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On Uniquely Colorable Planar Graphs*

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ABSTRACT

A labeled graph G with chromatic number n is called uniquely n-colorable or simply uniquely colorable if every two partitions of the point set of G into n color classes are the same. Uniquely colorable planar graphs are investigated. In particular, it is shown that uniquely 3-colorable planar graphs with at least four points contain at least two triangles, uniquely 4-colorable planar graphs are maximal planar, and uniquely 5-colorable planar graphs do not exist.

A *coloring of a graph* G is an assignment of colors to the points of G so that adjacent points are colored differently; an *n-coloring* uses n colors. The *chromatic number* $\chi(G)$ of G is the smallest n for which G admits an n-coloring.

Any $\chi(G)$-coloring of G induces a partition of the point set $\{v_1, v_2, \ldots, v_p\}$ of G into $\chi(G)$ subsets, called *color classes*, two points belonging to the same subset if and only if they are colored the same. If $\chi(G) = n$ and every two n-colorings of G induce the same partition, then G is called *uniquely n-colorable* or simply *uniquely colorable*. The graph G_1 of Figure 1 is uniquely 3-colorable while the graph G_2 may be 3-colored in five different ways, i.e., five different partitions of the point set of G_2 are possible from 3-colorings of G_2.

It is the study of uniquely colorable graphs, in particular uniquely colorable planar graphs, with which we are concerned. It might also be noted that uniquely colorable graphs may be defined in terms of their

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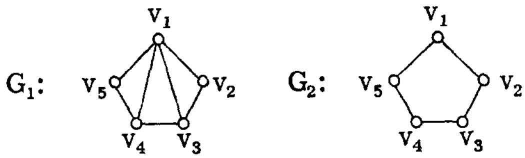

FIGURE 1

chromatic polynomials. The chromatic polynomial $f(G, t)$ of a labeled graph G is the number of distinct ways of coloring G with t colors, two t-colorings being considered distinct if there is at least one point colored differently in the two t-colorings. A graph G is then uniquely n-colorable if and only if $f(G, n) = n!$. (For a discussion of chromatic polynomials, see Read [7].)

Before beginning our investigation of the relationship between unique colorability and planarity, we consider some basic properties of uniquely colorable graphs in general.

PROPOSITION 1. *If a graph G is uniquely n-colorable, then in any n-coloring of G, every point v of G is adjacent with at least one point of every color different from that assigned to v.*

PROOF: The result follows simply by observing that if an n-coloring of G results in some point v colored α which is not adjacent with any point colored β, $\alpha \neq \beta$, then v may be recolored β, implying that G is not uniquely n-colorable.

A corollary to Proposition 1 is now obtained. By min deg G is meant the minimum among the degrees of the points of G.

COROLLARY 1a. *If G is uniquely n-colorable, then* min deg $G \geqslant n - 1$.

We next present a necessary condition for a graph G to be uniquely colorable. If U is a non-empty subset of the point set of a graph G, then the subgraph induced by U is that subgraph of G whose point set is U and where two points are adjacent if and only if these points are adjacent in G. The next result first appeared in [1].

THEOREM 2. *For any n-coloring of a uniquely n-colorable graph G, the subgraph induced by the union of any two color classes is connected.*

PROOF: Consider an n-coloring of a uniquely n-colorable graph G, and suppose there exist two color classes, say C_1 and C_2, of G such that the subgraph S of G induced by $C_1 \cup C_2$ is disconnected. Let S_1 and S_2 be two components of S. By Proposition 1, each of S_1 and S_2 contains points of both C_1 and C_2. An n-coloring different from the given one is produced if the color of the points in $C_1 \cap S_1$ is interchanged with the color of the points in $C_2 \cap S_1$. This implies G is not uniquely n-colorable, which is a contradiction.

From Theorem 2 it follows that a uniquely n-colorable graph, $n \geqslant 2$, must be connected. However, a more general result can be established. A graph G is *m-connected*, $m \geqslant 1$, if the removal of any k points, $0 \leqslant k < m$, neither disconnects G nor reduces it to the trivial graph consisting of a single point, the latter condition needed for complete graphs (in which every two points are adjacent) since such graphs cannot be disconnected by the removal of points.

THEOREM 3. *Every uniquely n-colorable graph is $(n-1)$-connected.*

PROOF: Let there be given an n-coloring of a uniquely n-colorable graph G. If G is complete, it necessarily has n points and so is $(n-1)$-connected. Assume that G is neither complete nor $(n-1)$-connected so that there exists a set U of $n-2$ points whose removal disconnects G. Thus there are at least two distinct colors α and β not assigned to any point of U. By Theorem 2, a point colored α is connected to any point colored β by a path all of whose points are colored α or β. Hence, the set of points of G colored α or β lies within the same component of $G - U$, say G_1. Another n-coloring of G can therefore be produced by recoloring any point of $G - U$ which is not in G_1 either α or β. This contradicts the fact that G is uniquely n-colorable; thus G is $(n-1)$-connected.

It therefore follows that the union of any k color classes of a uniquely n-colorable graph, $2 \leqslant k \leqslant n$, induces a uniquely k-colorable subgraph which is therefore $(k-1)$-connected.

We now consider unique colorability, as related to planar graphs. Since for any planar graph G, $\chi(G) \leqslant 5$ (see Heawood [5]), we need consider here only uniquely n-colorable graphs, where $1 \leqslant n \leqslant 5$. For $n = 1$, the situation is trivial. A graph is uniquely 1-colorable if and only if it has no lines, and all such graphs are obviously planar.

By a theorem of König [6, p. 151], a graph with lines has chromatic number 2 if and only if it has no odd cycles. It is then easy to show that a non-trivial graph is uniquely 2-colorable if and only if it is connected and has no odd cycles. A criterion involving planar graphs is given next.

THEOREM 4. *A connected non-trivial planar graph G is uniquely 2-colorable if and only if each interior region of G is bounded by an even number of lines.*

PROOF: From an earlier remark, a graph with chromatic number 2 has only even cycles; hence each interior region of a planar uniquely 2-colorable graph is necessarily bounded by an even number of lines.

Conversely, let G be a connected non-trivial planar graph in which each interior region is bounded by an even number of lines. To show G is uniquely 2-colorable it suffices to show that each cycle of G is even. Let C be a cycle of G. If C bounds an interior region of G, then, by hypothesis, C is even. Otherwise, there are several regions lying interior to C, say $R_1, R_2, \ldots, R_k$, where $k \geqslant 2$. Let q_i be the number of lines which bound R_i, $1 \leqslant i \leqslant k$. Denote the number of lines on C by Q_0 and the number of lines interior to C by Q_1. It thus follows that $\sum q_i = Q_0 + 2Q_1$, but since each q_i is even, $\sum q_i$ is even. Therefore, Q_0 is even and C is an even cycle.

No useful characterization of graphs having chromatic number 3 is known, although, of course, such graphs must have odd cycles. We now investigate some classes of uniquely 3-colorable planar graphs. Two distinct triangles in a graph are considered adjacent if they have a line in common.

THEOREM 5. *Let G be a planar graph having chromatic number* 3. *If G contains a triangle T such that for each point v of G there is a sequence $T, T_1, T_2, \ldots, T_m$ of triangles with v in T_m and having consecutive triangles in the sequence adjacent, then G is uniquely* 3*-colorable.*

PROOF: Let G be a graph satisfying the hypothesis of the theorem. Clearly, the points of the triangle T must be colored different from one another, say, α, β, γ. Since $\chi(G) = 3$ the only uncolored point of T_1 has its color determined by the other two points of T_1. Successively, all the points of $T_2, T_3, \ldots, T_m$ have their colors determined, implying the color of v is determined by the coloring of T. Hence, there is only one way, up to a permutation of the colors, to 3-color the points of G, that is, G is uniquely 3-colorable.

COROLLARY 5a. *If a* 2*-connected planar graph G has chromatic number* 3 *and at most one region of G (including the exterior region) is not a triangle, then G is uniquely* 3*-colorable.*

The converse of Corollary 5a is not true, that is, a uniquely 3-colorable planar graph may have more than one region which is not a triangle (see Figure 2). However, uniquely 3-colorable planar graphs must contain triangles.

THEOREM 6. *If G is a uniquely* 3*-colorable planar graph with $p \geqslant 4$ points, then G has at least two triangles.*

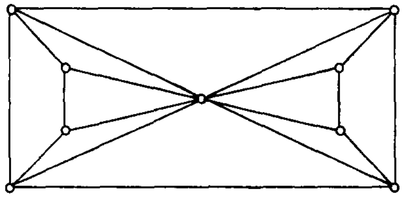

FIGURE 2

PROOF: Assume that G satisfies the hypotheses of the theorem but contains at most one triangle. By Theorem 3, G is 2-connected and thus contains a region, bounded by a cycle C, which is not a triangle. Let G be 3-colored. There are two cases to consider.

CASE 1. The cycle C is a 4-cycle and three colors are used in the coloring of C. Denote the points of C by u_i, $1 \leqslant i \leqslant 4$, and suppose that u_2 and u_4 are colored the same. Insert a new point v in the interior of C and add the lines u_1v, u_2v, and u_3v. The resulting graph G' is clearly planar. A triangle in G' which is not in G necessarily contains the point v. Two such triangles are vu_1u_2 and vu_2u_3. If there were a third triangle of G' which is not in G, then it would contain the lines vu_1 and vu_3 so that u_1u_3 is a line of G. However, the existence of this line in G implies that $u_1u_3u_4$ and $u_1u_2u_3$ are triangles in G, contrary to hypothesis. Hence G' has at most three triangles.

By a result of Grünbaum [3], every planar graph with three or fewer triangles can be 3-colored. Since $\chi(G) = 3$, it follows that $\chi(G') = 3$. In any 3-coloring of G', u_1 and u_3 belong to the same color class, for otherwise v would be adjacent to a point colored the same as v. However, any 3-coloring of G' induces a 3-coloring of G in which u_1 and u_3 belong to the same color class. This contradicts the assumption that G is uniquely 3-colorable so that G has at least two triangles.

CASE 2. The cycle C contains four points u_1, u_2, u_3, u_4 such that u_1 and u_3 have the same color as do u_2 and u_4 and u_1u_2 and u_3u_4 are lines of C. We insert the two new points v_1 and v_2 in the interior of C and add the lines u_1v_1, u_2v_1, u_3v_2, u_4v_2, and v_1v_2, denoting the resulting graph by G'. The graph G' has two more triangles than does G; thus, by Grünbaum's theorem, $\chi(G') = 3$. In any 3-coloring of G', the points v_1 and v_2 are colored differently. This implies, however, that it is impossible for u_1 and u_3 to belong to the same color class and for u_2 and u_4 to belong to the same color class. Since any 3-coloring of G' induces a 3-coloring of G, we arrive at a 3-coloring of G different from the given one. This again is contrary to hypothesis, so that G has at least two triangles.

Since the two cases are mutually exhaustive, this completes the proof.

We have already shown that the converse of Corollary 5a is not true. For a special class of planar graphs, however, the converse does follow. A graph is called *outerplanar* (see [2]) if it can be embedded in the plane so that every point lies on the exterior region. It can readily be shown that $\chi(G) \leqslant 3$ for every outerplanar graph G. A *maximal outerplanar* graph is an outerplanar graph G having the property that for every pair of non-adjacent points u and v the addition of the line uv results in a graph which is not outerplanar. A maximal outerplanar graph then is a 2-connected outerplanar graph having at most one region (namely the exterior) which is not a triangle. The following observation is of use.

REMARK. An outerplanar graph with $p \geqslant 2$ points is maximal outerplanar if and only if it has $2p - 3$ lines.

THEOREM 7. *An outerplanar graph G with $p \geqslant 3$ points is uniquely 3-colorable if and only if it is maximal outerplanar.*

PROOF: Let G be a uniquely 3-colorable outerplanar graph, and let the three color classes resulting from a 3-coloring of G be denoted by V_i, $1 \leqslant i \leqslant 3$, where $|V_i| = p_i$. Since, by Theorem 2, the union of every two color classes of G induces a connected subgraph, the number of lines in the subgraph induced by $V_i \cup V_j$, $i \neq j$, is at least $p_i + p_j - 1$. Hence the number of lines in G is at least $\sum(p_i + p_j - 1)$, where the sum is taken over all i and j such that $1 \leqslant i < j \leqslant 3$. This sum equals $2\sum p_i - 3 = 2p - 3$. Since the number of lines in G cannot exceed $2p - 3$, G has $2p - 3$ lines and so, by the preceding remark, is maximal outerplanar.

Conversely, if G is a maximal outerplanar graph with $p \geqslant 3$ points, then G has at most one region which is not a triangle. It is now a direct consequence of Theorem 5 that G is uniquely 3-colorable.

Returning once again to Corollary 5a, we obtain yet another corollary. A *maximal planar* graph is a planar graph for which the addition of a line results in a nonplanar graph. It is obvious that every region of maximal planar graph is a triangle.

COROLLARY 5b. *Every maximal planar graph having chromatic number* 3 *is uniquely* 3-*colorable.*

Not every maximal planar graph is uniquely 3-colorable since there are such graphs having chromatic number 4. (There may even be maximal planar graphs having chromatic number 5.) For example, the complete

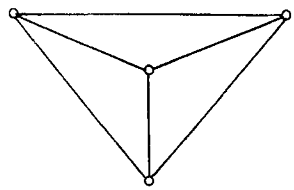

FIGURE 3

graph with 4 points (see Figure 3) has chromatic number 4; indeed, it is uniquely 4-colorable. This leads to our next result.

THEOREM 8. *Every uniquely 4-colorable planar graph is maximal planar.*

PROOF: Let G be a uniquely 4-colorable planar graph, which is 3-connected by Theorem 3. Thus every region of G is bounded by a cycle. Suppose there exists a region R of G which is not a triangle. By a theorem of Whitney [8], it is possible to embed G in the plane so that R is the exterior region. Let C be the cycle which bounds R and let there be given a 4-coloring of G.

First, consider the case in which two points of C, say u and v, have the same color α. Thus u and v are not adjacent. Let v_1 and v_2 be the points of C which are adjacent with v, and suppose they are assigned the colors α_1 and α_2, respectively (possibly $\alpha_1 = \alpha_2$). There exists a color β different from α, α_1, or α_2. By Theorem 2, there is a path P in G between u and v whose points are colored α or β. If $\alpha_1 \neq \alpha_2$, there is a path joining v_1 and v_2 all of whose points are colored α_1 or α_2, but this path must cross P, contradicting the planarity of G. If $\alpha_1 = \alpha_2$, then there exists a color γ different from α, α_1, or β, and there is a path joining v_1 and v_2 all of whose points are colored α_1 or γ. Again such a path must cross P contradicting the planarity of G. Therefore, this case cannot occur.

If no two points of C are colored the same, then C is a cycle of length 4, say $C = (v_1, v_2, v_3, v_4)$. Let v_i be colored with α_i, $1 \leqslant i \leqslant 4$, where $\alpha_i \neq \alpha_j$ for $i \neq j$. Again, by Theorem 2, there is a path P' between v_1 and v_3, all of whose points are colored α_1 or α_3. Similarly, there is a path P'' joining v_2 and v_4, all of whose points are colored α_2 or α_4. Since P' and P'' have no point in common, they must cross, but G is planar, and a contradiction arises here also. Thus, every region of G is a triangle.

It is well known that the maximum number of lines in a planar graph with $p \geqslant 3$ points is $3p - 6$, in which case the graph is maximal planar. Using this fact, one can give an alternative proof of the preceeding theorem. We employ this result, however, in the proof of the next theorem.[1]

[1] The authors thank Dr. Stephen Hedetniemi, who first formulated Theorem 9.

THEOREM 9. *No planar graph is uniquely 5-colorable.*

PROOF: Let G be a planar graph with p points and suppose G is uniquely 5-colorable. A 5-coloring of G produces five color classes V_i, $1 \leqslant i \leqslant 5$, where, say $|V_i| = p_i$. As in the proof of Theorem 7, the number of lines in the subgraph induced by $V_i \cup V_j$, $i \neq j$, is at least $p_i + p_j - 1$, and so the number of lines in G is at least $\sum(p_i + p_j - 1)$, where the sum is taken over all i and j such that $1 \leqslant i < j \leqslant 5$. This sum is $4\sum p_i - 10 = 4p - 10$; however, for $p \geqslant 5$, $4p - 10 > 3p - 6$ so that G is non-planar. This is a contradiction; hence G is not uniquely 5-colorable.

One of the consequences of Theorem 9 is that if the Four Color Conjecture is false, implying the existence of a planar graph G with chromatic number 5, then there is more than one way to 5-color G. The proof of the preceding theorem also implies that, in any 5-coloring of a planar graph, there are at least two color classes whose union does not induce a connected subgraph. We restate this as a corollary.

COROLLARY 9a. *In any 5-coloring of a planar graph G with chromatic number 5, there exist two points u and v which have the same color α such that for some color $\beta \neq \alpha$, there exists no path joining u and v, all of whose points are colored α or β.*

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