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Claw-free graphs with complete closure

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Abstract

We study some properties of the closure concept in claw-free graphs that was introduced by the first author. It is known that G is hamiltonian if and only if its closure is hamiltonian, but, on the other hand, there are infinite classes of non-pancyclic graphs with pancyclic closure. We show several structural properties of claw-free graphs with complete closure and their clique cutsets and, using these results, we prove that every claw-free graph on n vertices with complete closure contains a cycle of length $n-1$. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

We refer to [1] for terminology and notation not defined here and consider only finite undirected graphs $G=(V(G),E(G))$ without loops and multiple edges.

If G is a graph and $M\subset V(G)$, then the induced subgraph of G on M will be denoted by $\langle M\rangle_G$. We will simply write $G-M$ for $\langle V(G)\backslash M\rangle_G$ and $G-x$ for $G-\{x\}$ (where $x\in V(G)$). We will denote by $n_G=|V(G)|$ the order of G and by $c(G)$ the circumference of G (i.e. the length of a longest cycle in G). A graph G is *hamiltonian* if $c(G)=n_G$ and G is *pancyclic* if G contains a cycle of any length ℓ, $3\leqslant\ell\leqslant n_G$. By a *clique* we mean a (not necessarily maximal) complete subgraph of G. If $S\subset V(G)$ is a cutset of a connected graph G (i.e. $G-S$ is disconnected) such that $\langle S\rangle_G$ is a clique, we say that S is a *clique cutset* of G.

A graph G is *claw-free* if G does not contain a copy of *the claw* $K_{1,3}$ as an induced subgraph. Whenever we list vertices of an induced claw, its *center* (i.e. the only vertex of degree 3) is always the first vertex in the list.

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If C is a cycle in G with a fixed orientation and $u, v \in V(C)$, then by $u\overrightarrow{C}v$ ($v\overleftarrow{C}u$) we denote the consecutive vertices on C from u to v in the same (opposite) orientation with respect to the given orientation of C. The predecessor and successor of a vertex v on C will be denoted by v^- and v^+, respectively.

For any $x \in V(G)$, the set $N_G(x) = \{y \in V(G) \,|\, xy \in E(G)\}$ is called the *neighborhood of x in G*. For a set $M \subset V(G)$ we let $N_G(M) = \bigcup_{x \in M} N_G(x)$. We say that a vertex $x \in V(G)$ is *locally connected* if $\langle N_G(x)\rangle_G$ is a connected graph; otherwise x is said to be *locally disconnected*. A locally connected vertex x is said to be *eligible* if $\langle N_G(x)\rangle_G$ is not a clique; otherwise we say that x is *simplicial*. The set of all locally connected (eligible, simplicial, locally disconnected) vertices of G will be denoted by $V_{\rm LC}(G)$ ($V_{\rm EL}(G), V_{\rm SI}(G), V_{\rm LD}(G)$), respectively. Thus, the sets $V_{\rm EL}(G), V_{\rm SI}(G), V_{\rm LD}(G)$ are pairwise disjoint, $V_{\rm EL}(G) \cup V_{\rm SI}(G) = V_{\rm LC}(G)$ and $V_{\rm LC}(G) \cup V_{\rm LD}(G) = V(G)$. If $V_{\rm LC}(G) = V(G)$, we say that the graph G is locally connected.

Let $x \in V_{\rm EL}(G)$ be an eligible vertex and let $B_x = \{uv \,|\, u, v \in N_G(x), uv \notin E(G)\}$. Denote by G'_x the graph $G'_x = (V(G), E(G) \cup B_x)$ (i.e., G'_x is obtained from G by adding to $\langle N_G(x)\rangle_G$ all missing edges). The graph G'_x is called the *local completion of G at x*. The following proposition shows that the local completion operation preserves the claw-freeness and the value of circumference of G.

Proposition A (Ryjáček [3]). *Let G be a claw-free graph and let $x \in V_{\rm EL}(G)$ be an eligible vertex of G. Then*

(i) *the graph G'_x is claw-free,*
(ii) $c(G'_x) = c(G)$.

Apparently, if $x \in V_{\rm EL}(G)$, then x becomes simplicial in G'_x and, if $V_{\rm EL}(G'_x) \neq \emptyset$, the local completion operation can be applied repeatedly to another vertex. We thus obtain the following concept (introduced in [3]).

Let G be a claw-free graph. We say that a graph H is a closure of G, denoted $H = \mathrm{cl}(G)$, *if*

(i) *there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = H$, $x_i \in V_{\rm EL}(G_i)$ and $G_{i+1} = (G_i)'_{x_i}$, $i = 1, \ldots, t-1$,*
(ii) $V_{\rm EL}(H) = \emptyset$.

The following result summarizes basic properties of the closure operation.

Theorem B (Ryjáček [3]). *Let G be a claw-free graph. Then*

(i) *the closure* $\mathrm{cl}(G)$ *is well-defined,*
(ii) *there is a triangle-free graph H such that* $\mathrm{cl}(G)$ *is the line graph of H,*
(iii) $c(G) = c(\mathrm{cl}(G))$.

Remarks. (1) Part (i) of Theorem B says that $\mathrm{cl}(G)$ is uniquely determined, i.e., does not depend on the order of eligible vertices used during the construction.

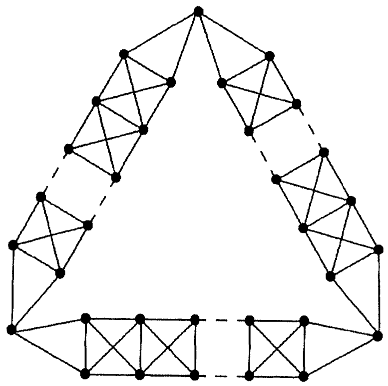

Fig. 1.

2. It is easy to see that cl(G) can be equivalently characterized as the minimum graph containing G, which does not contain an induced subgraph isomorphic to the diamond ($K_4 - e$).

Specifically, by part (iii) of Theorem B, a claw-free graph G is hamiltonian if and only if cl(G) is hamiltonian. On the other hand, the following theorem shows that this is not the case with the property of pancyclicity.

Theorem C (Brandt et al. [2]). *For every $k \geqslant 2$ there is a k-connected claw-free graph G such that G is not pancyclic but* cl(G) *is pancyclic.*

An example of an infinite family of such graphs for $k = 2$ is shown in Fig. 1. The graph in Fig. 1 is, moreover, an example of a non-pancyclic graph having a complete (and hence pancyclic) closure. This situation gives rise to the following question.

Problem. *Determine the maximum number $c_{\rm m}(n)$ of cycle lengths that can be missing in a claw-free graph on n vertices with complete closure.*

Let $k \geqslant 1$ and let G be the graph in Fig. 1 of order $n_G = 6k+3$. Then G is claw-free, cl(G) is complete and G contains no cycle of length ℓ for $2k+3 \leqslant \ell \leqslant 3k+2$, i.e. G misses $k = (n_G - 3)/6$ cycle lengths. This example shows that $c_{\rm m}(n) \geqslant (n-3)/6$. On the other hand, it is easy to see that a claw-free graph with complete closure on at least 4 vertices can miss neither a C_3 nor a C_4. Also, the main result of Section 3 shows that such a graph G cannot be missing a cycle of length $n_G - 1$.

More is likely to be true. No example is known when G has complete closure and large order but fails to contain one of all possible ‘short length’ and ‘long length’ cycles. We state this precisely as the following conjecture.

Conjecture. Let c_1, c_2 be fixed constants. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, where $3 \leqslant i \leqslant c_1$ and $n - c_2 \leqslant i \leqslant n$.

In Section 2 we prove several structural results about graphs with a clique cutset and their closures. In Section 3 we use these results to prove that every claw-free graph G with complete closure has a cycle of length $n_G - 1$.

2. Closure and clique cutsets

We begin with several simple observations.

Proposition 1. *Let G be a claw-free graph. Then* $V_{\rm SI}(G) \subset V_{\rm SI}({\rm cl}(G))$.

Proof. It is sufficient to show that, for any $x \in V_{\rm EL}(G)$, $V_{\rm SI}(G) \subset V_{\rm SI}(G'_x)$. Let $y \in V_{\rm SI}(G)$. If $xy \notin E(G)$, then no edge in B_x contains y and hence $N_{G'_x}(y) = N_G(y)$. If $xy \in E(G)$, then, since $\langle N_G(y)\rangle_G$ is a clique, $N_G(y) \subset N_G(x) \cup \{x\}$ and hence $\langle N_{G'_x}(y) \cup \{y\}\rangle_{G'_x} = \langle N_{G'_x}(x) \cup \{x\}\rangle_{G'_x}$. In both cases, $y \in V_{\rm SI}(G'_x)$.

Corollary 2. *For any claw-free graph G, the closure* ${\rm cl}(G)$ *is constructed in at most* $n_G = |V(G)|$ *local completions.*

Proposition 3. *Let G be a claw-free graph and let H be an induced subgraph of G. Then* $V_{\rm EL}(H) \subset V_{\rm EL}(G)$.

Proof. Let $x \in V_{\rm EL}(H)$ and let $z_1, z_2 \in N_H(x)$ be nonadjacent in $\langle N_H(x)\rangle_H$. If $x \in V_{\rm SI}(G)$, then $z_1z_2 \in E(G)$, implying $z_1z_2 \in E(H)$, a contradiction. If $x \in V_{\rm LD}(G)$, then, since x is eligible in H, the vertices z_1, z_2 are in the same component of $\langle N_G(x)\rangle_G$ and $z_1z_2 \notin E(G)$, but then, for any vertex z lying in the second component of $\langle N_G(x)\rangle_G$, $\langle\{x, z, z_1, z_2\}\rangle$ is a claw in G, which is again a contradiction. Hence $x \in V_{\rm EL}(G)$. □

Corollary 4. *Let H be an induced subgraph of a claw-free graph G. Then* ${\rm cl}(H) \subset \langle V(H)\rangle_{{\rm cl}(G)}$.

Proof. Let $H_1, \ldots, H_s$ and $x_1, \ldots, x_{s-1}$ be the sequences of graphs and corresponding eligible vertices that yield ${\rm cl}(H)$ (i.e., $H_1 = H$, $H_s = {\rm cl}(H)$, $x_j \in V_{\rm EL}(H_j)$ and $H_{j+1} = (H_j)'_{x_j}$, $j = 1, \ldots, s-1$). By Proposition 3, $x_1 \in V_{\rm EL}(G)$ and we can let $G_2 = G'_{x_1}$. Note that H_2 is an induced subgraph of G_2. By induction (and by Proposition 3), $x_j \in V_{\rm EL}(G_j)$ and we can let $G_{j+1} = (G_j)'_{x_j}$, $j = 2, \ldots, s-1$. Then ${\rm cl}(H) = \langle V(H)\rangle_{G_s}$. Since ${\rm cl}(G)$ is independent of the order of eligible vertices used during the construction, there are vertices $x_{s+1}, \ldots, x_t \in V(G)$ such that the sequence of local completions of G

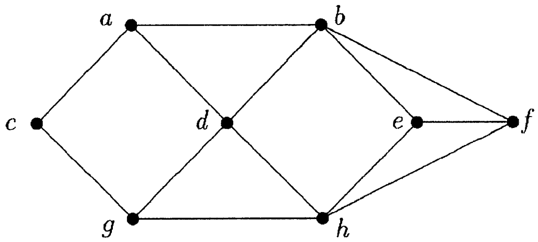

Fig. 2.

at $x_1,\ldots,x_s,x_{s+1},\ldots,x_t$ yields $\mathrm{cl}(G)$. Hence we have $\mathrm{cl}(H)=\langle V(H)\rangle_{G_s}\subset\langle V(H)\rangle_{G_t}=\langle V(H)\rangle_{\mathrm{cl}(G)}$. $\square$

Example. Let G be the graph in Fig. 2 and let $H=\langle\{a,c,d,g\}\rangle_G\subset G$. Then $\mathrm{cl}(H)\simeq C_4$, while $\langle V(H)\rangle_{\mathrm{cl}(G)}\simeq K_4$. Thus, it is possible that $\mathrm{cl}(H)$ is a proper subgraph of $\langle V(H)\rangle_{\mathrm{cl}(G)}$.

The following theorem is the main result of this section, giving structural information of the closure of the whole graph G in terms of the closures of its corresponding parts. Its corollaries will be useful in the next section for decomposition of $\mathrm{cl}(G)$ by means of clique cutsets.

Theorem 5. *Let $S\subset V(G)$ be a clique cutset of a claw-free graph G and let H_i, $i=1,\ldots,k$, be the components of $G-S$. For $i=1,\ldots,k$ let $S_i=N_G(V(H_i))\cap S$ and $G_i=\langle V(H_i)\cup S_i\rangle_G$. Let $I_0=\{i\mid |S_i|=1\}$ and $S_0=\bigcup_{i\in I_0}S_i$. Then*

(i) $V_{\mathrm{LD}}(\mathrm{cl}(G))=(\bigcup_{i=1}^k V_{\mathrm{LD}}(\mathrm{cl}(G_i)))\cup S_0$,
(ii) $\mathrm{cl}(G_i)=\langle V(G_i)\rangle_{\mathrm{cl}(G)}$.

Proof. Let K^i be the largest clique in $\mathrm{cl}(G_i)$ containing the clique $\langle S_i\rangle_G$, $i=1,\ldots,k$. Then, for every i and every $x\in V(K^i)$, either $\langle N_{\mathrm{cl}(G_i)}(x)\rangle_{\mathrm{cl}(G_i)}=K^i-x$ (and $x\in V_{\mathrm{SI}}(\mathrm{cl}(G_i))$), or $\langle N_{\mathrm{cl}(G_i)}(x)\rangle_{\mathrm{cl}(G_i)}$ consists of two disjoint cliques, one of them being K^i-x (and then $x\in V_{\mathrm{LD}}(\mathrm{cl}(G_i))$). Let $\tilde G$ be the graph obtained by taking a copy of each $\mathrm{cl}(G_i)$ and a copy of $\langle S\rangle_G$ and by identifying the vertices of every S_i with the corresponding vertices of S, $i=1,\ldots,k$. By Corollary 4, $\tilde G\subset\mathrm{cl}(G)$. Note that $\tilde G$ can contain induced claws centered at vertices of S (for example, if $S_1=\{a_1,a_2,a_3\}$, $\{b_1,b_2\}\subset V(H_1)$, $N_S(b_1)=\{a_1\}$ and $N_S(b_2)=\{a_2,a_3\}$, then we get $a_1b_2\in E(\mathrm{cl}(G))$ and, if $b_1b_2\notin E(\mathrm{cl}(G))$, then $\langle\{a_1,b_1,b_2,x\}\rangle_{\tilde G}$ is a claw for any $x\in S\backslash S_1$). It is straightforward to check that if $|S_{i_0}|=1$ for some $i_0\in I_0$, then $S_{i_0}\subset V_{\mathrm{LD}}(\mathrm{cl}(G))$ and $V_{\mathrm{LD}}(\mathrm{cl}(G_{i_0}))\cup S_{i_0}=V_{\mathrm{LD}}(\mathrm{cl}(G))\cap V(G_{i_0})$, and hence it is sufficient to verify the theorem in $G-V(H_{i_0})$. Hence we can suppose that $|S_i|\geqslant 2$ for every $i=1,\ldots,k$. Then the subgraph $\langle S\cup(\bigcup_{i=1}^k V(K^i))\rangle_{\tilde G}$ is locally connected. Let $\hat G$ be the graph obtained from $\tilde G$ by adding to $\langle S\cup(\bigcup_{i=1}^k V(K^i))\rangle_{\tilde G}$ all missing edges (i.e., the subgraph $K=\langle S\cup(\bigcup_{i=1}^k V(K^i))\rangle_{\hat G}$ is a clique). Since $\tilde G\subset\mathrm{cl}(G)$ and $\langle S\cup(\bigcup_{i=1}^k V(K^i))\rangle_{\tilde G}$ is locally connected, $\hat G\subset\mathrm{cl}(G)$. By the construction, it is now straightforward to verify the following facts:

(a) $\hat{G}$ is claw-free,
(b) if $x \in V(G_i)\backslash V(K)$, then $\langle N_{\mathrm{cl}(G_i)}(x)\rangle_{\mathrm{cl}(G_i)} = \langle N_{\hat{G}}(x)\rangle_{\hat{G}}$,
(c) if $x \in V(K^i)\backslash S$ for some $i = 1,\ldots,k$, then
(α) if $x \in V_{\mathrm{SI}}(\mathrm{cl}(G_i))$, then $\langle N_{\hat{G}}(x)\rangle_{\hat{G}} = K - x$ and hence $x \in V_{\mathrm{SI}}(\hat{G})$, and
(β) if $x \in V_{\mathrm{LD}}(\mathrm{cl}(G_i))$, then one component of $\langle N_{\hat{G}}(x)\rangle_{\hat{G}}$ is $K - x$ and the other component is the same in $\mathrm{cl}(G_i)$ and in $\hat{G}$, and hence $x \in V_{\mathrm{LD}}(\hat{G})$,
(d) if $x \in S$, then $x \in V_{\mathrm{LD}}(\mathrm{cl}(G_i))$ for at most one i, $1 \leqslant i \leqslant k$, since if $x \in V_{\mathrm{LD}}(\mathrm{cl}(G_{i_1})) \cap V_{\mathrm{LD}}(\mathrm{cl}(G_{i_2}))$ for some i_1, i_2 with $1 \leqslant i_1 < i_2 \leqslant k$, then x centers a claw in $\hat{G}$, contradicting (a), and
(α) if $x \in V_{\mathrm{SI}}(\mathrm{cl}(G_i))$ for all $i = 1,\ldots,k$, for which $x \in V(G_i)$, then $x \in V_{\mathrm{SI}}(\hat{G})$,
(β) if there is an i_0, $1 \leqslant i_0 \leqslant k$, such that $x \in V_{\mathrm{LD}}(\mathrm{cl}(G_{i_0}))$, then $x \in V_{\mathrm{LD}}(\hat{G})$.

(Note that (dα) includes the case when $x \notin \bigcup_{i=1}^{k} V(G_i)$). This immediately implies that $V(\hat{G}) = V_{\mathrm{SI}}(\hat{G}) \cup V_{\mathrm{LD}}(\hat{G})$, i.e., $V_{\mathrm{EL}}(\hat{G}) = \emptyset$. Since $\hat{G} \subset \mathrm{cl}(G)$, we have $\hat{G} = \mathrm{cl}(G)$, and by (b), ($c\beta$) and (dβ), $V_{\mathrm{LD}}(\hat{G}) = \bigcup_{i=1}^{k} V_{\mathrm{LD}}(\mathrm{cl}(G_i))$.

Proof of part (ii) follows immediately from the construction of $\hat{G} = \mathrm{cl}(G)$. $\square$

Example. Let G be the graph in Fig. 2 and put $S = \{b,h\}$, $G_1 = \langle\{a,b,c,d,g,h\}\rangle_G$, $G_2 = \langle\{b,e,f,h\}\rangle_G$. Then $V_{\mathrm{LD}}(\mathrm{cl}(G_1)) = \{a,c,d,g\}$, but $V_{\mathrm{LD}}(\mathrm{cl}(G)) = \emptyset$. This example shows that Theorem 5 fails if $\langle S\rangle_G$ is not a clique.

Corollary 6. *Let G be a claw-free graph and let $S \subset V(G)$ be a clique cutset of G. Denote by $H_1,\ldots,H_k$ the components of $G - S$, let $S_i = N_G(V(H_i)) \cap S$ and let $G_i = \langle V(H_i) \cup S_i\rangle_G$. Suppose that $|S_i| \geqslant 2$, $i = 1,\ldots,k$. Then* $\mathrm{cl}(G)$ *is complete if and only if* $\mathrm{cl}(G_i)$ *is complete for every $i = 1,\ldots,k$.*

Proof. If $\mathrm{cl}(G)$ is complete, then all $\mathrm{cl}(G_i)$ are complete by part (ii) of Theorem 5. Conversely, suppose that all $\mathrm{cl}(G_i)$ are complete and let K^i, $K, \tilde{G}$ and $\hat{G}$ be the same as in the proof of Theorem 5. Then $K^i = G_i$, $\tilde{G}$ is locally connected and $\hat{G} = \mathrm{cl}(G) = K$. $\square$

Corollary 7. *Let G be a claw-free graph and let $x \in V_{\mathrm{SI}}(G)$. Then* $\mathrm{cl}(G)$ *is complete if and only if* $\mathrm{cl}(G - x)$ *is complete.*

Proof. If $x \in V_{\mathrm{SI}}(G)$, then $\langle N_G(x)\rangle_G$ is a clique cutset. The rest of the proof follows immediately from Corollary 6 by setting $S = N_G(x)$. $\square$

3. Cycle of length $n_G - 1$

In the main result of this section, Theorem 12, we prove that every claw-free graph G with complete closure contains a cycle of length $n_G - 1$. Before we present this result, we first prove several auxiliary statements. The first of them is of importance in its own right.

We say that a set $S \subset V(G)$ is *cyclable in G* if there is a cycle $C \subset G$ such that $V(C) = S$.

Theorem 8. *Let G be a claw-free graph and let $G_0, G_1, \ldots, G_t$, $t \geqslant 1$, be a sequence of graphs such that $G_0 = G$ and $G_i = (G_{i-1})'_{x_{i-1}}$ for some $x_{i-1} \in V_{\mathrm{EL}}(G_{i-1})$, $i = 1, \ldots, t$. Let $B_i = E(G_i) \backslash E(G_{i-1})$ $(i = 1, \ldots, t)$ and $B_0 = E(G_0)$. For every cycle $C \subset G_t$ set $b_i(C) = |E(C) \cap B_i|$, $i = 0, 1, \ldots, t$. Then for every cyclable set S in G_t there is a cycle C in G_t with $V(C) = S$ such that*

(i) $b_i(C) \leqslant 2$ *for every* $i = 1, \ldots, t$,
(ii) *if* $x_{i-1}x_i \in E(G_{i-1})$ *and* $b_{i+1}(C) \geqslant 1$, *then* $b_i(C) \leqslant 1$ $(1 \leqslant i \leqslant t-1)$.

Proof. Since every edge $e \in E(G_t)$ is in exactly one B_k $(0 \leqslant k \leqslant t)$, we can define a weight function $w(e)$ on $E(G_t)$ by $w(e) = k$ if $e \in B_k$. For any cycle $C \subset G_t$ we define the weight of C by $w(C) = \sum_{e \in E(C)} w(e)$. Let $S \subset V(G)$ be cyclable in G_t and let C be a cycle in G_t such that $V(C) = S$ and $w(C)$ is as small as possible.

(i) Let, to the contrary, $b_i(C) \geqslant 3$ for some i, $1 \leqslant i \leqslant t$, and let e_1, e_2, e_3 be distinct edges in $E(C) \cap B_i$. Let $e_j = u_j v_j$ $(1 \leqslant j \leqslant 3)$, and assume the notation is chosen such that u_1, v_1, u_2, v_2, u_3 and v_3 appear in this order along C. Then u_1, u_2, u_3 are distinct vertices in $N_{G_{i-1}}(x_{i-1})$. Since $\langle \{x_{i-1}, u_1, u_2, u_3\} \rangle_{G_{i-1}}$ cannot be an induced claw, $\{u_1u_2, u_1u_3, u_2u_3\} \cap E(G_{i-1}) \neq \emptyset$. By symmetry, we can suppose that $u_1u_2 \in E(G_{i-1})$. Let $C' = v_2 \overrightarrow{C} u_1 u_2 \overleftarrow{C} v_1 v_2$. Then C' is a cycle in G_i with $V(C') = V(C) = S$ (recall that $v_1v_2 \in V(G_i)$ since $v_1, v_2 \in N_{G_{i-1}}(x_{i-1})$), and $E(C') = E(C) \backslash \{u_1v_1, u_2v_2\} \cup \{u_1u_2, v_1v_2\}$. By the assumption, $w(u_1v_1) = w(u_2v_2) = i$. On the other hand, since $u_1u_2 \in E(G_{i-1})$ and $v_1v_2 \in E(G_i)$, $w(u_1u_2) \leqslant i-1$ and $w(v_1v_2) \leqslant i$. Therefore, we have $w(C') \leqslant w(C) - (i+i) + (i-1+i) = w(C) - 1$, contradicting the minimality of C.

(ii) Assume that $b_i(C) \geqslant 2$ and $b_{i+1}(C) \geqslant 1$. Let $e_1, e_2 \in E(C) \cap B_i$, $e_1 \neq e_2$, setting $e_j = u_j v_j$ $(j = 1, 2)$ and let $e = uv \in E(C) \cap B_{i+1}$. Suppose that the notation is chosen such that u, v, u_1, v_1, u_2 and v_2 appear in this order along C. By the definition, $\{u_1, v_1, u_2, v_2\} \subset N_{G_{i-1}}(x_{i-1})$ and $\{u, v\} \subset N_{G_i}(x_i)$. Apparently, $u_1 \neq u_2$. If $u_1u_2 \in E(G_{i-1})$, then let $C' = v_2 \overrightarrow{C} u_1 u_2 \overleftarrow{C} v_1 v_2$. Then C' is a cycle in G_t with $V(C') = V(C) = S$ and $E(C') = E(C) \backslash \{u_1v_1, u_2v_2\} \cup \{u_1u_2, v_1v_2\}$. Since $w(u_1v_1) = w(u_2v_2) = i$, $w(u_1u_2) \leqslant i-1$ and $w(v_1v_2) \leqslant i$, we have $w(C') \leqslant w(C) - 2i + 2i - 1 = w(C) - 1$, a contradiction. Therefore, $u_1u_2 \notin E(G_{i-1})$. Similarly, $v_1v_2 \notin E(G_{i-1})$.

Next consider u and u_1. Apparently $u \neq u_1$, and we show that $uu_1 \notin E(G_{i-1})$. Let $uu_1 \in E(G_{i-1})$ and set $C' = v_1 \overrightarrow{C} uu_1 \overleftarrow{C} vv_1$. First suppose $v_1 \neq x_i$. Then, since $v_1, x_i \in N_{G_{i-1}}(x_{i-1})$, we have $v_1x_i \in E(G_i)$. Since $v \neq v_1$, this implies $vv_1 \in E(G_{i+1})$. Hence C' is a cycle in $G_{i+1} \subset G_t$ with $V(C') = V(C) = S$ and with $E(C') = E(C) \backslash \{uv, u_1v_1\} \cup \{uu_1, vv_1\}$. Since $w(uv) = i+1$, $w(u_1v_1) = i$, $w(uu_1) \leqslant i-1$ and $w(vv_1) \leqslant i+1$, we have $w(C') \leqslant w(C) - i - (i+1) + (i-1) + (i+1) = w(C) - 1$, a contradiction. Let thus $v_1 = x_i$. Then $vv_1 = vx_i \in E(G_i)$, and since again $E(C') = E(C) \backslash \{uv, u_1v_1\} \cup$

$\{uu_1, vv_1\}$ and $w(uv) = i + 1$, $w(u_1v_1) = i$, $w(uu_1) \leqslant i - 1$ and $w(vv_1) \leqslant i$, we obtain $w(C') \leqslant w(C) - i - (i + 1) + (i - 1) + i = w(C) - 2$, which is again a contradiction. Hence $uu_1 \notin E(G_{i-1})$. Similarly, $uu_2 \notin E(G_{i-1})$, $vv_1 \notin E(G_{i-1})$ and $vv_2 \notin E(G_{i-1})$. Hence $\{u, u_1, u_2\}$ and $\{v, v_1, v_2\}$ are independent sets in G_{i-1}. This implies that $x_{i-1}u \notin E(G_{i-1})$ (since otherwise $\langle\{x_{i-1}, u, u_1, u_2\rangle\}_{G_{i-1}}$ is a claw) and hence $x_iu \notin B_i$, which implies $x_iu \in E(G_{i-1})$. Similarly we have $x_{i-1}v \notin E(G_{i-1})$ and $x_iv \in E(G_{i-1})$. Since $u_1x_{i-1} \in E(G_{i-1})$ but $u_1u \notin E(G_{i-1})$, we have $x_{i-1} \neq u$, and similarly $x_{i-1} \neq v$, but then $\langle\{x_i, x_{i-1}, u, v\}\rangle_{G_{i-1}}$ is a claw. This contradiction proves the theorem. $\square$

Let C be a cycle in a graph G. An edge $uv \in E(G)\backslash E(C)$ with $u, v \in E(C)$ will be called a *chord of* C. A *2-chord* of a cycle C is a chord xy of C such that $x\overrightarrow{C}y$ or $x\overleftarrow{C}y$ has exactly one interior vertex. If $u_1v_1, u_2v_2 \in E(G)\backslash E(C)$ are such that $u_1, v_1 \in V(C)$ and either $\{u_2, v_2\} = \{u_1^-, v_1^+\}$ or $\{u_2, v_2\} = \{u_1^+, v_1^-\}$, then we say that the edges u_1v_1 and u_2v_2 are a *pair of parallel chords of* C.

Lemma 9. *Let G be a claw-free graph on n_G vertices such that* $\mathrm{cl}(G)$ *is complete and G has no cycle of length $n_G - 1$. Let C be a hamiltonian cycle in G and let $xy \in E(G)\backslash E(C)$ be a chord of C. Then there is a pair of parallel chords uv, u^-v^+ of C such that $x \in \{u^-, u\}$ and $y \in \{v, v^+\}$.*

Proof. Since G has no cycle of length $n_G - 1$, C has no 2-chord, and hence all the vertices x^-, x^+, y^-, y^+ exist and are distinct. Since $\langle\{x, x^-, x^+, y\}\rangle_G$ cannot be a claw, we have $x^-y \in E(G)$ or $x^+y \in E(G)$; from $\langle\{y, y^-, y^+, x\}\rangle_G \not\simeq K_{1,3}$ similarly $xy^- \in E(G)$ or $xy^+ \in E(G)$. If $x^-y \in E(G)$ and $xy^- \in E(G)$ or $x^+y \in E(G)$ and $xy^+ \in E(G)$, then we are done; thus suppose that $x^-y \in E(G)$ and $xy^+ \in E(G)$ or $x^+y \in E(G)$ and $xy^- \in E(G)$. In the first case, since $x^-y^- \notin E(G)$ (otherwise $xy^+\overrightarrow{C}x^-y^-\overleftarrow{C}x$ is a cycle of length $n_G - 1$), from $\langle\{y, y^-, y^+, x^-\}\rangle_G \not\simeq K_{1,3}$ we get $x^-y^+ \in E(G)$. The second case is symmetric. $\square$

Lemma 10. *Let G be a claw-free graph having no cycle of length $n_G - 1$. Let C be a hamiltonian cycle in G and $\{x, y\}$ a cutset of G such that $\langle\{x^-, x, y, y^+\}\rangle_G \simeq K_4$. Then*

(i) $N_G(x) \cap (y^+\overrightarrow{C}x^-) = N_G(y) \cap (y^+\overrightarrow{C}x^-)$,
(ii) $\langle(N_G(x) \cap (y^+\overrightarrow{C}x^-)) \cup \{x, y\}\rangle_G$ *is a clique.*

Proof. By symmetry, it is sufficient to show that $N_G(y) \cap (y^+\overrightarrow{C}x^-) \subset N_G(x) \cap (y^+\overrightarrow{C}x^-)$. Let thus $z \in N_G(y) \cap (y^+\overrightarrow{C}x^-)$. If $z = y^+$ or $z = x^-$, then obviously $z \in N_G(x)$. Hence we may assume $z \in y^{++}\overrightarrow{C}x^{--}$. Considering $\langle\{z, z^-, z^+, y\}\rangle_G$ we have $z^-y \in E(G)$ or $z^+y \in E(G)$. Suppose without loss of generality that $z^-y \in E(G)$ (otherwise we change the notation). Since $\{x, y\}$ is a cutset, $y^-z^- \notin E(G)$ and $y^-z \notin E(G)$. From $\langle\{y, y^-, y^+, z\}\rangle_G \not\simeq K_{1,3}$ and $\langle\{y, y^-, y^+, z^-\}\rangle_G \not\simeq K_{1,3}$ we then get $y^+z \in E(G)$ and $y^+z^- \in E(G)$, i.e., $\langle\{y, y^+, z^-, z\}\rangle_G \simeq K_4$. From $\langle\{y^+, y^{++}, z, x\}\rangle_G \not\simeq K_{1,3}$ we now get

$zx \in E(G)$ (since if $y^{++}x \in E(G)$, then $xy^{++}\overrightarrow{C}x^{-}y\overrightarrow{C}x$, and if $y^{++}z \in E(G)$, then $y^{++}z\overrightarrow{C}yz^{-}\overleftarrow{C}y^{++}$ is a cycle of length $n_G - 1$). Now, since $z^{+}x \notin E(G)$ (otherwise $xz^{+}\overrightarrow{C}x^{-}y^{+}\overrightarrow{C}z^{-}y\overleftarrow{C}x$ is a cycle of length $n_G - 1$), from $\langle\{z, z^{-}, z^{+}, x\}\rangle_G \not\simeq K_{1,3}$ we get also $z^{-}x \in E(G)$. Hence $N_G(y) \cap (y^{+}\overrightarrow{C}x^{-}) \subset N_G(x) \cap (y^{+}\overrightarrow{C}x^{-})$.

If some $u, v \in N_G(x) \cap (y^{+}\overrightarrow{C}x^{-})$ are nonadjacent, then $\langle\{x, x^{+}, u, v\}\rangle_G$ is a claw. Hence $\langle(N_G(x) \cap (y^{+}\overrightarrow{C}x^{-})) \cup \{x, y\}\rangle_G$ is a clique. □

Lemma 11. *Let G be a minimal (with respect to $n_G = |V(G)|$) claw-free graph with complete closure and without a cycle of length $n_G - 1$. Let C be a hamiltonian cycle in G and let $\{x, y\}$ be a cutset of G such that $\langle\{x, x^{-}, y, y^{+}\}\rangle_G$ is a clique. Then $|x\overrightarrow{C}y| = |y^{+}\overrightarrow{C}x^{-}| = n_G/2$.*

Proof. Let $G_1 = \langle x\overrightarrow{C}y\rangle_G$ and $G_2 = \langle y\overrightarrow{C}x\rangle_G$. Let H_1 be the graph obtained by taking two vertex disjoint copies of G_1 and by adding the edges $x^1x^2, y^1y^2, x^1y^2, x^2y^1$ (where by x^i, y^i we denote the vertices corresponding to the vertices x and y in the ith copy of G_1, $i = 1, 2$), and let H_2 be the graph obtained by identifying the vertices corresponding to the vertices x and y in two vertex disjoint copies of G_2. Then, by Corollary 6, both H_1 and H_2 have complete closure. If some H_i, $i \in \{1, 2\}$, has a cycle of length $n_{H_i} - 1$, then, by the construction and since $\{x, y\}$ is a cutset, we apparently have a cycle of length $n_G - 1$ in G. Hence, by the minimality of G, $|V(H_i)| \geqslant n_G$, $i = 1, 2$. If we show that, moreover, $|V(H_2)| \geqslant n_G + 2$, then we have $|V(H_1)| = 2|x\overrightarrow{C}y| \geqslant n_G$ and $|V(H_2)| - 2 = 2|y^{+}\overrightarrow{C}x^{-}| \geqslant n_G$. Since $|x\overrightarrow{C}y| + |y^{+}\overrightarrow{C}x^{-}| = n_G$, this implies $|x\overrightarrow{C}y| = |y^{+}\overrightarrow{C}x^{-}| = n_G/2$.

Hence it remains to show that $|V(H_2)| \geqslant n_G + 2$. Suppose, to the contrary, $|V(H_2)| \leqslant n_G + 1$, and let $H = (H_2)'_x$. Since $\{x, y\}$ is a cutset of H_2, by Lemma 10, y is simplicial in H. The graph $\hat{H} = H - \{x, y\}$ is obviously claw-free and, by Corollary 7, $\mathrm{cl}(\hat{H})$ is complete. Since $|V(\hat{H})| = |V(H_2)| - 2 \leqslant n_G + 1 - 2 = n_G - 1$, by the minimality of G, $\hat{H}$ has a cycle $C_{\hat{H}}$ of length $n_{\hat{H}} - 1$. Let $B = E(H) \backslash E(H_2)$. Since $\{x, y\}$ is a cutset of H_2, $|E(C_{\hat{H}}) \cap B| \geqslant 2$. By Theorem 8(*i*), $C_{\hat{H}}$ can be chosen such that $|E(C_{\hat{H}}) \cap B| = 2$. Let $e_1 = u_1v_1$, $e_2 = u_2v_2$ be these edges. Since $\{x, y\}$ is a cutset of H_2, each of e_1, e_2 has its endvertices in different components of $H_2 - \{x, y\}$. By Lemma 10(*ii*), replacing in $C_{\hat{H}}$ the edges u_1v_1 and u_2v_2 by the paths u_1xv_1 and u_2yv_2, we get a cycle C_{H_2} in H_2 of length $n_{H_2} - 1$. Let P be the shorter of the paths $y\overrightarrow{C_{H_2}}x$ and $y\overleftarrow{C_{H_2}}x$. Then the cycle $x\overrightarrow{C}yPx$ is a cycle in G of length $n_G - 1$. This contradiction proves the lemma. □

Now we can proceed to the main result of this section.

Theorem 12. *Let G be a claw-free graph such that* $\mathrm{cl}(G)$ *is complete. Then G contains a cycle of length $n_G - 1$.*

Proof. Suppose the theorem fails and let G be a counterexample with minimum $n_G = |V(G)|$. Let C be a hamiltonian cycle in G. We first make two general observations.

(i) The cycle C has no 2-chords, i.e., for any chord uv of C, both $u\overrightarrow{C}v$ and $u\overleftarrow{C}v$ have at least two interior vertices.
(ii) If a vertex x has two nonadjacent neighbors u, v lying in the same component of $\langle N_G(x)\rangle_G$, then $x \in V_{\mathrm{EL}}(G)$ (since if x is locally disconnected, then for any z in the other component of $\langle N_G(x)\rangle_G$, $\langle x, u, v, z\rangle_G$ is a claw).

These observations will be often used implicitly throughout the proof.

For any hamiltonian cycle C and an eligible vertex x we say that the vertex x is *of the first type with respect to C*, if there is an x^-,x^+-path of length 2 in $\langle N_G(x)\rangle_G$. In the other case (i.e., if all x^-,x^+-paths in $\langle N_G(x)\rangle_G$ have length at least 3), we say that x is *of the second type with respect to C*.

First suppose that the hamiltonian cycle C can be chosen such that there is a vertex $x \in V_{\mathrm{EL}}(G)$ of the first type with respect to C. Let y be a common neighbor of x^- and x^+ in $\langle N_G(x)\rangle_G$. If $x^-y^- \in E(G)$, then $x^-y^-\overleftarrow{C}x^+y\overrightarrow{C}x^-$ is a cycle of length $n_G - 1$; thus $x^-y^- \notin E(G)$. From $\langle\{y, y^-, y^+, x^-\}\rangle_G$ we get $x^-y^+ \in E(G)$ and, by symmetry, $x^+y^- \in E(G)$. Since $\langle\{y, y^-, y^+, x\}\rangle_G$ cannot be a claw, we have $xy^- \in E(G)$ or $xy^+ \in E(G)$. By symmetry, we can suppose that $xy^+ \in E(G)$. Then $\langle\{x^-, x, y, y^+\}\rangle_G \simeq K_4$. We consider the conditions under which $\{x, y\}$ can be a cutset of G.

By Lemma 9, it is sufficient to verify the nonexistence of all possible pairs of parallel chords uv, u^+v^- such that $u, u^+ \in y\overrightarrow{C}x$ and $v^-, v \in x\overrightarrow{C}y$.

Case	Cycle of length $n_G - 1$
$u, u^+ \in y\overrightarrow{C}x^-$; $v^-, v \in x^+\overrightarrow{C}y^-$	$uv\overrightarrow{C}y^-x^+\overrightarrow{C}v^-u^+\overrightarrow{C}x^-y\overrightarrow{C}u$
$u^+ = x$; $v^-, v \in x^+\overrightarrow{C}y^-$	$xy^+\overrightarrow{C}x^-v\overrightarrow{C}y^-x^+\overrightarrow{C}v^-x$
$u, u^+ \in y^+\overrightarrow{C}x^-$; $v = y$	$uyx^+\overrightarrow{C}y^-u^+\overrightarrow{C}x^-y^+\overrightarrow{C}u$

We thus have the following observation.

($*$) The only possible pair of parallel chords uv, u^+v^- such that at least one of them crosses the edge xy, is for $v^- = x$, $v = x^+$; $u, u^+ \in y^+\overrightarrow{C}x^-$.

(This observation will be used several times in what follows.)

We show that $xy^- \notin E(G)$. Indeed, if $xy^- \in E(G)$, then, by symmetry and by the previous observations, $\{x, y\}$ is a cutset of G. But then, since $\langle\{x, y, x^+, y^-\}\rangle_G \simeq \langle\{x, y, x^-, y^+\}\rangle_G \simeq K_4$, by Lemma 11 we have $|x^+\overrightarrow{C}y^-| = |y\overrightarrow{C}x| = n_G/2$ and $|x\overrightarrow{C}y| = |y^+\overrightarrow{C}x^-| = n_G/2$, from which $n_G = |x^+\overrightarrow{C}y^-| + |y^+\overrightarrow{C}x^-| + |\{x, y\}| = n_G/2 + n_G/2 + 2 > n_G$, a contradiction. Hence $xy^- \notin E(G)$. Considering $\langle\{x^+, x, x^{++}, y^-\}\rangle_G$ we then have $x^{++}y^- \in E(G)$.

We now prove that $x^{++}y \in E(G)$. Thus suppose, to the contrary, $x^{++}y \notin E(G)$. Then from $\langle\{y^-, y, y^{--}, x^{++}\}\rangle_G$ we have $x^{++}y^{--} \in E(G)$. We show that $\{x, y\}$ is again a cutset. Suppose, to the contrary, $u, u^+ \in y^+\overrightarrow{C}x^-$ and $x^+u, xu^+ \in E(G)$ (see the observation ($*$)). If $u = y^+$, then $x^+y^+\overrightarrow{C}x^-y\overleftarrow{C}x^+$ is a cycle of length $n_G - 1$; thus $u \neq y^+$. If $x^{++}u \in E(G)$, then $x^{++}\overrightarrow{C}yxu^+\overrightarrow{C}x^-y^+\overrightarrow{C}ux^{++}$ is a cycle of length $n_G - 1$. Thus, since $\langle\{x^+, x^{++}, y, u\}\rangle_G$ cannot be a claw, we have $yu \in E(G)$. From

$\langle\{u, u^-, u^+, x^+\}\rangle_G$ then $u^-x^+ \in E(G)$ or $u^+x^+ \in E(G)$, but then in the first case $x^+\overrightarrow{C}yu\overrightarrow{C}x^-y^+\overrightarrow{C}u^-x^+$ and in the second case $x^+u^+\overrightarrow{C}x^-y^+\overrightarrow{C}uy\overleftarrow{C}x^+$ is a cycle of length n_G-1. Hence $\{x, y\}$ is a cutset.

We show that x and y have no other neighbors except x^+ and y^- on $x^+\overrightarrow{C}y^-$. Thus, first let, by Lemma 9, $xv \in E(G)$ and $x^+v^- \in E(G)$ for $v^-, v \in x^{++}\overrightarrow{C}y^{--}$. Then $xv\overrightarrow{C}y^{--}x^{++}\overrightarrow{C}v^-x^+y\overrightarrow{C}x$ is a cycle of length n_G-1. Secondly, let $yv^- \in E(G)$ and $y^-v \in E(G)$ for some $v^-, v \in x^{++}\overrightarrow{C}y^{--}$. From $\langle\{y, y^+, x^+, v^-\}\rangle_G$ we have $v^-x^+ \in E(G)$. Considering $\langle\{v^-, v, v^{--}, y\}\rangle_G$ we now get $vy \in E(G)$ or $v^{--}y \in E(G)$, but then $x^{++}\overrightarrow{C}v^-x^+\overleftarrow{C}yv\overrightarrow{C}y^{--}x^{++}$ in the first case and $x^{++}\overrightarrow{C}v^{--}y\overrightarrow{C}x^+v^-\overrightarrow{C}y^{--}x^{++}$ in the second case, respectively, is a cycle of length n_G-1. Hence $N_G(x)\cap(x^+\overrightarrow{C}y^-) = \{x^+\}$ and $N_G(y)\cap(x^+\overrightarrow{C}y^-) = \{x^+, y^-\}$.

Since, by Lemma 10, $N_G(x)\cap(y^+\overrightarrow{C}x^-) = N_G(y)\cap(y^+\overrightarrow{C}x^-)$ and obviously $y \in V_{\mathrm{EL}}(G)$, $x \in V_{\mathrm{SI}}(G'_y)$. Then, similarly as in the proof of Lemma 11, the graph $H = G'_y - \{x, y\}$ is claw-free, $\mathrm{cl}(H)$ is complete and hence H has a cycle C_H of length $n_H - 1 = n_G - 3$ such that $E(C_H)\cap(E(G'_y)\backslash E(G)) = \{e_1, e_2\}$ for some $e_1 = u_1v_1$ and $e_2 = u_2v_2$ having endvertices in different components of $G - \{x, y\}$. Since $N_G(y)\cap(x^+\overrightarrow{C}y^-) = \{x^+, y^-\}$ and $N_G(x)\cap(x^+\overrightarrow{C}y^-) = \{x^+\}$, we can suppose that $u_1 = x^+$ and $u_2 = y^-$. Then, replacing u_1v_1 by u_1xv_1 and u_2v_2 by u_2yv_2, we get a cycle in G of length n_G-1. This contradiction proves that $x^{++}y \in E(G)$. Hence $\langle\{x, y, y^+, x^-\}\rangle_G \simeq \langle\{x^+, x^{++}, y^-, y\}\rangle_G \simeq K_4$.

We show that $\{x, y\}$ or $\{x^+, y\}$ is a cutset of G. Indeed, if not, then, by the observation (∗), there are $u, u^+ \in y^+\overrightarrow{C}x^-$ and $v^-, v \in x^{++}\overrightarrow{C}y^-$ such that $\{xv, xu^+, x^+v^-, x^+u\} \subset E(G)$, but then $xu^+\overrightarrow{C}x^-y^+\overrightarrow{C}ux^+v^-\overleftarrow{C}x^{++}y^-\overleftarrow{C}vx$ is a cycle of length n_G-1. Thus, by symmetry, we can suppose that $\{x, y\}$ is a cutset of G.

Now, $\{x^+, y\}$ cannot be also a cutset of G, since otherwise Lemma 11 implies $|x^{++}\overrightarrow{C}y^-| = |y^+\overrightarrow{C}x^-| = n_G/2$, from which $n_G = |x^{++}\overrightarrow{C}y^-| + |y^+\overrightarrow{C}x^-| + |\{x, x^+, y\}| = 2n_G/2 + 3 > n_G$, a contradiction. Thus, by the observation (∗), there are $v^-, v \in x^{++}\overrightarrow{C}y^-$ such that $xv \in E(G)$ and $x^+v^- \in E(G)$. Apparently $|x^{++}\overrightarrow{C}v^-| \geqslant 4$ and $|v\overrightarrow{C}y^-| \geqslant 4$ (otherwise we easily obtain a cycle of length n_G-1). If $xv^+ \in E(G)$, then $xv^+\overrightarrow{C}y^-x^{++}\overrightarrow{C}v^-x^+y\overrightarrow{C}x$, and if $x^+v^{--} \in E(G)$, then $xv\overrightarrow{C}y^-x^{++}\overrightarrow{C}v^{--}x^+y\overrightarrow{C}x$ is a cycle of length n_G-1. Hence both $xv^+ \notin E(G)$ and $x^+v^{--} \notin E(G)$, from which, considering $\langle\{v, v^-, v^+, x\}\rangle_G$ and $\langle\{v^-, v^{--}, v, x^+\}\rangle_G$, we have $xv^- \in E(G)$ and $x^+v \in E(G)$, i.e. $\langle\{x, x^+, v^-, v\}\rangle_G \simeq K_4$.

Let $K_1 = \langle N_G(x)\cap(x^+\overrightarrow{C}y^-)\rangle_G$ and $K_2 = \langle N_G(y)\cap(x^+\overrightarrow{C}y^-)\rangle_G$. Since $\{x, y\}$ is a cutset of G, both K_1 and K_2 is a clique (otherwise some two nonadjacent vertices together with x^- or y^+ form a claw centered at x or at y). Since $x^+ \in V(K_1)\cap V(K_2)$, $N_G(x^+)\cup\{x^+\}\backslash\{x, y\} \supset (V(K_1)\cup V(K_2))$.

We show that $N_G(x^+)\cup\{x^+\}\backslash\{x, y\} = (V(K_1)\cup V(K_2))$. Suppose, to the contrary, $z \in N_G(x^+)\backslash(\{x, y\}\cup V(K_1)\cup V(K_2))$. Since $\{x, y\}$ is a cutset, $z \in x^+\overrightarrow{C}y^-$. By the definition of K_1 and K_2 and by symmetry, we can suppose that $z \in v^+\overrightarrow{C}y^{--}$. If $z = v^+$, then $xv^- \overleftarrow{C} x^+z \overrightarrow{C} x$, and if $z = y^{--}$, then $x^+z \overleftarrow{C} x^{++} y \overrightarrow{C} x^+$ is a cycle of length $n_G - 1$, hence $v^+ \neq z \neq y^{--}$. From $\langle\{z, z^-, z^+, x^+\}\rangle_G$ we have $z^-x^+ \in E(G)$ or $z^+x^+ \in E(G)$. By symmetry, suppose that $z^+x^+ \in E(G)$. Then, similarly as above, $z^+ \neq y^{--}$. Since $z, y^- \notin N_G(x)$, from $\langle\{x^+, z, y^-, x\}\rangle_G$ we have $zy^- \in E(G)$. Since $z, y^{--} \notin N_G(y)$, from $\langle\{y^-, y, y^{--}, z\}\rangle_G$ we have $zy^{--} \in E(G)$, but then $x^+z^+ \overrightarrow{C} y^{--}z \overleftarrow{C} x^{++} y \overrightarrow{C} x^+$ is a cycle of length $n_G - 1$. This contradiction proves that $N_G(x^+) \cup \{x^+\} \backslash \{x, y\} = (V(K_1) \cup V(K_2))$.

Let $H_1 = G'_{x^+}$ and $H_2 = (H_1)'_y$. Since $N_G(x^+) \cup \{x^+\} \backslash \{x, y\} = (V(K_1) \cup V(K_2))$ and, by Lemma 10, $N_G(x) \cap (y^+ \overrightarrow{C} x^-) = N_G(y) \cap (y^+ \overrightarrow{C} x^-)$, implying $N_G(x) \subset N_G(y) \cup N_G(x^+)$, we have $\{x, y, x^+\} \subset V_{\mathrm{SI}}(H_2)$. The graph $H = H_2 - \{x, y, x^+\}$ thus has a complete closure. Let $B_1 = E(H_1) \backslash E(G)$ and $B_2 = E(H_2) \backslash E(H_1)$. Then, by the minimality of G and by Theorem 8(ii), H has a cycle C_H of length $n_H - 1 = n_G - 4$ such that either $|E(C_H) \cap B_1| \leqslant 2$ and $|E(C_H) \cap B_2| = 0$, or $|E(C_H) \cap B_1| \leqslant 1$ and $|E(C_H) \cap B_2| \leqslant 2$. Since $\{x, y\}$ is a cutset of G, at least two edges of $E(C_H) \cap (B_1 \cup B_2)$ have an endvertex in $y^+ \overrightarrow{C} x^-$. Since $N_G(x^+) \subset x \overrightarrow{C} y$, this implies $|E(C_H) \cap B_2| \geqslant 2$. Hence $|E(C_H) \cap B_1| \leqslant 1$ and $|E(C_H) \cap B_2| = 2$. Let $e_1 = u_1v_1$, $e_2 = u_2v_2$ be the two edges in $E(C_H) \cap B_2$ and (if nonempty), $e_3 = u_3v_3$ be the only edge in $E(C_H) \cap B_1$. By the above, we can suppose that $\{u_1, u_2, u_3, v_3\} \subset x^{++} \overrightarrow{C} y^-$ and $\{v_1, v_2\} \subset y^+ \overrightarrow{C} x^-$.

If $u_1 \in V(K_1)$ and $u_2 \in V(K_2)$, then, replacing in C_H the edge u_1v_1 by the path u_1xv_1, the edge u_2v_2 by the path $u_2x^+yv_2$ (if $E(C_H) \cap B_1 = \emptyset$) or by the path u_2yv_2 (if $E(C_H) \cap B_1 \neq \emptyset$) and the edge u_3v_3 (if any) by the path $u_3x^+v_3$, we obtain a cycle of length $n_G - 1$ in G. If $u_1, u_2 \in V(K_1)$ and $B_1 = \emptyset$, then we analogously replace in C_H the edges u_1v_1 and u_2v_2 by the paths u_1xv_1 and $u_2x^+yv_2$. Since $N_G(x^+) \cup \{x^+\} \backslash \{x, y\} = (V(K_1) \cup V(K_2))$, it remains to consider (up to symmetry) the case when $u_1, u_2 \in V(K_1)$ and $B_1 \neq \emptyset$. Since $u_3, v_3 \in N_G(x^+)$ and $u_3v_3 \notin E(G)$, we have $u_3 \in V(K_1)$ and $v_3 \in V(K_2)$, or $v_3 \in V(K_1)$ and $u_3 \in V(K_2)$. Let P_1, P_2, P_3 be the three paths that C_H splits into by deleting e_1, e_2, e_3 and suppose the notation is chosen such that, in the path system obtained by deleting e_1, e_2, e_3 from C_H, u_3 and u_1 are endvertices of the same path (this is always possible since there can be no path joining u_3, v_3 and since $\{x, y\}$ is a cutset of G). Then, replacing in C_H the edges e_1, e_2, e_3 by u_1xv_1, u_2u_3 and $v_2yx^+v_3$ if $u_3 \in V(K_1)$ and $v_3 \in V(K_2)$, or by u_2xv_2, u_1v_3 and $u_3x^+yv_1$ if $v_3 \in V(K_1)$ and $u_3 \in V(K_2)$, respectively, we obtain a cycle of length $n_G - 1$ in G.

This contradiction proves that for any choice of a hamiltonian cycle C in G, no eligible vertex of G is of the first type with respect to C.

Let now C be a hamiltonian cycle in G and x an eligible vertex (of second type with respect to C). Let P be a shortest x^-, x^+-path in $\langle N_G(x)\rangle_G$. Since G is claw-free, P is of length 3. Let $V(P) = x^-y_1y_2x^+$. Then either $y_1 \in x \overrightarrow{C} y_2$, or $y_2 \in x \overrightarrow{C} y_1$.

Case 1: $y_1 \in x \overrightarrow{C} y_2$. We consider $\langle\{y_1, y_1^-, y_1^+, x^-\}\rangle_G$ and $\langle\{y_2, y_2^-, y_2^+, x^+\}\rangle_G$. If both $x^-y_1^+ \in E(G)$ and $x^+y_2^- \in E(G)$, then the cycle $x^-y_1^+ \overrightarrow{C} y_2^-x^+ \overrightarrow{C} y_1y_2 \overrightarrow{C} x^-$ is a cycle of length $n_G - 1$ in G. Hence we can suppose (by symmetry) that $x^-y_1^- \in E(G)$. Then, on the cycle $C' = xy_1 \overrightarrow{C} x^-y_1^- \overleftarrow{C} x$, the predecessor of x is x^+ and the successor is y_1. Since y_1 and x^+ have a common neighbor $y_2 \in N_G(x)$, x is of type 1 with respect to C'-a contradiction.

Case 2: $y_2 \in x\overrightarrow{C}y_1$. We first show that x can be chosen such that y_2, y_1 are not consecutive on C. Suppose, to the contrary, that this is not the case and choose x such that $x\overrightarrow{C}y_2$ is shortest possible. Since x is of type 2, $x^+y_1 \notin E(G)$, and from $\langle\{y_2, y_1, y_2^-, x^+\}\rangle_G$ we have $x^+y_2^- \in E(G)$. Similarly $xy_2^- \notin E(G)$ (otherwise y_2 is of type 1 with respect to C) and from $\langle\{x^+, x, x^{++}, y_2^-\}\rangle_G$ we have $x^{++}y_2^- \in E(G)$. But then the path $xy_2y_2^-x^{++}$ in $\langle N_G(x^+)\rangle_G$ contradicts the choice of x. Hence we may assume that $y_2^+ \neq y_1$.

Suppose now that $x^-y_1^- \in E(G)$ and let $C' = x\overrightarrow{C}y_1^-x^-\overleftarrow{C}y_1x$. Then the predecessor y_1 and successor x^+ of x on C' have a common neighbor $y_2 \in N_G(x)$ and hence x is of type 1 with respect to C', a contradiction. Hence $x^-y_1^- \notin E(G)$ and, by symmetry, $x^+y_2^+ \notin E(G)$. Considering $\langle\{y_1, y_1^-, y_1^+, x^-\}\rangle_G$ and $\langle\{y_2, y_2^-, y_2^+, x^+\}\rangle_G$ we then get $y_1^+x^- \in E(G)$ and $y_2^-x^+ \in E(G)$.

We show that $xy_2^- \in E(G)$. If $xy_2^- \notin E(G)$, then from $\langle\{y_2, y_2^-, y_2^+, x\}\rangle_G$ we have $xy_2^+ \in E(G)$, and since we already know that $x^+y_2^+ \notin E(G)$, from $\langle\{x, x^-, x^+, y_2^+\}\rangle_G$ we get $x^-y_2^+ \in E(G)$. Considering $\langle\{y_1, y_1^-, y_1^+, y_2\}\rangle_G$ we then have $y_2y_1^- \in E(G)$ or $y_2y_1^+ \in E(G)$, but in the first case the cycle $C' = x\overrightarrow{C}y_2y_1^-\overleftarrow{C}y_2^+x^-\overleftarrow{C}y_1x$ and in the second case the cycle $C' = x\overrightarrow{C}y_2y_1^+\overrightarrow{C}x^-y_2^+\overrightarrow{C}y_1x$ yields a contradiction, since in both these cases x is of type 1 with respect to C'. Hence $xy_2^- \in E(G)$ and, by symmetry, $xy_1^+ \in E(G)$, which implies that $\langle\{x, x^+, y_2^-, y_2\}\rangle_G \simeq \langle\{x, x^-, y_1^+, y_1\}\rangle_G \simeq K_4$.

Now consider $\langle\{y_2, y_2^+, y_1, x^+\}\rangle_G$. If $x^+y_1 \in E(G)$, then x is of first type with respect to C; thus $x^+y_1 \notin E(G)$. Since we already know that $x^+y_2^+ \notin E(G)$, we have $y_1y_2^+ \in E(G)$. Since $y_2^-xy_1y_2^+$ is a path in $\langle N_G(y_2)\rangle_G$ and $y_2^-y_2^+ \notin E(G)$, by the observation (ii) we have $y_2 \in V_{\mathrm{EL}}(G)$. Thus, by the previous argument, $\langle\{y_2, y_2^+, y_1^-, y_1\}\rangle_G \simeq K_4$.

We show that $\{y_1, y_2\}$ is a cutset of G. Suppose, to the contrary, that (recall Lemma 9) uv, u^+v^- is a pair of parallel chords such that at least one of them crosses y_1y_2, i.e. such that $u, u^+ \in y_2\overrightarrow{C}y_1$ and $v^-, v \in y_1\overrightarrow{C}y_2$.

Case	Cycle	Vertex of type 1
$u, u^+ \in y_2^+\overrightarrow{C}y_1^-; v^-, v \in y_1\overrightarrow{C}y_2$	$y_1\overrightarrow{C}v^-u^+\overrightarrow{C}y_1^-y_2^+\overrightarrow{C}uv\overrightarrow{C}y_2y_1$	y_1
$u = y_2; v = y_1^+$	C	y_1
$u = y_2; v^-, v \in y_1^+\overrightarrow{C}x^-$	$y_1x\overrightarrow{C}y_2v\overrightarrow{C}x^-y_1^+\overrightarrow{C}v^-y_2^+\overrightarrow{C}y_1$	y_1
$u = y_2; v = x$	$x\overrightarrow{C}y_2y_1^-\overleftarrow{C}y_2^+x_1^-\overleftarrow{C}y_1x$	x
$u = y_2; v = x^+$	C	y_2
$u = y_2; v^-, v \in x^+\overrightarrow{C}y_2^-$	$xy_2v\overrightarrow{C}y_2^-x^+\overrightarrow{C}v^-y_2^+\overrightarrow{C}x$	x

Since these are, up to symmetry, all possibilities, $\{y_1, y_2\}$ is a cutset of G. By symmetry, $\{x, y_1\}$ and $\{x, y_2\}$ are also cutsets of G. But then, by Lemma 11, $|x^+\overrightarrow{C}y_2^-| = |y_2^+\overrightarrow{C}y_1^-| = |y_1^+\overrightarrow{C}x^-| = n_G/2$, from which $n_G = |x^+\overrightarrow{C}y_2^-| + |y_2^+\overrightarrow{C}y_1^-| + |y_1^+\overrightarrow{C}x^-| + |\{x, y_1, y_2\}| = 3n_G/2 + 3 > n_G$, a contradiction.

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