitute into (23), use (25), and isolate those terms involving \( p \) only. This gives
\[
P(p) \frac{d}{dp} \left( \frac{\alpha_1'(p)}{\alpha_2'(p)} \right)^2 = \frac{-2\gamma^2(S)}{AF^2(\psi, S)} = C,
\]
where \( C \) is a negative separation constant. We are thus led to the two conditions
\[
\frac{d}{dp} \left[ \frac{\alpha_1'(p)}{\alpha_2'(p)} \right]^2 = \frac{C}{P(p)} \quad (C < 0),
\]
\[
\gamma^2(S) = -\frac{AC}{2} F^2(\psi, S),
\] (28)
where it is clear from (28) that \( F \) is actually independent of \( \psi \).

In summary, we gather up those requirements which must be met to have a solution, of the type we are considering, to the dynamical equations.

(i) The Jacobian, \( J = \gamma_3(x_{\psi} x_\varphi - x_\varphi x_\psi) \neq 0 \). From (24), we see this is equivalent to \( \beta_1'(\psi) \beta_2(\psi) - \beta_1(\psi) \beta_2'(\psi) \neq 0 \).

(ii) \( \beta_1'(\psi) + \beta_2^2(\psi) = A \), where \( A \) is a constant.

(iii) \( [P(p) \alpha_1(p) \alpha_1'(p)/\alpha_2(p)/\alpha_2'(p)] = D \), where \( D \) is a constant.

(iv) \( (d/dp)(\alpha_1'(p)/\alpha_2'(p))^2 = C/P(p) \). \( (C < 0) \).

(v) \( \gamma^2(S) = -(AC/2) F^2(S) \).

To obtain a flow we could proceed according to the following steps.

(I) Pick a positive constant \( A \), a negative constant \( C \), and an arbitrary function \( F(S) \).
(II) Determine $\gamma(S)$ from the equation in requirement (v).

(III) Pick $\beta_1(\psi)$ such that $\beta_1^2(\psi) < A$, and then determine $\beta_2(\psi)$. This can be done, for example, by choosing $A = 1$ and $\beta_1(\psi) = \cos \psi$.

(IV) Satisfy requirements (iii) and (iv) as indicated below.

Requirement (iv) implies that

$$\frac{\alpha_1'(p)}{\alpha_2'(p)} = \pm (CI(p) + C_1)^{1/2},$$

where $C_1$ is an arbitrary constant chosen such that $CI(p) + C_1 > 0$ in the region of flow, and the function $I(p)$ is an integral of $1/P(p)$. Thus, (iii) and (iv) lead to the equations

$$\frac{\alpha_1'(p)}{\alpha_2'(p)} = k(p), \quad (29)$$

$$\alpha_1(p) \alpha_2(p) = h(p), \quad (30)$$

where

$$k(p) = \pm (CI(p) + C_1)^{1/2} \quad \text{and} \quad h(p) = \frac{D}{P(p) k(p)}.$$  

From (29) and (30) we obtain a single differential equation with $\alpha_2$ as dependent variable, solve this differential equation, and then obtain $\alpha_1$ from equation (30). With $Y = \alpha_2^2$, the differential equation we obtain is

$$\frac{dY}{dp} = \frac{2h'(p)Y}{h(p) + k(p)Y}.$$

The substitution $1/r = h(p) + k(p)Y$ leads to the differential equation

$$\frac{dr}{dp} = f_1(p) r + f_2(p) r^2 + f_3(p) r^3 \quad (31)$$

with

$$f_1(p) = -\frac{k'(p)}{k(p)},$$

$$f_2(p) = \frac{h(p) k'(p)}{k(p)} - 3h'(p),$$

$$f_3(p) = 2h(p) h'(p).$$

Methods for solving (31) for various forms of the coefficients are given in E. Kamke [12, p. 25].

In conclusion, the problem of finding a solution to system (C) where $x$, $y$, and $z$ are of the form (18) and for which streamlines are geodesics on Lamb
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surfaces has been reduced to the problem of solving a single first-order differential equation.

It might be of interest to note in closing that if one chooses $\beta_1(\psi) = \cos \psi$, $\beta_2(\psi) = \sin \psi$ and $A = 1$, the Lamb surfaces are surfaces of revolution with the $y$-axis as their axis of symmetry, the streamlines are meridian curves of these surfaces, and the parallels are isobars.

REFERENCES