JOURNAL OF FUNCTIONAL ANALYSIS 65, 368–405 (1986)

Hamilton–Jacobi Equations in Infinite Dimensions. II. Existence of Viscosity Solutions*

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Communicated by the Editors

Received April 1985

INTRODUCTION

This is the second paper of a series devoted to the study of Hamilton–Jacobi equations in infinite dimensions. Part I [10] was concerned with the uniqueness of viscosity solutions of general first order equations of the form

$$F(x, u, Du) = 0 \qquad \text{in } \Omega \tag{HJ}$$

in which Ω is an open subset of some (real) Banach space V, the unknown function $u\colon \Omega \to \mathbf{R}$ is continuous, and $Du(x)$ denotes the Fréchet derivative of u at x; thus $Du(x) \in V^*$, where V^* is the dual of V. The nonlinear function F defining the equation is a continuous mapping $F\colon \Omega \times \mathbf{R} \times V^* \to \mathbf{R}$. The notion of a viscosity solution for (HJ) considered in [10] is a straightforward adaptation of the notion of a viscosity solution first used in obtaining existence and uniqueness results in the finite dimensional case (i.e., $V = \mathbf{R}^n$) in M. G. Crandall and P. L. Lions [7] (see also M. G. Crandall, P. L. Lions, and L. C. Evans [5]). One of the equivalent forms of this notion is recalled in Section 1 below.

Here we prove general existence results for two typical problems, namely, the Cauchy problem

$$\begin{aligned} u_t + H(x, t, u, Du) &= 0 \qquad \text{in } V \times]0, T[, \\ u(x, 0) &= \varphi(x) \qquad \text{in } V \end{aligned} \tag{CP}$$

* Sponsored by the United States Army under Contract DAAG29-80-C-0041 and partially supported by the National Science Foundation under Grant MCS-8002946.

0022-1236/86 $3.00

and the stationary problem

$$u + H(x, u, Du) = 0 \qquad \text{in } V, \tag{SP}$$

where the functions H and φ are given and satisfy conditions detailed in Section 1. In either case we call H the "Hamiltonian." Here the Du in (CP) denotes the derivative of the map $x \to u(x, t)$; i.e., D is the gradient in the "space variable" x. The equation in (CP) is regarded as a special case of (HJ) by regarding the pair (x, t) in (CP) as x in (HJ) and $V \times \mathbf{R}$ as V.

The existence results established here are precisely formulated in Section 1. It is quite striking that these results are obtained under much the same assumptions used to obtain the corresponding results in the finite dimensional case in H. Ishii [16, 17] and M. G. Crandall and P. L. Lions [9]. (See [3, 7, 18, 19, 21] for earlier existence results.) However, even though the finite and infinite dimensional theories have similar formulations, the proofs must be modified substantially in the infinite dimensional case.

There are three main difficulties in the passage from finite to infinite dimensions. First, the finite dimensional theory relies everywhere on the fact that continuous functions on closed and bounded sets attain maximum and minimum values, and this is false in infinite dimensions. A way to deal with this in infinite dimensions was demonstrated in [10] in the course of proving uniqueness. Next, in finite dimensions the method of vanishing viscosity can be used to solve (CP) or (SP) in a simple way if the data H and φ are "nice." That is, a term $-\varepsilon\Delta$ is added to the equations; the resulting problem is solved; and then ε is sent to zero using a priori estimates, the Arzela–Ascoli (or AA) theorem, and properties of viscosity solutions to pass to a limit. Having obtained existence for a restricted class of H and φ, a priori estimates and the AA theorem are used again to obtain solutions for H and φ of the generality desired. In infinite dimensions we have neither the Δ nor the AA theorem available to us. This first difficulty is circumvented by the use of explicit formulae from the theory of differential games. We obtain solutions for a restricted class of Hamiltonians by forming ad hoc differential games (following the finite dimensional discussion in L. C. Evans and P. E. Souganidis [14]) whose value functions are shown to provide the desired solutions. Next, to deal with less regular Hamiltonians by limiting arguments in the absence of the AA theorem, we must prove a sharp constructive result concerning the convergence of solutions of the approximate problems. This result is new and interesting even in finite dimensions.

As mentioned above, we formulate the hypotheses for the Existence Theorem in Section 1 and also review and present preliminaries on viscosity solutions in infinite dimensions. In Section 2 we formulate and prove the Convergence Theorem mentioned above. It is used later (several times) to show that solutions of approximate problems converge to a solution of the desired problems. This result is one of the main ones of this paper; it and variants should prove useful in other situations, even in finite dimensions. In order to use the Convergence Theorem one needs control of moduli of continuity, and a priori estimates of such moduli are given in Section 2 as well. Here we adapt the arguments of Ishii [17] to deal with the current generality. Section 3 concerns several reduction processes which are used to reduce, via the Convergence Theorem, the general existence theorem to simpler and simpler cases. Eventually we are interested in existence for Lipschitz continuous Hamiltonians and this is taken up in Section 4 by the differential game method. Section 4 is independent of Sections 2 and 3 and could be read independently. Section 5 is devoted to illuminating remarks, examples, variants, and extensions.

In order to keep this paper to a reasonable size we have had to rely on the reader to supply a variety of routine (in the theory of viscosity solutions) arguments at various places. As a consequence, the details of the proofs are not really accessible to inexperienced readers. In particular, we assume familiarity with M. G. Crandall and P. L. Lions [10] and enough prerequisites to read this paper as well as earlier works on existence in finite dimensions like H. Ishii [17]. The following remarks likewise assume some knowledge on the part of the reader.

We do not impose explicit conditions at ∞ on the solutions we obtain for (CP) and (SP) as the behaviour at ∞ will be controlled by the requirement that our solutions be uniformly continuous in x (uniformly in t in (CP)). However, under conditions on H other than those we impose here other behaviours at ∞ of the solutions are appropriate, just as in the finite dimensional case. In this regard, one can obtain infinite dimensional results with other behaviours at ∞ like the finite dimensional results of M. G. Crandall and P. L. Lions [11] and H. Ishii [16]. Similarly, one may use G. Barles' idea in [3] to treat boundary value problems on the basis of the current results by reducing the boundary problem to an equivalent one in all of V.

We conclude this introduction by mentioning topics related to the current investigation to be taken up in Part III of this series. In view of the complexity of the arguments needed to establish the existence results, it is natural to seek other approaches. In particular, the idea of Galerkin-type approximations (in which one projects the equation on a sequence of increasing finite dimensional subspaces) is tempting. However, as we will explain in Part III, simple examples show that this method does not converge in general and cast doubt that it can ever be made the basis of the existence theory. Next, one of the principal motivations for studying Hamilton–Jacobi equations in infinite dimensions is that they arise as equations satisfied by value functions of optimal control problems with infinite dimensional state variables. Important problems of this sort arise in engineering applications where the state equation is a linear partial differential equation like the heat or wave equation, and this leads to Hamilton–Jacobi equations in which the Hamiltonian incorporates terms like $(Ax, Du(x))$, where A is an unbounded linear operator in V (like $-\Delta$ in L^2). Barbu and Da Prato [1] derive such equations and in, e.g., [1, 2] study them (see also the references therein). In our context, the unboundedness of A seems to present substantial difficulties for the viscosity theory. However, in Part III we will explain how to accommodate some equations of this type within variations of the theory presented here and in Part I. Part III will also demonstrate existence results in spaces "less smooth" than those treated here using an ε-formulation of viscosity solutions as discussed in the Appendix of Part I.

We mention, finally, that the main results proved here were announced in M. G. Crandall and P. L. Lions [12].

1. Preliminaries and Statements of Results

In all that follows, unless otherwise stated, V is a real Banach space with the Radon–Nikodym property (or "V is RN"), V^* is its dual space, and Ω is an open subset of V. We will use the same notation $|\ |$ for the norm of V, the dual norm on V^*, and the absolute value on $\mathbf{R}$. The value of $p \in V^*$ at $x \in V$ will be written (p, x).

There are many equivalent ways to say that V has the Radon–Nikodym property—see, for example, [4]. We will use the following form: V is RN if and only if whenever φ is a continuous mapping of a closed ball B in V into $\mathbf{R}$ which is bounded below (above) and $\varepsilon > 0$, then there is an element x^* of V^* such that $|x^*| < \varepsilon$ and $\varphi + x^*$ attains its minimum (respectively, maximum) value over B.

Both (SP) and (CP) may be regarded as equations of the general form $H(x, u, Du) = 0$ by using $V \times \mathbf{R}$ in place of V in the case of (CP). Let $H \in C(V \times \mathbf{R} \times V^*)$. The notion of viscosity sub- and supersolutions of an equation $H = 0$ in a set Ω were defined in [10]. One of the equivalent forms of this definition is:

Definition 1.1. Let $u \in C(\Omega)$. Then u is a viscosity subsolution $H = 0$ in Ω if

Whenever $\varphi \in C(\Omega)$, $y \in \Omega$, φ is differentiable at y and $u - \varphi$ has a local maximum at y, then $H(y, u(y), D\varphi(y)) \leqslant 0$. (1.1)

Similarly, u is a viscosity supersolution of $H=0$ in Ω if

> Whenever $\varphi \in C(\Omega)$, $y \in \Omega$, φ is differentiable at y and $u-\varphi$ has a local minimum at y, then $H(y, u(y), D\varphi(y)) \geq 0$. (1.2)

Finally, u is a viscosity solution of $H=0$ if it is both a viscosity subsolution and a viscosity supersolution.

We will also refer to a viscosity subsolution of $H=0$ as a viscosity solution of $H \leq 0$, etc. We next formulate conditions on the Hamiltonians H in (SP) and (CP) which will be among the hypotheses under which we will prove the existence of solutions. These conditions will involve two auxiliary functions $v\colon V \to [0, \infty)$ and $d\colon V \times V \to [0, [0, \infty)$ which are required to satisfy the following conditions (C):

(C) For every $y \in V$ the nonnegative function $x \to d(x, y)$ is Fréchet differentiable at every point except y and the derivative is denoted by $d_x(x, y)$. Similarly, $y \to d(x, y)$ is differentiable except at x and its derivative is $d_y(x, y)$. The function v is nonnegative and differentiable everywhere. Moreover, there is a constant $K>0$ such that

$$|d_x(x, y)|, |d_y(x, y)|, |Dv(x)| \leq K \tag{1.3}$$

whenever the quantities on the left are defined,

$$\liminf_{|x| \to \infty} \frac{v(x)}{|x|} > 1, \tag{1.4}$$

and

$$|x-y| \leq d(x, y) \leq K|x-y| \qquad \text{for} \quad x, y \in V. \tag{1.5}$$

We continue. A function m: $[0, \infty) \to [0, \infty)$ will be called a *modulus* if it is continuous, nondecreasing, nonnegative, and subadditive and satisfies $m(0)=0$. We will use m, m_H, etc., to denote such functions. We will also say that a function $\sigma\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ is a *local modulus* if $r \to \sigma(r, R)$ is a modulus for each $R \geq 0$ and $\sigma(r, R)$ is continuous and nondecreasing in both variables. We next formulate conditions on the Hamiltonian $H\colon V \times [0, T] \times \mathbf{R} \times V^* \to \mathbf{R}$ in (CP). These conditions are interpreted in the obvious way for time independent Hamiltonians $H\colon V \times \mathbf{R} \times V^* \to \mathbf{R}$ as in (SP). Throughout the statements it is assumed that conditions (C) hold and d, v are the functions in (C). We also let $B_R(x) = \{z \in V\colon |z-x| \leq R\}$ be the closed ball centered at x of radius R in V; $B_R^*(p)$ be the ball of radius R in V^* centered at $p \in V^*$; and $B_R = B_R(0)$, $B_R^*(0) = B_R^*$.

(H0) There is a local modulus σ_0 such that

$$|H(x, t, r, p) - H(y, s, v, q)| \leqslant \sigma_0(|x-y| + |t-s| + |r-v| + |p-q|, R)$$

for $R > 0$, $x, y \in B_R$, $t, s \in [0, T]$, $|r|, |v| \leqslant R$, and $p, q \in B_R^*$.

(H1) For each $(x, t, p) \in V \times [0, T] \times V^*$ the map $r \to H(x, t, r, p)$ is nondecreasing.

(H2) There is a local modulus σ_H such that

$$H(x, t, r, p) - H(x, t, r, p + \lambda Dv(x)) \leqslant \sigma_H(\lambda, |p| + \lambda)$$

whenever $0 \leqslant \lambda$, $(x, r, t, p) \in V \times [0, T] \times \mathbf{R} \times V^*$.

(H3) There is a modulus m_H such that

$$H(y, t, r, -\lambda d_y(x, y)) - H(x, t, r, \lambda d_x(x, y)) \leqslant m_H(\lambda d(x, y) + d(x, y))$$

for $x, y \in V$ with $x \neq y$, $t \in [0, T]$, $r \in \mathbf{R}$ and $\lambda \geqslant 0$.

The existence results will be proved for (CP) under (H0)–(H3). For (SP), (H3) will need to be augmented and sometimes we will invoke the additional condition:

(H4) There is a function F: $[0, \infty) \times [0, \infty) \to \mathbf{R}$ nondecreasing in its arguments such that

$$H(y, r, -\lambda d_y(x, y)) - H(x, r, \lambda d_x(x, y)) \leqslant F(\lambda, d)$$

for $x, y \in V$, $r \in \mathbf{R}$, and $\lambda \geqslant 0$ and a nonnegative nondecreasing uniformly continuous map G: $[0, \infty) \to \mathbf{R}$ which is continuously differentiable on $(0, \infty)$ and satisfies

$$G(r) \geqslant F(G'(r), r) \qquad \text{on} \quad r > 0.$$

In order to appreciate the need for (H4) for (SP), as well as for further insight into the nature of the other conditions above, we invite the reader to refer ahead to Section 5 at this time and to see the discussion in [10, Remarks 2]. The reader will observe some differences in the formulations of (C) (in which, for example, a constant called k in [10] has been put equal to 1 here, as can be done without loss of generality) and (H0)–(H3) from the corresponding formulations in [10]. In particular, (H0) is stronger in several respects than its analogue due to the more stringent requirements of the existence theory. We also recall that for (CP), (H1) can be weakened to the monotonicity of $r \to H(x, t, r, p) + \lambda r$ for some $\lambda \in \mathbf{R}$ by means of the change of variable $u \to e^{-\lambda t}u$.

Finally we introduce various function classes. If X is any metric space, the space of uniformly continuous real-valued functions on X is $\mathrm{UC}(X)$ and the subspace of bounded functions in $\mathrm{UC}(X)$ is $\mathrm{BUC}(X)$. The space $\mathrm{UC}_s(V \times [0, T])$ consists of those functions $u: V \times [0, T] \to \mathbf{R}$ which are uniformly continuous in x uniformly in t and uniformly continuous on bounded sets. This amounts to asking that there be a modulus m and a local modulus σ such that

$$|u(x, t) - u(y, s)| \leqslant m(|x - y|) + \sigma(|t - s|, |y|).$$

$\mathrm{BUC}_s(V \times [0, T])$ is the subspace of $\mathrm{UC}_s(V \times [0, T])$ consisting of bounded functions.

EXISTENCE THEOREM 1.1. (i) *Let* (H0)–(H3) *hold and* $\varphi \in \mathrm{UC}(V)$. *Then there is a unique* $u \in \mathrm{UC}_s(V \times [0, T])$ *which is a viscosity solution of* $u_t + H(x, t, u, Du) = 0$ *on* $V \times (0, T)$ *and satisfies* $u(x, 0) = \varphi(x)$ *on* V.

(ii) *Let* (H0)–(H4) *hold. Then there is a unique viscosity solution* $u \in \mathrm{UC}(V)$ *of* $u + H(x, u, Du) = 0$ *on* V.

The program of proof of the Existence Theorem is quite long and involved and will occupy the next three sections. There are infinitely many variants of this result and we discuss some of them in Section 5. This section concludes with some results of general interest concerning viscosity solutions which will be used in various parts of the proof of the Existence Theorem. In the statements of these results the reader should think of the equations $H = 0$, etc., which are involved as including both (SP) and (CP).

In what follows we will assume the existence of a function $N: V \to [0, \infty)$ with the following properties:

The nonnegative function N is Lipschitz continuous on V and differentiable on $V \backslash \{0\}$. Moreover, $N(0) = 0$ and $N(x) \geqslant |x|$ in some neighborhood of 0 in V. (1.6)

For example, if the conditions (C) hold, then $N(x) = d(x, 0)$ has the desired properties. In addition, if X is reflexive, then it has an equivalent norm which may serve as a function N satisfying (1.6). When (1.6) holds, the requirements defining viscosity sub- and supersolutions may be weakened without modifying the notion. We formulate and prove a result to this effect (see also [10, Remarks 1]).

PROPOSITION 1.2. *Let* (1.6) *hold. Let* $u \in C(\Omega)$. *Then* u *is a viscosity solution of* $H \leqslant 0$ $(H \geqslant 0)$ *in* Ω *if and only if whenever* $\varphi \in C(\Omega)$ *is everywhere differentiable,* $y \in \Omega$ *is a point of continuity of* $D\varphi$ *and there is an* $r > 0$ *such that* $u(x) - \varphi(x) \leqslant u(y) - \varphi(y) - |x - y|^2$ *for* $|x - y| \leqslant r$ *(respec-*

tively, $u(y)-\varphi(y)\leqslant u(x)-\varphi(x)-|x-y|^2$ *for* $|x-y|\leqslant r$), *then* $H(y, u(y), D\varphi(y))\leqslant 0$ (*respectively,* $H(y, u(y), D\varphi(y))\geqslant 0$).

Sketch of Proof. Assume that u satisfies the conditions the proposition asserts are equivalent to being a vicosity solution of $H\leqslant 0$, $\varphi\in C(\Omega)$ and $y\in\Omega$ is both a local maximum point of $u-\varphi$ and a point of differentiability of φ. In order to deduce that $H(y, u)(y), D\varphi(y))\leqslant 0$ we first remark that:

LEMMA 1.3. *Let* (1.6) *hold. Let* $\varphi\in C(\Omega)$ *be differentiable at* $y\in\Omega$. *Then there is a function* $\psi\in C(\Omega)$ *which is everywhere differentiable on* Ω *and an* $r>0$ *such that* $\psi(y)=\varphi(y)$, $D\psi(y)=D\varphi(y)$, $D\psi$ *is continuous at* y, *and* $\psi(x)+|x-y|^2\leqslant\varphi(x)$ *for* $|x-y|\leqslant r$.

Proof of Lemma 1.3. Set $p=D\varphi(y)$. By assumption and (1.6) there is an $r>0$ and an $h\in C(\mathbf{R})$ satisfying $h(0)=h'(0)=0$ and

$$\varphi(x)\geqslant\varphi(y)+(p, x-y)+h(N(x-y)) \qquad \text{for} \quad |x-y|\leqslant r.$$

Lemma I.4 of [7] (due to Evans) provides us with a continuously differentiable function g on $\mathbf{R}$ which satisfies $g(0)=g'(0)=0$ and $g(s)\leqslant h(s)$ for small s. Clearly $\psi(x)=\varphi(y)+(p, x-y)+g(N(x-y))-N(x-y)^2$ has the desired properties.

End of Proof of Proposition 1.2. Since y is a maximum point of $u-\varphi$, if ψ is related to φ as in the lemma, y is also a strict maximum point of $u-\psi$ in the sense of the assumptions. But then, by the assumption on u and the properties of ψ, $H(y, u(y), D\varphi(y))=H(y, u(y), D\psi(y))\leqslant 0$, completing the proof. The case of supersolutions is treated in a parallel way.

The next result presents a key stability property of viscosity solutions.

THEOREM 1.4. *Let* (1.6) *hold. Let* $u_n\in C(\Omega)$ *and* $H_n\in C(V\times\mathbf{R}\times V^*)$, $n=1, 2,\ldots$, *converge to* u, H *as* $n\to\infty$ *in the following way*:

$$\text{For every } x\in\Omega \text{ there is an } R>0 \text{ such that } u_n\to u \text{ uniformly on } B_R(x) \text{ as } n\to\infty, \tag{1.7}$$

and

$$\text{If } (x, r, p), (x_n, r_n, p_n)\in\Omega\times\mathbf{R}\times V^* \text{ for } n=1, 2,\ldots, \text{ and } (x_n, r_n, p_n)\to(x, r, p) \text{ as } n\to\infty, \text{ then } H_n(x_n, r_n, p_n)\to H(x, r, p). \tag{1.8}$$

If u_n *is a viscosity subsolution* (*respectively, supersolution*) *of* $H_n=0$ *in* Ω, *then* u *is a viscosity subsolution*) *of* $H=0$ *in* Ω.

Proof. We treat the case in which the u_n are solutions of $H_n\leqslant 0$; the case of supersolutions is entirely similar. Let $\varphi \in C(\Omega)$ be differentiable and $y \in \Omega$ be a point of continuity of $D\varphi$. Assume, moreover, that

$$u(x) - \varphi(x) \leqslant u(y) - \varphi(y) - |y-x|^2 \qquad \text{for} \quad |y-x| \leqslant r \tag{1.9}$$

for some $r > 0$. We may also assume, due to the nature of the convergence of the u_n, that

$$u_n(x) \leqslant u(x) + \varepsilon_n \qquad \text{for} \quad |y-x| \leqslant r, \tag{1.10}$$

where $\varepsilon_n \to 0$. If $p \in V^*$, $|p| \leqslant \alpha$, and $|x-y| = \delta \leqslant r$, we deduce from (1.9) and (1.10) that

$$\begin{aligned} u_n(x) - (\varphi(x) + (p, x-y)) &\leqslant u(x) - \varphi(x) - (p, x-y) + \varepsilon_n \\ &\leqslant u(y) - \varphi(y) + \alpha\delta + \varepsilon_n - \delta^2 \\ &\leqslant u_n(y) - \varphi(y) + \alpha\delta + 2\varepsilon_n - \delta^2. \end{aligned}$$

Hence if

$$\alpha\delta + 2\varepsilon_n < \delta^2, \tag{1.11}$$

any maximum of the function $u_n(x) - \varphi(x) - (p, x-y)$ over $B_\delta(y)$ must occur at an interior point. Let $\alpha_n, \delta_n > 0$ be sequences convergent to 0 and satisfying (1.11). According to Stegall [20] we may choose $p_n \in V^*$ such that $|p_n| \leqslant \alpha_n$ and $u_n(x) - \varphi(x) - (p_n, x-y)$ has a maximum y_n with respect to $B_{\delta_n}(y)$. By the above, y_n is an interior point of $B_{\delta_n}(y)$ and so y_n is a local maximum. Then, by the definition of viscosity solutions, $H_n(y_n, u_n(y_n), D\varphi(y_n) + p_n) \leqslant 0$. Since $y_n \to y$, $p_n \to 0$, and $D\varphi$ is continuous, the assumed convergences yield $H(y, u(y), D\varphi(y)) \leqslant 0$ in the limit.

As mentioned above, conditions much like (H0)–(H3) were introduced in [10], where the question of uniqueness was studied and some continuity of solutions with respect to the equations was proved. We will in fact need to supplement these results of [10] somewhat to achieve full generality below. This task is taken up in the next section.

2. Convergence Theorems and Moduli of Continuity

The results of this section, which are both technical and general, concern the continuity of solutions of Hamilton–Jacobi equations in the data of the problem. These data are here taken to be the equation and, in the case of (CP), the initial-value. Results of this sort were given in [10]; however, they have proven to be inadequate for the full existence program treated in this paper. Roughly speaking, the results of [10] were formulated to correspond to uniform estimates on all of V, while we will need to deal with estimates uniform on bounded sets, but not on V. We will, however, make full use of the proofs in [10] by referring to them rather than repeating arguments when appropriate. This minimizes repetition and shortens the presentation. We ask the reader's forbearance as we launch ourselves into the discussion.

We consider sequences of Cauchy problems

$$\begin{aligned} u_{nt} + H_n(x, t, u_n, Du_n) &= 0 \qquad \text{in } V \times (0, T], \\ u_n(x, 0) &= \varphi_n(x) \qquad \text{in } V, \end{aligned} \tag{CP$_n$}$$

and stationary problems

$$u_n + H_n(x, u_n, Du_n) = 0 \qquad \text{in } V \tag{SP$_n$}$$

indexed by $n = 1, 2, 3,\ldots$. In each case the Hamiltonians will converge to a limit H in the sense that

$$\lim_{n \to \infty} H_n(x, t, r, p) = H(x, t, r, p) \text{ uniformly}$$
$$\text{on bounded subsets of } V \times [0, T] \times \mathbf{R} \times V^* \tag{2.1}$$

(interpreted in the obvious way if H_n is independent of t). When a sequence of functions f_n converges to a limit f uniformly on bounded subsets of its domain we will simply write

$$f_n \to f \text{ UB},$$

and say f_n converges UB to f. The point of the main convergence result is that if H_n (and φ_n for (SP)) converge UB to a limit H (and φ) which obeys (H0)–(H3) and we have some additional information like (but not exactly) a uniform modulus of continuity of the u_n, then the u_n converge UB to a viscosity solution u of the limit problem. The convergence theorem will be invoked later to assert that solutions of approximate problems converge to solutions of limit problems. In order to verify the hypotheses in these applications, we will need to obtain something like uniform moduli of continuity. Results in this direction are given in Theorems 2.2 and 2.3 below.

CONVERGENCE THEOREM 2.1. *Let H_n, $n = 1, 2,\ldots$, be Hamiltonians which are uniformly continuous on bounded sets and satisfy* (H0) *and H be a Hamiltonian which satisfies* (H0)–(H3). *Let H_n converge* UB *to H.*

(i) *Let* $u_n \in \mathrm{UC_s}(V \times [0, T])$ *be a viscosity solution of* $(\mathrm{CP})_n$ *for* $n = 1, 2, \ldots$. *Assume that there are constants A and B such that*

$$|u_n(x, t)| \leqslant A_0 + B_0 v(x) \qquad \textit{for} \quad (x, t) \in V \times [0, T] \tag{2.2}$$

for $n = 1, 2, \ldots$. *Assume, moreover, that*

$$\lim_{r \downarrow 0} \limsup_{n \to \infty} \sup \{|u_n(x, t) - u_n(y, t)|: d(x, y) \leqslant r, 0 \leqslant t \leqslant T\} = 0. \tag{2.3}$$

Then there is a $u \in \mathrm{UC_s}(V \times [0, T])$ *such that* $u_n \to u$ UB.

(ii) *Let the above assumptions holds. Let* $u_n \in \mathrm{UC}(V)$ *be viscosity solutions of* $(\mathrm{SP})_n$ *for* $n = 1, 2, \ldots$, *and* (2.2)–(2.3) *hold* (*where the* u_n *are now independent of* t). *Then there exists a* $u \in \mathrm{UC}(V)$ *such that* $u_n \to u$ UB.

Of course, it follows from Theorem 1.4 that in both cases (i) and (ii) above u is a viscosity solution of the limiting problem. Before beginning the proof of the Convergence Theorem we will formulate two theorems which provide, by giving estimates on viscosity solutions and their moduli of continuity, a way to verify its hypotheses. Later we will need to expand the range of application still further.

The following "data" of (CP) or (SP) will be referred to in addition to the data in (C) and (H0)–(H4): A modulus m_0 such that

$$|\varphi(x) - \varphi(y)| \leqslant m_0(d(x, y)) \qquad \text{for} \quad x, y \in V \tag{2.4}$$

and positive constants A_H, B_H, A_0 B_0 such that

$$|H(x, t, 0, 0)| \leqslant A_H + B_H v(x) \qquad \text{for} \quad (x, t) \in V \times [0, T] \tag{2.5}$$

and

$$|\varphi(x)| \leqslant A_0 + B_0 v(x), \tag{2.6}$$

and the function

$$b_H(R) = \sup\{|H(x, t, r, p)|: v(x), |p|, |r| \leqslant R \text{ and } t \in [0, T]\}. \tag{2.7}$$

We begin with (CP) since the hypotheses needed for (SP) are more restrictive.

THEOREM 2.2. (i) *Let H in* (CP) *satisfy* (H0)–(H3) *and* $\varphi \in \mathrm{UC}(V)$ *satisfy* (2.4)–(2.7). *Let* $u \in \mathrm{UC_s}(V \times [0, T])$ *be a viscosity solution of* (CP). *Then there are constants* A, B *depending on* A_0, B_0, A_H, B_H, σ_H such that

$$|u(x, t)| \leqslant (A + Bv(x)), \qquad (x, t) \in V \times [0, T], \tag{2.8}$$

and a modulus m depending on m_0 and m_H such that

$$|u(x, t) - u(y, t)| \leqslant m(d(x, y)) \qquad \textit{for } x, y \in V \textit{ and } t \in [0, T]. \tag{2.9}$$

Moreover, there is a local modulus σ depending on A, B in (2.8), *m in* (2.9), *b_H, v, and d such*

$$|u(x, t) - u(x, s)| \leqslant \sigma(|t - s|, |x|) \qquad \textit{for } t, s \in [0, T]. \tag{2.10}$$

(ii) *Let H in* (SP) *satisfy* (H0)–(H4) *and $u \in \mathrm{UC}(V)$ be a viscosity solution of* (SP). *Then there are constants A and B depending on A_H, B_H, and σ_H such that*

$$|u(x)| \leqslant A + Bv(x) \qquad \textit{for} \quad x \in V, \tag{2.11}$$

and a modulus m, depending on G in (H4), *A and B in* (2.11), *and m_H, such that*

$$|u(x) - u(y)| \leqslant m(d(x, y)). \tag{2.12}$$

As a simple example of the use of these results, observe that by combining the Convergence Theorem and Theorem 2.2 we learn, in particular, that if H satisfies (H0)–(H3) and u_n is a solution of (CP) with the initial data $\varphi_n \in \mathrm{UC}(V)$, $n = 1, 2, \ldots$, the φ_n admit a common modulus of continuity, $\varphi_n(0)$ is bounded, and $\varphi_n \to \varphi$ UB, then $u_n \to u$ UB and $u \in \mathrm{UC}_s(V \times [0, T])$ is the viscosity solution of (CP) for the initial-value φ. This result employs only the special case of the Convergence Theorem in which the modulus is uniform in the u_n and is interesting and new in the classical finite dimensional theory.

We remark that the additional assumption (H4) (or some variant) is necessary for (SP) in the sense that it is possible to give examples (there is one in Section 5) of stationary problems $u + H(x, Du) = 0$ in $V = \mathbf{R}$ in which H satisfies (H0)–(H3) but there are no uniformly continuous viscosity solutions.

We next sketch the proof of the Convergence Theorem. After this is complete we establish Theorem 2.2.

Proof of the Convergence Theorem 2.1. We begin with the case (i). As a first step we replace u_n by $e^{-t}u_n$ in the usual way, the effect being that we may assume that u_n is a viscosity solution of the problem

$$u_{nt} + u_n + H_n(x, t, u_n, Du_n) = 0, \tag{2.13}$$

and the remaining assumptions are still satisifed. We seek to estimate the difference $u_n(x, t) - u_m(x, t)$ on the set $v(x) < R$. To this end, let $G: \mathbf{R} \to \mathbf{R}$ be continuously differentiable and satisfy

$$G(r) = 0 \qquad \text{if} \quad r \leqslant 0,\ G(1) = 1, \text{ and } 0 \leqslant G'(r) \leqslant 2. \tag{2.14}$$

Let $R, R', \alpha, \beta > 0$, and put

$$\Phi(x, y, t) = u_n(x, t) - u_m(y, t) - \left(\frac{d(x, y)^2}{\alpha} + \beta G\left(\frac{v(x) - R}{R'}\right)\right). \tag{2.15}$$

The parameters will be chosen for various purposes later. Roughly speaking, we will first produce a bound on $u_n - u_m$ on $v(x) \leqslant R$ which is independent of R and then we will use this to sharpen the estimates and show the convergence. Let

$$M_{nm} = \sup\{\Phi(x, y, t): (x, y, t) \in S\}, \tag{2.16}$$

where $S = \{(x, y, t) \in V \times V \times [0, T]: v(x), v(y) \leqslant R + R' \text{ and } d(x, y) \leqslant 1\}$ and $(x_k, y_k, t_k) \in S$, be such that

$$\begin{aligned} &\text{(i)} && \Phi(x_k, y_k, t_k) \uparrow M_{nm}, \\ &\text{(ii)} && \Phi(x_k, y_k, t_k) \geqslant \Phi(x_k, x_k, t_k). \end{aligned} \tag{2.17}$$

Notice that, by (2.14),

$$M_{nm} \geqslant \sup\{u_n(x, t) - u_m(x, t): v(x) < R, t \in [0, T]\}. \tag{2.18}$$

The relation (2.17) (ii) implies that

$$d(x_k, y_k)^2/\alpha \leqslant u_m(x_k, t_k) - u_m(y_k, t_k). \tag{2.19}$$

Now we set

$$\begin{aligned} \omega_n(r) &= \sup\{|u_n(x, t) - u_n(y, t)|: d(x, y) \leqslant r, 0 \leqslant t \leqslant T\}, \\ \omega(r) &= \limsup_{n \to \infty} \omega_n(r) \qquad \text{for} \quad 0 \leqslant r. \end{aligned} \tag{2.20}$$

Using (2.19) and (2.20) we have

$$d(x_k, y_k)^2 \leqslant \alpha \omega_m(d(x_k, y_k)). \tag{2.21}$$

We consider three possible situations: Either

(I) $t_k \to 0$ or

(II) $\max(v(x_k), v(y_k)) \to R + R'$ or

(III) for some $\eta > 0$, $t_k > \eta$ and $v(x_k), v(y_k) \leqslant R + R' - \eta$ for large k.

By passing to a subsequence of $\{\Phi(x_k, y_k, t_k)\}$ if necessary, we can always reduce to a case in which one of (I)–(III) holds. Using the bound (2.21) and $d(x_k, y_k) \leqslant 1$ we find

$$d(x_k, y_k) \leqslant (\alpha\omega_m(1))^{1/2} \tag{2.22}$$

and so

$$d(x_k, y_k)/\alpha \leqslant (\omega_m(1)/\alpha)^{1/2}. \tag{2.23}$$

Using (2.22) in (2.21)

$$d(x_k, y_k)^2/\alpha \leqslant \omega_m((\alpha\omega_m(1))^{1/2}). \tag{2.24}$$

Let us now assume we are in case (I). In this event, (2.17), (2.22), and $u_n(x, 0) = \varphi_n(x)$, etc., imply that

$$M_{nm} \leqslant \sup\{|\varphi_n(x) - \varphi_m(x)| : v(x) \leqslant R + R'\} + \omega_m((\alpha\omega_m(1))^{1/2}). \tag{2.25}$$

If we are in case (II), we use (2.2) to conclude that

$$M_{nm} \leqslant 2(A_0 + B_0(R + R')) - \beta \limsup_{k \to \infty} G((v(x_k) - R)/R'). \tag{2.26}$$

Now either $v(x_k) \to R + R'$ or $v(y_k) \to R + R'$. Since $v(x_k) \geqslant v(y_k) - K|x_k - y_k| \geqslant v(y_k) - Kd(x_k, y_k)$ and G' is bounded by 2, in both cases we conclude that

$$M_{nm} \leqslant 2(A + B(R + R')) - \beta(1 - 2(K/R')(\alpha\omega_m(1))^{1/2}). \tag{2.27}$$

In case (III) we may use the arguments of [10] to conclude that the estimates we arrive at below by assuming that in fact $\Phi(x_k, y_k, t_k) = M_{nm}$ and using the equations satisfied by u_n and u_m are valid if we keep the parameters in the range where (2.22) gurantees $d(x_k, y_k)$ remains strictly away from 1 (as we will). Hence we simply assume (x_k, y_k, t_k) is a maximum point of Φ on S. We write $(x_k, y_k, t_k) = (\bar{x}, \bar{y}, \bar{t})$ to have a nicer appearance. Recall ([10]) that the function $z(x, y, t) = u_n(x, t) - u_m(y, t)$ satisfies the equation

$$z_t + z + H_n(x, t, u_n(x, t), D_x z) - H_m(y, t, u_m(y, t), -D_y z) = 0$$

on $V \times V \times (0, T]$ in the viscosity sense. Therefore we have

$$\begin{aligned} &u_n(\bar{x}, \bar{t}) - u_m(\bar{y}, \bar{t}) \\ &\quad \leqslant H_m(\bar{y}, \bar{t}, u_m(\bar{y}, \bar{t}), 2dd_y/\alpha) \\ &\qquad - H_n(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha + (\beta/R')\, G'((v(\bar{x}) - R)/R')\, Dv(\bar{x})), \end{aligned} \tag{2.28}$$

where $d = d(\bar{x}, \bar{y})$, etc. We also have

$$
\begin{aligned}
&-H_n(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha + (\beta/R')\, G'((v(\bar{x}) - R)/R')\, Dv(\bar{x}))\\
&\quad = H(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha + (\beta/R')\, G((v(\bar{x}) - R)/R'\, Dv(\bar{x}))\\
&\qquad - H_n(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha + (\beta/R')\, G'((v(\bar{x}) - R)/R')\, Dv(\bar{x}))\\
&\qquad - H(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha + (\beta/R')\, G'((v(\bar{x}) - R)/R')\, Dv(\bar{x}))\\
&\qquad + H(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha) - H(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha).
\end{aligned}
\tag{2.29}
$$

Let us introduce the functions

$$k_n(R) = \sup\{|H_n(x, t, r, p) - H(x, t, r, p)| : v(x), |r|, |p| \leqslant R\}.$$

Using (2.22)–(2.29), and (H2) we deduce that

$$
\begin{aligned}
&-H_n(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha + (\beta/R')\, G'((v(\bar{x}) - R)/R')\, Dv(\bar{x}))\\
&\quad \leqslant k_n(A + (1 + B)(R + R') + 2K((\omega_m(1)/\alpha)^{1/2} + \beta/R'))\\
&\qquad + \sigma_H(2K\beta/R', 2K((\omega_m(1)/\alpha)^{1/2}) + \beta/R')) - H(\bar{x}, \bar{t}, u_n(\bar{x}, \bar{t}), 2dd_x/\alpha).
\end{aligned}
\tag{2.30}
$$

In a similar way we find that

$$
\begin{aligned}
&H_m(\bar{y}, \bar{t}, u_m(\bar{y}, \bar{t}), 2dd_y/\alpha)\\
&\quad \leqslant k_m(A + (1 + B)(R + R') + 2K((\omega_m(1)/\alpha)^{1/2}))\\
&\qquad + H(\bar{y}, \bar{t}, u_m(\bar{y}, \bar{t}), -2dd_y/\alpha).
\end{aligned}
\tag{2.31}
$$

Putting (2.30) and (2.31) together with (2.28) and using (H1) and (H3) yields

$$
\begin{aligned}
&u_n(\bar{x}, \bar{t}) - u_m(\bar{y}, \bar{t})\\
&\quad \leqslant (k_m + k_n)(A_0 + (1 + B_0)(R + R') + 2K((\omega_m(1)/\alpha)^{1/2} + \beta/R'))\\
&\qquad + \sigma_H(2K\beta/R', 2K((\omega_m(1)/\alpha)^{1/2} + \beta/R'))\\
&\qquad + m_H(2K(\omega_m((\alpha\omega_m(1))^{1/2}) + (\alpha\omega_m(1))^{1/2}))
\end{aligned}
\tag{2.32}
$$

and so

$$
\begin{aligned}
M_{nm} &\leqslant (k_m + k_n)(A_0 + (1 + B_0)(R + R') + 2K((\omega_m(1)/\alpha)^{1/2} + (\beta/R'))\\
&\quad + \sigma_H(2K\beta/R', 2K((\omega_m(1)/\alpha)^{1/2})) + \beta/R'))\\
&\quad + m_H(2K(\omega_m((\alpha\omega_m(1))^{1/2}) + (\alpha\omega_m(1))^{1/2})).
\end{aligned}
\tag{2.33}
$$

Let

$$L = \omega(1)^{1/2}. \tag{2.34}$$

To estimate $\limsup_{n,m \to \infty} M_{nm}$ we can assume that we are in one of the cases (I)–(III) for each n, m. In case (I) we use (2.3), (2.20), (2.25), (2.34), and the assumed UB convergence of the φ_n to conclude that

$$\limsup_{n,m \to \infty} M_{nm} \leqslant \omega(\alpha^{1/2} L) \tag{2.35}$$

(where we should write $\omega(\alpha^{1/2} L+)$ on the right but won't).

In case (II) we use (2.27) and (2.24) to conclude

$$\limsup_{n,m \to \infty} M_{nm} \leqslant 2(A + B(R + R')) - \beta(1 - 2(K/R')\, \alpha^{1/2} L). \tag{2.36}$$

Letting

$$R' = R \geqslant 1 \tag{2.37}$$

and assuming hereafter that α is so small that

$$\alpha^{1/2} L \leqslant \min\{\tfrac{1}{2}, 1/(4K)\} \tag{2.38}$$

we see that the right-hand side of (2.36) is negative if

$$\beta = 4(A + 2BR) + 1, \tag{2.39}$$

and we may therefore disregard case (II) when (2.37)–(2.39) hold.

In case (III) we use (2.33) and (2.37)–(2.39) together with $k_n \to 0$ UB to conclude that when these conditions hold

$$\begin{aligned} \limsup_{n,m \to \infty} M_{nm} \leqslant\ & m_H(2K\omega((\alpha L^{1/2}) + \tfrac{1}{2}) \\ & + \sigma_H(2K(4(A + 2B) + 1), 2K(L\alpha^{-1/2} + 4(A + 2B) + 1)). \end{aligned} \tag{2.40}$$

Notice that the right-hand sides of (2.35) and (2.40) are independent of R, so from (2.18) we deduce that for all $R > 0$

$$\sup\{u_n(x, t) - u_m(x, t) \colon v(x) < R, t \in [0, T]\} \leqslant \varepsilon_{nm} + B_1, \tag{2.41}$$

where

$$\varepsilon_{nm} \to 0 \qquad \text{as} \quad n, m \to \infty \tag{2.42}$$

and B_1 is some constant independent of R.

The estimate (2.35) is sufficient for our purposes in case (I), but we need to use the above information to sharpen the bounds in cases (II), (III). First of all, we argue that (2.41) holds with a constant B_1 independent of R *in general.* This follows from the above by choosing $R = R'$ and β as in (2.39) to reduce to either case (I) or (III) so that (2.41) or (2.35) holds. Since (2.35) implies (2.41), we conclude that (2.41) holds for all R. Since (2.41) is independent of β, α we are free to assume (2.41) holds and to choose β, α anew in further analysis.

To analyze case (II) further, we use (2.41) (with $R + R'$ in place of R) to deduce

$$u_n(x_k, t_k) - u_m(y_k, t_k) \leqslant \varepsilon_{nm} + B_1 + \omega_m((\alpha\omega_m(1))^{1/2})$$

and so in case (II), (2.27) can be replaced by

$$M_{nm} \leqslant \varepsilon_{nm} + B_1 + \omega_m((\alpha\omega_m(1))^{1/2}) - \beta(1 - 2(K/R')(\alpha\omega_m(1))^{1/2}). \quad (2.43)$$

Hence, by (2.42),

$$\limsup_{n,m \to \infty} M_{n,m} \leqslant B_1 + L^2 - \beta(1 - 2(K/R')\,\alpha^{1/2}L). \quad (2.44)$$

We see that we may *fix* β sufficiently large and independent of R, $R' \geqslant 1$ in such a way that the right-hand side of (2.44) is negative for all sufficiently small $\alpha > 0$. With β so fixed, we may disregard case (II). The estimate (2.35) is still sufficient for our purposes in case (I). In case (III) we use (2.33) with the β just fixed by case (II) to conclude that

$$\limsup_{n,m \to \infty} M_{nm} \leqslant \sigma_H(2K\beta/R', 2K(L\alpha^{-1/2} + \beta/R')) + m_H(2K\omega(\alpha^{1/2}L) + \alpha^{1/2}L). \quad (2.45)$$

By (2.3), the right-hand side of (2.35) can be made as small as desired by choosing α small. Similarly, the right-hand side of (2.45) yields 0 in the iterated limit $R' \to \infty$ and then $\alpha \to 0$. In view of (2.18) we conclude the UB convergence of the u_n to a limit u. It follows easily from (2.3) that u is uniformly continuous in x uniformly in t. Since each u_n is uniformly continuous on bounded sets, their UB limit u also has the property and we conclude that $u \in \mathrm{UC_s}(V \times [0, T])$.

The proof in the case (ii) of (SP) is given in an entirely analogous way. One still uses (2.17) (now independent of t) and proceeds through the same steps. We leave it to the reader.

We turn now to the proof of Theorem 2.2.

Proof of Theorem 2.2. We adapt the comparison function technique of

Ishii [16, 17] to the current case. First of all, we bound u. Let A, $B \geqslant 0$. Using (H1) and (H2) we have

$$-H(x, t, v, (1+t)\,BDv) \leqslant -H(x, t, 0, 0) + \sigma_H((1+T)\,B, (1+T)B) \quad (2.46)$$

for $v \geqslant 0$ and so we see that the function $v(x, t) = (A + Bv(x))(1+t)$ is a viscosity solution of $v_t + H(x, t, v, Dv) \geqslant 0$ if

$$A + Bv(x) \geqslant H(x, t, 0, 0) + \sigma_H((1+T)\,B, (1+T)B). \quad (2.47)$$

In making this claim, we are using the obvious remark that inequations in the viscosity sense for everywhere differentiable functions are equivalent to the corresponding pointwise statements. If also $A + Bv(x) \geqslant A_0 + B_0 v(x)$, then $v(x, 0) \geqslant u(x, 0) = \varphi(x)$ and we conclude, using [10, Theorem 2] (the global nature of the assumption (H0) in r there is not necessary in $\Omega = V$), that if

$$\begin{aligned} B &= \max(B_0, B_H), \\ A &= \max\{A_0, A_H\} + \sigma_H((1+T)B, (1+T)B) \end{aligned} \quad (2.48)$$

then

$$u(x, t) \leqslant (A + Bv(x))(1+t).$$

To begin the estimate on the modulus, we recall [10] that the function $z(x, y, t) = u(x, t) - u(y, t)$ on $V \times V \times [0, T]$ satisfies the equation

$$z_t + H(x, t, u(x, t), D_x z) - H(y, t, u(y, t), -D_y z) = 0 \quad (2.49)$$

in the viscosity sense. Therefore, by (H1), on the set $\{(x, y, t) \in V \times V \times (0, T]: z(x, y, t) > 0\}$, z is a viscosity solution of

$$z_t + \bar{H}(x, y, t, u(y, t), D_x z, D_y z) \leqslant 0, \quad (2.50)$$

where

$$\bar{H}(x, y, t, r, p, q) = H(x, t, r, p) - H(y, t, r, -q). \quad (2.51)$$

We seek a function $w(x, y, t)$ of the form

$$w(x, y, t) = (E_0 + F_0\, d(x, y))\, e^{\lambda t}, \quad (2.52)$$

which is a viscosity solution of

$$w_t + \bar{H}(x, y, t, r, D_x w, D_y w) \geqslant 0 \quad (2.53)$$

for all $r \in \mathbf{R}$. To this end, we make the following simplifying remark:

Remark 2.3. Even though $d(x, y)$ is not differentiable with respect to x or y on the diagonal $x = y$, all of the formal calculations below with trial subsolutions and supersolutions involving $d(x, y)$ can be made rigorous by first doing the calculations with $d_\varepsilon(x, y) = (\varepsilon + d(x, y)^2)^{1/2}$ for $\varepsilon > 0$, then letting $\varepsilon \to 0$ and invoking Theorem 1.4. Observe that $d_\varepsilon(x, y)$ is everywhere differentiable with respect to x and y and, for example,

$$d_{\varepsilon x}(x, y) = (\varepsilon + d(x, y)^2)^{-1/2}\, d(x, y)\, d_x(x, y)$$

(interpreted as zero on $x = y$) and $(\varepsilon + d(x, y)^2)^{-1/2}\, d(x, y) \leqslant 1$. For everywhere differentiable functions the pointwise and viscosity notions of solutions obviously coincide.

In order that (2.52) solve (2.53) we need that

$$\lambda(E_0 + F_0\, d(x, y))\, e^{\lambda t} \geqslant H(y, t, r, -e^{\lambda t} F_0\, d_y) - H(x, t, r, e^{\lambda t} F_0\, d_x) \tag{2.54}$$

for all r. In view of (H3), it suffices to have

$$\lambda(E_0 + F_0\, d(x, y)) \geqslant e^{-\lambda t} m_H(e^{\lambda t} F_0\, d + d).$$

Because m_H is a modulus, $m_H(r) \leqslant m_H(1) + m_H(1) r$, and it follows that (2.54) holds as soon as λ is large enough. Moreover, by (2.4), for E_0, $F_0 \geqslant m_0(1)$

$$w(x, y, 0) = E_0 + F_0\, d(x, y) \geqslant u(x, 0) - u(y, 0) = \varphi(x) - \varphi(y),$$

We claim that then $z(x, y, t) \leqslant w(x, y, t)$. This follows from the proof of [10, Theorem 1] (but not quite from the theorem itself). To see this, observe that for fixed r, $\bar{H}(x, y, t, r, p, q)$ satisfies conditions (H0)–(H3) of [10] (while $H(x, t, u(x, t), p, q)$ may not). In checking that this is so, one uses

$$\bar{v}(x, y) = v(x) + v(y) \qquad \text{and} \qquad \bar{d}((x, y), (\bar{x}, \bar{y})) = (d(x, \bar{x})^2 + d(\bar{y}, y)^2)^{1/2}$$

as the functions for conditions (C) on $V \times V$ and in (H2)–(H3) for $\bar{H}$. In order to prove $z \leqslant w$ we will only need to discuss z where it is nonnegative. Then proceeding as in the proof of the comparison theorem and using (2.50) and (2.53) one encounters an upper bound roughly of the form

$$\bar{H}(x, y, t, r, p, q) - \bar{H}(x, y, t, u(y, t), p, q) + \text{terms which go to zero},$$

where (x, y, t) is chosen to maximize a certain functional. Since r is at our disposal, we may use $r = u(y, t)$ and the comparison argument succeeds. Fixing $1 > \alpha$, $\gamma > 0$, we next seek a supersolution w of (2.53) on the set

$$S = \{(x, y, t) \in V \times V \times [0, T] \colon d(x, y) < 1\}$$

of the form

$$w = (E + F(\alpha + d(x, y))^{\gamma})(1 + t), \tag{2.55}$$

where $E \geqslant 0$ and F satisfies

$$F \geqslant e^{\lambda T}(E_0 + F_0) \tag{2.56}$$

with λ, E_0, F_0 from above. This guarantees that $w(x, y, t) \geqslant u(x, t) - u(y, t)$ on $d(x, y) = 1$ and on $t = 0$, $d(x, y) \leqslant 1$. A calculation reveals that (2.55) is a viscosity supersolution on S if

$$E + F(\alpha + d(x, y))^{\gamma} \geqslant m_H((1 + T)\,\gamma F(\alpha + d)^{\gamma} + d). \tag{2.57}$$

Fix

$$F = \max(m_H(1) + 1, m_0(1), E_0 + F_0) \tag{2.58}$$

so that (2.56) holds and put

$$E(\gamma) = \max\{m_H((1 + T)\,\gamma F(\alpha + r)^{\gamma} + r) - E(\alpha + r)^{\gamma} : 0 \leqslant r, \alpha < 1\}. \tag{2.59}$$

One easily shows that $E(0+) = 0$ (see, e.g., [9, Lemma 1]). As above, comparison implies that for $0 < \alpha$, $\gamma < 1$

$$|u(x, t) - u(y, t)| \leqslant (E(\gamma) + F(\alpha + d(x, y))^{\gamma})(1 + t)$$

and sending $\alpha \to 0$ and taking the infimum over γ on the right produces the modulus $m(d(x, y))$ on $d < 1$.

In order to exhibit the local modulus in time, fix $\bar{x} \in V$ with $v(\bar{x}) \leqslant R$ and $\bar{t} \in [0, T]$ and seek a supersolution v of

$$v_t + H(x, t, v, Dv) \geqslant 0 \tag{2.60}$$

on $\{(x, t) \in V \times (\bar{t}, T]\}$ of the form

$$v(x, t) = u(\bar{x}, \bar{t}) + A + Bd(x, \bar{x}) + C(t - \bar{t}), \tag{2.61}$$

which further satisfies

$$v(x, t) \geqslant u(x, t) \qquad \text{if } v(x) = R + 1 \text{ and } t \in [\bar{t}, T] \tag{2.62}$$

and

$$v(x, \bar{t}) \geqslant u(x, \bar{t}) \qquad \text{for} \quad v(x) \leqslant R + 1. \tag{2.63}$$

Using (2.8) we see that

$$|u(x, t)| \leqslant C_R \qquad \text{if } v(x) \leqslant R + 1 \text{ and } t \in [0, T],$$

where C_R depends only on the data. Let $d(x, y) \geqslant L$ if $|v(x) - v(y)| \geqslant 1$. We will have (2.62) provided only that

$$BL \geqslant 2C_R.$$

Using (2.9) we see that (2.63) holds provided that

$$A + Bd(x, \bar{x}) \geqslant m(d(x, \bar{x})) \qquad \text{on } v(x) \leqslant R + 1.$$

Since m is a modulus,

$$m(d) \leqslant m(\varepsilon) + (m(\varepsilon)/\varepsilon)d \qquad \text{for} \quad \varepsilon, d > 0,$$

and we can thus choose

$$B = \max\{2C_R/L, m(\varepsilon)/\varepsilon\}, \qquad A = m(\varepsilon), \tag{2.64}$$

for any $\varepsilon > 0$ and have that (2.62) and (2.63) hold. Fixing A, B as in (2.64) and $C \geqslant 0$ (so $v \geqslant u(\bar{x}, \bar{t}) \geqslant -C_R$)) the inequation (2.50) holds provided that

$$C + H(x,t, -C_R, Bd_x(x, \bar{x})) \geqslant 0 \qquad \text{on } v(x) \leqslant R + 1, \bar{t} < t \leqslant T,$$

and for this it is enough that

$$C = b_H(R + 1 + 2C_R/L + m(\varepsilon)/\varepsilon). \tag{2.65}$$

By comparison, we conclude that for each $\varepsilon > 0$

$$u(\bar{x}, \bar{t}) \leqslant u(\bar{x}, t) + m(\varepsilon) + C(t - \bar{t}),$$

where C is given by (2.65). Arguing in a similar way one obtains analogous estimates from below and for $t \leqslant \bar{t}$, establishing the desired results.

The result for (SP) is proved in a similar way. First, we see that $v(x, t) = A + Bv(x)$ is a supersolution of (SP) if

$$A + Bv(x) \geqslant -H(x, 0, 0) + \sigma_H(B, B)$$

so it is enough to put $B = B_H$ and $A = A_H + \sigma_H(B, B)$ to guarantee (2.11).

To begin the estimate on the modulus we set $z(x, y) = u(x) - u(y)$ so that

$$z + \bar{H}(x, y, u(y, t), D_x z, D_y z) \leqslant 0$$

on the set $z > 0$, where $\bar{H}$ is given by (2.51) as before. It is in getting a first bound on z in $V \times V$ that we need to invoke (H4), which is designed just for that purpose. Indeed, set $w(x, y) = G(d(x, y))$. Then, by (H4), w is a solution of

$$w(x, y) + \bar{H}(x, y, r, D_x w, D_y w) \geqslant 0.$$

This follows from the computation

$$\begin{aligned} H(x, r, D_x w) - H(y, r, -D_y w) &= H(x, r, G'(d)d_x) - H(y, r, -G'(d)d_y) \\ &\geqslant -F(G'(d), d) \geqslant -G(d) = -w, \end{aligned}$$

showing that w is a supersolution. We conclude that $w \geqslant z$. Now we can proceed as in (CP) once more. Since $z \leqslant G(1)$ on $\{(x, y) \in V \times V: d(x, y) \leqslant 1\}$, and $E_0 + F_0(\alpha + d(x, y))^\gamma$ is a supersolution on $d < 1$ if

$$E_0 + F_0(\alpha + d(x, y))^\gamma \geqslant m_H(\gamma F_0(\alpha + d)^\gamma + d),$$

one proceeds in the same way as for the (CP).

3. Reduction to the Case of Lipschitz Continuous Hamiltonians

We will carry out the discussion below for the Cauchy and stationary problems simultaneously. Appropriate distinctions between the cases will be made at those times when it is necessary—otherwise the discussion proceeds as if H depends on t and (SP) is understood to be included by allowing H to be independent of t.

First Reduction—To Lipschitz continuous initial-values

The first reduction is only relevant for (CP). Let $H(x, t, r, p)$ satisfy (H0)–(H3) and $\varphi \in \mathrm{UC}(V)$. We consider (CP):

$$u_t + H(x, t, u, Du) = 0, \qquad u(x, 0) = \varphi(x). \tag{3.1}$$

Then φ may be approximated by its "inf convolution"

$$\varphi_n(x) = \inf\{\varphi(y) + n\,|x - y| : y \in V\} \tag{3.2}$$

for $n > 0$. If m_0 is a modulus for φ, i.e.,

$$|\varphi(x) - \varphi(y)| \leqslant m_0(|x - y|) \qquad \text{for} \quad x, y \in V, \tag{3.3}$$

then one can easily demonstrate that

$$\varphi_n(x) \leqslant \varphi(x) \leqslant \varphi_n(x) + \varepsilon_n, \tag{3.4}$$

where

$$\varepsilon_n = \inf\{m_0(r) - nr \colon 0 \leqslant r\}$$

satisfies $\varepsilon_n \to 0$ as $n \to \infty$. (In particular, ε_n is finite if $n > m(1)$, and φ_n is well-defined in this range.) Moreover, m_0 is a modulus for φ_n. In all, φ_n is Lipschitz continuous for n large (with n as a Lipschitz constant) and converges uniformly to φ as $n \to \infty$ and has the same modulus of continuity as φ. Using the Convergence Theorem (or the simpler result which bounds the difference of solutions of (CP) by the supremum of the difference of the initial values), we conclude that to prove existence of a solution $u \in \mathrm{UC}_s(V \times [0, T])$ of (CP), we may assume that φ is Lipschitz continuous. We could (but do not need to) further approximate by putting (assuming now that φ is already Lipschitz) $\varphi_n(x) = (1 - |x|/n)^+ \varphi(x)$ (where $r^+ = \max(r, 0)$). The φ_n have a common modulus and converge UB to φ, so we may use the Convergence Theorem to assert that it is enough to solve (CP) in the case of Lipschitz continuous initial-values of bounded support.

Next we begin a sequence of approximations of the Hamiltonian.

Second Reduction—To bounded Hamiltonians

For $n > 0$ set

$$H_n = \max(\min(H, n), -n). \tag{3.5}$$

Since H is uniformly continuous on bounded sets by (H0), it is bounded on bounded sets and therefore $H_n \to H$ UB. Moreover, it is easy to see that H_n satisfies (H0)–(H3) with the same functions σ_H, m_H as H. In the stationary case, H will satisfy (H4) and the H_n's do also with the same F, G. Therefore, if we can establish the existence assertions with H replaced by H_n, the Convergence Theorem and Theorem 2.2 can be invoked to establish the existence assertions for H. We have now reduced our considerations to bounded Hamiltonians satisfying the assumptions.

Third Reduction–$H(x, t, r, p)$ is also uniformly continuous in (x, r) for p bounded

Now let H be a bounded Hamiltonian satisfying (H0)–(H3). For $n > 0$ put

$$H_n(x, t, r, p) = (1 - |x|/n)^+ H(x, t, r_n, p), \tag{3.6}$$

where r_n denotes r truncated at the level n as in (3.5). Then H_n is supported on the bounded set $|x| \leqslant n$ and is independent of r on $r \geqslant n$ and on $r \leqslant -n$. In the case of the Cauchy problem, one easily checks that the H_n converge

UB to H and satisfy (H0)–(H3) uniformly in such a way that the Convergence Theorem and Theorem 2.2 may be invoked to reduce the existence assertions for H to the case in which H is bounded, has a bounded support in x, and is independent of large r. For (SP) we would need to have (H4) satisfied uniformly in n. This is so, but not because H in (3.6) satisfies (H4): Instead we use the uniform bound C on $|H_n|$ provided by the bound on H and put $F=2C$, $G=2C$. Thus we may assume hereafter that, in addition to (H0)–(H3) (and (H4) for (SP)), $H(x, t, r, p)$ is now bounded and (jointly) uniformly continuous in (x, r) uniformly in $t \in [0, T]$ and bounded p.

Fourth Reduction—$H(x, t, r, p)$ is also independent of r

We remark that the reader will be better served on an initial reading to assume H is independent of r from the beginning, skip this reduction, and then return to it when it is convenient. To begin the reduction to the case in which H is independent of r, assume the conditions through the third reduction and put

$$H_n(x, t, r, p) = \inf\{H(x, t, s, p) + n\,|r-s| : s \in \mathbf{R}\}. \tag{3.7}$$

If H has all the properties achieved through the third reduction, then H_n does as well and uniformly in n. Moreover, H_n is Lipschitz continuous in r and converges to H UB. Thus it is enough to solve (CP) (or (SP)) with H_n in order to have the solution in general. To do this we will use a fixed point argument based on the solvability (yet to be established) for the case in which H is independent of r. We describe the program in the case of (SP). One begins by verifying that if $H(x, r, p)$ satisfies the conditons achieved through the third reduction and is Lipschitz continuous in r, then for $w \in \mathrm{UC}(V)$ the Hamiltonian $H(x, w(x), p)$ satisfies the same conditions and the solvability of (SP) with this Hamiltonian would be guaranteed if we had settled the case in which H is independent of r. If the map $w \to u$ is then shown to have a fixed point we would be done. A slight modification of this outline indeed succeeds, as we now establish.

Let H have all the properties of H_n above,

$$|H(x, r, p) - H(x, s, p)| \leqslant L\,|r-s|, \tag{3.8}$$

and $w \in \mathrm{BUC}(V)$. Let $w, \bar{w} \in \mathrm{UC}(V)$ and $u, \bar{u} \in \mathrm{UC}(V)$ be viscosity solutions of

$$u + H(x, w(x), Du) = 0 \qquad \text{and} \qquad \bar{u} + H(x, \bar{w}(x), D\bar{u}) = 0.$$

Because H is bounded, so are u and $\bar{u}$. Using the comparison result of [10] we easily find that

$$|u(x)-\bar{u}(x)| \leqslant \mathrm{L}\sup\{|w(z)-\bar{w}(z)|: z \in V\}.$$

Thus the map $w \to u$ is a strict contraction of BUC(V) if $L<1$, and there is a fixed point u which solves $u+H(x, u, Du)=0$. If $L \geqslant 1$, we proceed by choosing $v \in \mathrm{BUC}(V)$ and using the result just obtained to uniquely solve

$$u+\lambda H(x, u, Du)-\eta v=0$$

when $\lambda L<1$ (the Hamiltonian now being $\lambda H(x, r, p)-\eta v(x)$. The self-map $v \to u$ of BUC(V) has $|\eta|$ as a Lipschitz constant and therefore has a fixed point if $0 \leqslant \eta<1$. The fixed point u satisfies $(1-\eta)u+\lambda H(x, u, Du)=0$. Putting, for example, $\lambda=1-\eta=1/(2L)$ we satisfy all the conditions; the fixed point u therefore exists and it solves $u+H(x, u, Du)=0$, so we are done.

The Cauchy problem is treated in an analogous way, with the details being somewhat more complex. We consider the map $w \to K(w)=$ solution u of the Cauchy problem $u_t+H(x, t, w(x, t), Du)=0$, $u(x, 0)=\varphi(x)$ as a self-map of $\mathrm{BUC}_s(V \times [0, T])$ under the assumptions which we now have available on H and the assumed solvability if H is independent of r. Via the arguments of [10] one shows that

$$|K(w)(x, t)-K(\hat{w})(x, t)| \leqslant L \int_0^t \sup\{|w(y, s)-\hat{w}(y, s)|: y \in V\}\, ds,$$

where L is a Lipschiptz constant for H in r, and thus concludes that K has a fixed point. We leave it to the reader to verify this inequality—it is not quite explicit in [10, Theorem 2, and following remarks]. There is another way to deal with u dependence of H, as is remarked at the end of Section 4.

Fifth Reduction—H is also Lipschitz continuous in (x, p)

At the next stage, we begin with a bounded Hamiltonian $H(x, t, p)$ (the r-dependence having been taken care of by the fourth reduction) which is uniformly continuous in x uniformly in t and bounded p and satisfies (H0)–(H3) (and (H4) for (SP)). We then set

$$H_n(x, t, p)=\inf\{H(y, t, q)+(|y-x|+|p-q|)\, n: (y, q) \in V \times V^*\} \tag{3.9}$$

and wish to argue that it is enough to solve (CP) or (SP) with H_n in place of H. H_n is well-defined (since H is bounded) and H_n is Lipschitz continuous in x and p. Moreover, H_n satisfies (H0) and there is a continuous nondecreasing function ε_n such that

$$|H_n(x, t, p) - H(x, t, p)| \leqslant \varepsilon_n(|p|) \tag{3.10}$$

and

$$\lim_{n \to \infty} \varepsilon_n(R) = 0 \qquad \text{for} \quad R \geqslant 0. \tag{3.11}$$

In particular, $H_n \to H$ UB. However, it does not follow from the assumptions that H_n satisfies (H1)–(H3) with d, v from (C), let alone uniformly in n. It does, however, satisfy (H4) uniformly in n since H is bounded—see the third reduction. We will need the full force of the Convergence Theorem.

We continue the discussion in the context of (CP). Assume that $u_n \in \mathrm{UC}_s(V \times [0, T])$ is a viscosity solution of

$$\begin{aligned} u_{nt} + H_n(x, t, Du_n) &= 0, \\ u_n(x, 0) &= \varphi(x). \end{aligned} \tag{3.12}$$

We seek to estimate $u_n(x, t) - u_n(y, t) = z(x, y, t)$. Using the above and (3.10) we find that

$$z_t + H(x, t, D_x z) - H(y, t, -D_y z) \leqslant \varepsilon_n(|D_x z|) + \varepsilon_n(|D_y z|). \tag{3.13}$$

We construct some supersolutions of (3.13). First, we observe that if w as given by (2.52) solves (2.54), then

$$w_n(x, y, t) = (E_0 + F_0\, d(x, y))\, e^{\lambda t} + 2\varepsilon_n(e^{\lambda T} K F_0) t$$

is a supersolution of (3.13) since $e^{\lambda T} K F_0$ is a Lipschitz constant for w in x and y. We claim that $z \leqslant w_n$. The inequations solved by w_n and z do not satisfy the hypotheses of the comparison theorem in [10], but going through the proof given in [10] and regarding the terms involving $\varepsilon_n(\cdot)$ as a perturbation one easily justifies the above claim. We leave the tedious verification to the interested reader.

In particular, $u_n(x, t) - u_n(y, t)$ is bounded independently of n on the set $d(x, y) \leqslant 1$, say by M. Next let $w(x, y, t)$ be the supersolution $E(\gamma) + F(\alpha + d(x, y))^\gamma (1 + t)$ of (2.53) constructed in (2.55)–(2.59) with, however,

$$F \geqslant \max(m_H(1) + 1, m_0(1), M)$$

in place of (2.58) (where m_0 is a modulus for φ as in (2.4)) so that

$w \geqslant u_n(x, t) - u_n(y, t)$ on $t = 0$ and $d \leqslant 1$ and on $d = 1$. Then, as above, we conclude that

$$u_n(x, t) - u_n(y, t) \leqslant (E(\gamma) + F(\alpha + d(x, y))^{\gamma})(1 + T) + 2\varepsilon_n(L(\gamma, \alpha))T, \qquad (3.14)$$

where L is a Lipschitz constant for w in x and y. It is now clear that the condition (2.3) in the Convergence Theorem holds. Our conclusion is that if we can solve (CP) with bounded Lipschitz continuous Hamiltonians, then we have proved the Existence Theorem. This last step is taken up in the next section. The arguments above need only minor adaptations to cover the case of (SP).

4. Existence in the Lipschitz Continuous Case

We will now prove the existence in the Lipschitz continuous case to which the above considerations have reduced the proof of the Existence Theorem.

PROPOSITION 4.1. (i) *Let* $H: V \times [0, T] \times V^* \to \mathbf{R}$ *satisfy* (H0) *and* L_1, L_2 *be constants such that*

$$|H(\bar{x}, t, \bar{p}) - H(x, t, p)| \leqslant L_1 |\bar{x} - x| + L_2 |\bar{p} - p| \qquad (4.1)$$

for $x, \bar{x} \in V$, $p, \bar{p} \in V^*$, *and* $t \in [0, T]$. *Assume that* $\varphi \in \mathrm{UC}(V)$ *is Lipschitz continuous with constant* L. *Then there is a viscosity solution* u *of* $u_t + H(x, t, Du) = 0$ *on* $V \times (0, T)$ *which is Lipschitz continuous on bounded sets and satisfies*

$$\begin{aligned} |u(\bar{x}, t) - u(x, t)| &\leqslant (L_1 T + L) |\bar{x} - x| && \text{for } x, \bar{x} \in V, t \in [0, T], \\ u(x, 0) &= \varphi(x) && \text{for } x \in V. \end{aligned} \qquad (4.2)$$

(ii) *If* H *in* (SP) *satisfies* (4.1), *then there is a Lipschitz continuous viscosity solution* u *of* (SP) *with* L_1 *as a Lipschitz constant.*

Proof. We will mainly treat the case of (CP) and relegate (SP) to remarks. It follows from (4.1) that

$$H(x, t, p) = \min_{q \in B_R^*} (H(x, t, q) + L_2 |p - q|) \qquad \text{for} \quad R \geqslant |p| \qquad (4.3)$$

and so

$$H(x, t, p) = \min_{q \in B_R^*} \sup_{z \in B_{L_2}} (-(p, z) + (q, z) + H(x, t, q)) \qquad \text{for} \quad R \geqslant |p|. \quad (4.4)$$

So motivated, we introduce the following Hamiltonians:

$$H_M(x, t, p) = \inf_{q \in B_M^*} \sup_{z \in B_{L_2}} (-(p, z) + (q, z) + H(x, t, q)). \tag{4.5}$$

Clearly $H_M(x, p) = H(x, p)$ for $|p| \leqslant M$. We will produce a viscosity solution u_M of

$$u_{Mt} + H_M(x, t, Du_M) = 0 \text{ in } V \times (0, T], \qquad u_M(x, 0) = \varphi(x) \text{ in } V, \tag{4.6}$$

such that u_M is Lipschitz continuous in x with Lipschitz constant $(L_1 T + L)$ and u_M is Lipschitz continuous in t uniformly on bounded sets of V. In this event, if $(p, a) \in V^* \times \mathbf{R}$ and $(p, a) \in D^+ u_M(x, t) \cup D^- u_M(x, t)$ (D^+ and D^- here taking values in $V \times \mathbf{R}$), then it is easy to see that $|p| \leqslant (L_1 T + L)$ and so u_M is also a viscosity solution of $u_{Mt} + H(x, t, Du_M) = 0$ if $M \geqslant (L_1 T + L)$.

For $0 \leqslant s \leqslant t \leqslant T$, we set $Q_{s,t} = \{\text{strongly measurable } q\colon [s, t] \to B_M^*\}$ and $Z_{s,t} = \{\text{strongly measurable } z\colon [s, t] \to B_{L_2}\}$. The set of strategies on $[s, t]$, $\Xi_{s,t}$ is defined by $\Xi_{s,t} = \{\text{nonanticipating maps } \xi\colon Q_{s,t} \to Z_{s,t}\}$, where nonanticipating means that if $p, q \in Q_{s,t}$ agree almost everywhere on an interval $[a, t]$, $s \leqslant a \leqslant t$, then so do $\xi(p)$ and $\xi(q)$.

Let $0 \leqslant s \leqslant t \leqslant T$. We define $U(t, s)\varphi \in \mathrm{UC}(V)$ for $\varphi \in \mathrm{UC}(V)$ by

$$\begin{aligned} U(t, s)\, \varphi(x) &\\ = \inf_{\xi \in \Xi_{s,t}} \sup_{q \in Q_{s,t}} &\left(\int_s^t f(X(\tau, t, x), \tau, \xi(q)(\tau), q(\tau))\, d\tau + \varphi(X(s, t, x)) \right), \end{aligned} \tag{4.7}$$

where the "state process" $X(t, s, x)$ is given by

$$X(t, s, x) = x + \int_s^t \xi(q)(\tau)\, d\tau, \tag{4.8}$$

and

$$f(x, t, z, q) = -(q, z) - H(x, t, q) \qquad \text{for } x \in V, z \in B_{L_2}, \text{and } q \in B_M^*. \tag{4.9}$$

It is easy to see that $U(t, s)$ is in fact a self-map of $\mathrm{UC}(V)$ (see below). We are abusing notation a bit by not expressing the dependence of X on ξ and q, which should be kept in mind. For those without experience with differential games, let us mention that the key relation in what follows is that if $g(X, t, z, p) = z$ (so that (4.8) amounts to $X' = g(X, t, \xi(q), q)$, $X(s, x) = x$), then (4.5) may be written

$$H_M(x, t, p) = \inf_{q \in B_M^*} \sup_{z \in B_{L_2}} \{-(p, g(x, t, z, q)) - f(x, t, z, q)\}. \tag{4.10}$$

Indeed, all that follows generalizes to suitable representations of this sort.

LEMMA 4.2. *The value function u_M given by $u_M(x, t) = (U(t, 0)\varphi)(x)$ is a viscosity solution of $u_t + H_M(x, t, Du) = 0$ in $V \times (0, T)$. Moreover, u_M is Lipschitz continuous on bounded sets and is Lipschitz continuous in x with constant $(L_1 T + L)$. Finally, $u_M(x, 0) = \varphi(x)$.*

Proof. It will simplify the notation if we agree to write u in place of u_M hereafter. With ξ and q fixed and $x, \bar{x} \in V$,

$$X(t, s, x) - X(t, s, \bar{x}) = x - \bar{x}.$$

Moreover, from (4.1) and (4.9) we have $|f(x, t, z, q) - f(\bar{x}, t, z, q)| \leqslant L_1 |x - \bar{x}|$. It then follows immediately from the definitions and assumptions that

$$|u(x, t) - u(\bar{x}, t)| \leqslant (L_1 T + L) |x - \bar{x}| \tag{4.11}$$

for $x, \bar{x} \in V$ and $t \in [0, T]$, and the asserted Lipschitz continuity in x holds. Moreover, u is easily seen to be Lipschitz continuous in t uniformly for bounded x.

To prove that u is a viscosity solution of $u_t + H_M(x, t, Du) = 0$ we will use the optimality conditions of the dynamic programming principle. In this situation, this just amounts to the statement that U is an evolution operator. That is, if $0 \leqslant r \leqslant s \leqslant t \leqslant T$, then

$$U(t, r) = U(t, s)\, U(s, r), \tag{4.12}$$

where juxtaposition denotes composition of mappings. This relation may be verified in the usual way—see [14] for the finite dimensional case. In order to verify that u is a viscosity supersolution of $u_t + H(x, t, u, Du) = 0$ we assume the contrary and reach a contradiction. Assume that $(p, a) \in V^* \times \mathbf{R}$, $(y, b) \in V \times (0, T)$, and

$$u(x, t) \leqslant u(y, b) + (p, x - y) + a(t - b) + o(|x - y| + |b - t|); \tag{4.13}$$

i.e., (p, a) lies in the superdifferential of u at (y, b). Assume, moreover, that

$$4\gamma = a + H(y, b, p) > 0, \tag{4.14}$$

where γ is defined by (4.14). Using (4.5) we conclude that

$$a + \sup_{z \in B_{L_2}} (-(p, z) + (q, z) + H(y, b, q)) \geqslant 4\gamma \qquad \text{for} \quad q \in B_M^*. \tag{4.15}$$

Using (4.15) and a partition of unity argument, one easily concludes the existence of a continuous function $\xi \colon B_M^* \to B_{L_2}$ such that

$$a - (p, \xi(q)) + (q, \xi(q)) + H(y, b, q) \geqslant 3\gamma \qquad \text{for} \quad q \in B_M^*. \tag{4.16}$$

Next choose an arbitrary $q \in Q_{s,b}$ and $s < b$. From the continuity properties of H and (4.16), we deduce that if $s \leqslant t \leqslant b$ and $b-s$ is small enough, then

$$a - (p, \xi(q(t))) + (q(t), \xi(q(t))) + H(X(t, s, y), t, q(t)) \geqslant 2\gamma, \quad (4.17)$$

where X is given by (4.8) and $\xi(q)(t) = \xi(q(t))$ denotes the strategy associated with the mapping of B_M^* into B_{L_2} discussed above. Integration of (4.17) with respect to t over the range $s \leqslant t \leqslant b$ and use of the relation

$$\int_s^b (p, \xi(q(\tau)))\, d\tau = (p, X(b, s, y) - y) \quad (4.18)$$

yield

$$a(b-s) + (p, y - X(b, s, y)) + \int_s^b (H(X(t, s, y), t, q(t)) + (q(t), \xi(q(t))))\, dt \geqslant 2\gamma(b-s). \quad (4.19)$$

Use of (4.13) in conjunction with (4.19) and obvious considerations of continuity yield

$$-u(X(b, s, y), s) + u(y, b) + \int_s^b (H(X(t, s, y), t, q(t)) + (q(t), \xi(q(t))))\, dt \geqslant \gamma(b-s)$$

for $b-s$ sufficiently small. Hence, using (4.12),

$$\begin{aligned} u(y, b) > \inf_{\xi \in \Xi_{s,b}} \sup_{q \in Q_{s,b}} &\Big(u(X(b, s, y), s) \\ &- \int_s^b (H(X(t, s, y), t, \xi(q)(t)) + (q(t), \xi(q)(t)))\, dt\Big) \\ = (U(b, s)\, u(\cdot, s))(y) &= (U(b, s)\, U(s, 0)\varphi)(y) \\ = (U(b, 0)\varphi)(y) &= u(y, b), \end{aligned}$$

a contradiction. Thus u is a subsolution.

The proof that u is a supersolution is similar. Assume not, so that there is a $(y, b) \in V \times (0, T]$ and a $(p, a) \in V^* \times \mathbf{R}$ such that both

$$u(x, t) \geqslant u(y, b) + (p, x-y) + a(t-b) + o(|x-y| + |t-b|) \quad (4.20)$$

and there is a $\gamma > 0$ and a $\bar{q} \in B_M^*$ such that

$$a + H(y, b, \bar{q}) + (\bar{q}, \xi) - (p, \xi) \leqslant -3\gamma \qquad \text{for all} \quad \xi \in B_{L_2}. \quad (4.21)$$

Proceeding as above, we find that for every strategy ξ

$$a + H(X(t, s, y), t, \bar{q}) + (\bar{q}, \xi(\bar{q})) - (p, \xi(\bar{q})) \leqslant -2\gamma, \tag{4.22}$$

where $\bar{q}$ denotes the corresponding constant element of Q, provided only that $s \leqslant t \leqslant b$ and $b - s$ is sufficiently small. Integration of (4.22) over $s \leqslant t \leqslant b$ and use of (4.8), (4.20), and then (4.12) yield

$$\begin{aligned} u(y, b) &\leqslant u(X(t, s, y), s) - \int_s^b (H(X(t, s, y), t, \bar{q}) + (\bar{q}, \xi(\bar{q})(t)))\, dt - \gamma \\ &< \inf_{\xi \in \Xi_{s,b}} \sup_{q \in Q_{s,b}} (u(X(b, s, y), s) \\ &\quad - \int_s^b (H(X(t, s, y), t, q(t)) + (q(t), \xi(q(t)))\, dt) \\ &= (U(b, s)\, u(\cdot, s))(y) = (U(b, s)\, U(s, 0)\varphi)(y) = u(y, b), \end{aligned}$$

a contradiction. Thus u is a viscosity solution and has all the claimed properties in the case of (CP).

In the case of (SP), the analogous considerations succeed. One still has (4.4) (independent of t) and defines H_M as in (4.5). The value function

$$u(x) = \inf_{\xi \in \Xi} \sup_{q \in Q} \int_0^\infty e^{-\tau} f(X(\tau, x), \xi(q)(\tau), q(\tau))\, d\tau$$

now obeys the dynamic programming principle in the form

$$u(x) = \inf_{\xi \in \Xi} \sup_{q \in Q} \left(\int_0^t e^{-\tau} f(X(\tau, x), \xi(q(\tau)), q(\tau))\, d\tau + u(X(t, x))\, e^{-t} \right),$$

where $X(t, x) = X(t, 0, x)$, Q is the set of controls on $[0, \infty)$, and Ξ is the set of strategies on $[0, \infty)$ (where ξ is a strategy if whenever controls p and q agree on an interval $[0, t)$ then so do $\xi(q)$ and $\xi(p)$). Then one (for example) writes the statement that u is not a viscosity subsolution, multiplies this by e^{-t}, integrates over a small interval $0 \leqslant t \leqslant b$, and reaches a contradiction as above.

We remark here about the case in which $H(x, t, r, p)$ depends on r. If H is Lipschitz in (x, r, p) and nondecreasing in r then

$$\begin{aligned} &H(x, t, r, p) \\ &\quad = \inf\{H(x, t, s, q) + L(r - s)^+ + L\,|p - q| : s \in R, q \in V^*\} \\ &\quad = \inf_{(s,q) \in R \times V^*} \sup_{(\theta,z) \in [0,L] \times B_L} \{H(x, t, s, q) + \theta(r - s) - (z, p - q)\}. \end{aligned} \tag{4.23}$$

One can use this formula to obtain a solution for (CP) and (SP) via differential games as was done above in the r-independent case, but the complexity becomes unpleasant.

We conclude this section with a variety of comments. All of the above presentation holds in a general Banach space V—no geometrical assumptions were invoked and conditions (C) played no role. In this generality, however, the value of the results is not clear. No uniqueness results are available. If one wants to pass beyond the context of RN spaces, it seems likely that the notion of strict viscosity solutions [10, Appendix] or a variant is appropriate rather than the notion used here. Indeed, in the language of [10], $D^+u(x)$ and $D^-u(x)$ may be empty for all x; hence the simple notion becomes useless. Moreover, the uniqueness of the solutions constructed above even when V is RN and (C) holds does not follow at once, since (H3) may fail. For example, for $H(x, p) = |p|$, (H3) is equivalent to the existence of a constant C such that $|d_x + d_y| \leqslant C\,|x-y|$ for $x, y \in V$, and this is not implied by the assumptions. However, one can replace d by $\bar{d}(x, y) = d(x-y, 0)$, and $\bar{d}$ does satisfy this condition. Using the uniform approximability of uniformly continuous $H(x, p)$ by Lipschitz continuous functions, one can extend existence to the case of uniformly continuous Hamiltonians.

5. Variants, Examples, and Remarks

In this section we attempt to provide some feeling for the assumptions used in this paper and [10]. Let us begin by reviewing what various assumptions mean in the event that V is a real Hilbert space, $d(x, y) = |x-y|$, and

$$H(x, p) = (b(x), p) - f(x) \tag{5.1}$$

is an affine Hamiltonian. Here (,) denotes the inner product on V and we have identified V and V^*. Note that in this case

$$d_x(x, y) = -d_y(x, y) = |x-y|^{-1}(x-y).$$

With this choice of d, (H3) is equivalent to requiring that $f \in \mathrm{UC}(V)$ and that there be a $c \geqslant 0$ such that

$$(b(x)-b(y), x-y) \geqslant -c\,|x-y|^2 \qquad \text{for} \quad x, y \in V; \tag{5.2}$$

that is, $x \to b(x) + cx$ is monotone. The condition (H1) is automatically satisfied since H is independent of u, and (H0) is equivalent to the uniform continuity of b on bounded sets. Finally, (H2) is equivalent to the existence of a $c_1 \in \mathbf{R}$ such that

$$(b(x), Dv(x)) \geqslant -c_1 \qquad \text{for} \quad x \in V, \tag{5.3}$$

where v is a function satisfying (1.4).

It is easy to see that some condition like (5.3) is necessary for uniqueness even in the linear case. Indeed, set $b(x) = -\lambda x$ for some $\lambda > 0$ and $f = 0$. Clearly (5.2) holds with $c = -\lambda$, the other assumptions hold as well, and (SP) becomes

$$-\lambda(x, Du) + u = 0 \qquad \text{in } V. \tag{5.4}$$

If $\lambda > 1$, (5.4) has the distinct uniformly continuous viscosity solutions $u = 0$ and $u = |x|^{1/\lambda}$.

We turn to the condition (H4) and its role in the study of (SP). First, let us give two examples of functions F. If we take

$$F(a, b) = C(1 + b) + \Phi(a), \tag{5.5}$$

where $C \geqslant 0$ and $b \to \Phi(b)$ is continuous and nondecreasing, then we may choose

$$G(r) = C(1 + r) + \Phi(C).$$

Observe that this choice of F is appropriate for the situation in which V is a Hilbert space, $d(x, y) = |x - y|$, and $H(x, p) \in \mathrm{UC}(V \times B_R^*)$ for all $R > 0$; indeed, we then have

$$H(y, 0) - H(x, 0) \leqslant C(1 + |x - y|)$$

and

$$|H(x, p) - H(x, 0)| \leqslant \Phi(|p|) \qquad \text{for} \quad x, y \in V$$

for a suitable constant C and a nondecreasing function Φ, and so

$$\begin{aligned} &H(y, -\lambda d_y) - H(x, \lambda d_x) \\ &\quad \leqslant H(y, -\lambda d_y) - H(y, 0) + H(y, 0) - H(x, 0) + H(x, 0) - H(x, \lambda d_x) \\ &\quad \leqslant C(1 + |x - y|) + 2\Phi(K\lambda). \end{aligned}$$

A second F of interest for which (H4) is verifiable is

$$F(a, b) = c_0 ab + c_1 b^{\alpha} + c_2 \qquad \text{for} \quad a, b \geqslant 0, \tag{5.6}$$

where $\alpha \in (0, 1]$, and $c_0, c_2 \geqslant 0$. Then (H4) holds if $c_0 \alpha < 1$, for we can then use $G(r)=c_1(1-c_0\alpha)^{-1}r^\alpha+c_2$. In fact, the curious relation $c_0\alpha<1$, which was used without further explanation in Ishii [17] (a somewhat related condition also occurs in [8]) is necessary here, as the next example shows. In this example, (H0)–(H3) hold, and the estimate of (H4) holds with $F(\lambda, d)=\lambda d+d$, but $c_0\alpha=1$ in this case and there are no uniformly continuous solutions of (SP).

Let $V=\mathbf{R}$, $H(x, p)=b(x)p-|x|$, and choose b as follows: set $x_n=(n+1)(n+2)$ for integral $n\geqslant 0$ and

$$\begin{aligned} b(x)&=0 && \text{if } \; x\leqslant 0,\\ &=x-x_n && \text{if } \; x_n\leqslant x\leqslant x_n+(n+1),\\ &=x_{n+1}-x && \text{if } \; x_n+n+1\leqslant x\leqslant x_{n+1}. \end{aligned} \tag{5.7}$$

Clearly (H0)–(H3) hold and the first inequality of (H4) holds with $F(\lambda, d)=\lambda d+d$. Here $c_0\alpha=1$ and we can show that (SP) does not have a uniformly continuous solution. Indeed, if such a solution exists, it is easy to show that it is given by

$$u(x)=\int_0^\infty |X(t, x)|\, e^{-t}\, dt,$$

where $X(t, x)$ is the solution of the Cauchy problem $\dot{X}=b(X)$, $X(0)=x$. Obviously $u(x_n)=0$ while for $h\in(0, 1)$

$$u(x_n+h)\geqslant \int_0^s X(t, x_n+h)\, e^{-t}\, dt,$$

where $s=\log((n+1)/h)$. Since $X(t, x_n+h)=x_n+e^t h$ for $0\leqslant t\leqslant s$ we deduce that

$$u(x_n+h)-u(x_n)\geqslant hs=h\log(n+1)/h)$$

and this contradicts the uniform continuity of u.

Part of the complexity of the proofs we have presented here and in [10] arises from the need to first bound (in the case of (SP)) quantities like $u(x)-u(y)$ (in the case of moduli) or $u(x)-v(y)$ (for comparison purposes) for $d(x, y)\leqslant 1$ where u, v are solutions of (SP) before obtaining the final estimates. Thus when dealing with bounded solutions the arguments can be simplified and assumptions weakened. For instance, in order to guarantee the existence of a unique solution in BUC(V) for (SP) or in BUC$_s$($V\times[0, T]$) for (CP), (H2) and (H3) need only be assumed on bounded r-sets. More interestingly, we can weaken the requirements on v in (1.4) of (C) to

$$v(x) \to \infty \qquad \text{as} \quad |x| \to \infty. \tag{1.4)'$$

Let us denote this variant of (C) by (C)$'$. If (H2) and (H3) hold on bounded r sets and in (H2) the function v satisfies (C)$'$ rather than (C), we will say (H2)$'$ and (H3)$'$ are satisfied. These conditions guarantee uniqueness of such bounded solutions. The existence of bounded solutions will depend on further assumptions. A simple case arises if

$$M = \sup\{|H(x, t, 0, 0)| : x \in V, 0 \leqslant t \leqslant T\} < \infty. \tag{5.8}$$

Under these conditions we have:

THEOREM 5.1. *Let* (H0), (H1), (H2)$'$, (H3)$'$, *and* (5.8) *hold. Then*

(i) *For* $\varphi \in \mathrm{BUC}(V)$, (CP) *has a unique solution* $u \in \mathrm{BUC}_s(V \times [0, T])$.

(ii) *The problem* (SP) *has a unique solution* $u \in \mathrm{BUC}(V)$.

There are a variety of ways to prove this result—the uniqueness follows from [10, Theorem 3] and we just remark here that our entire existence program can be carried out as before, using at all stages an a priori bound from above by $M' = MT + \sup\{|\varphi(x)| : x \in V\}$ (which is always a supersolution) and from below by $-M'$ (which is a subsolution) for (CP) and by $\pm M$ for (SP).

As is seen from this sketch of proof, it is not (5.8) which is crucial to the existence of bounded solutions, but rather the ability to find suitable bounded sub- and supersolutions.

To illustrate what is gained by replacing (C) by (C)$'$ we consider again the case of the linear Hamiltonian given by (5.1) in a real Hilbert space V. Let $b(x)$ be uniformly continuous on bounded sets and satisfy (5.2). As the example of nonuniqueness shows, we cannot deduce from these assumptions that (5.3) holds for some v satisfying (1.4). However, $v(x) = (1/2)\log(1 + |x|^2)$ satisfies (C)$'$ and (5.2) with $y = 0$ implies.

$$(b(x), Dv(x)) = (b(x), x/(1 + |x|^2)) \geqslant -c + (b(0), x)/(1 + |x|^2)$$

and the right-hand side of this expression is clearly bounded. We conclude that BUC solutions of $u + (b(x), Du) - f(x) = 0$ are unique whenever b satisfies the above conditions. This assertion does not contradict the nonuniqueness of solutions of (5.4) exhibited above, since the second solution was unbounded.

We want to examine this situation further. We next present a class of examples that indicate that even in the case of BUC solutions some assumption like (H2)′ is needed—indeed, the example shows that this condition is rather sharp. Consider the equation

$$-g(x)\,|u'|^{\alpha}+u=0, \tag{5.9}$$

where $0<\alpha<1$ and

$$g\in \mathrm{UC}(\mathbf{R}) \text{ is odd, nondecreasing, } g(s)>0 \text{ for } s>0 \tag{5.10}$$

and

$$\int_0^1 \frac{1}{g(s)^{1/\alpha}}\,ds=\infty. \tag{5.11}$$

The uniqueness of $\mathrm{BUC}(\mathbf{R})$ solutions of (5.9) is determined by whether or not

$$I=\int_1^{\infty} \frac{1}{g(s)^{1/\alpha}}\,ds \tag{5.12}$$

is finite. Indeed, if I is finite one easily checks that the odd function given by

$$u(x)=\left(\frac{1-\alpha}{\alpha}\int_x^{\infty} \frac{1}{g(s)^{1/\alpha}}\,ds+1\right)^{-\alpha/(1-\alpha)} \qquad \text{for} \quad x>0 \tag{5.13}$$

is a BUC solution together with $u=0$, so BUC solutions are nonunique. For example, if $g(x)=x$ and $\alpha=\frac{1}{2}$, then $u(x)=x/(1+|x|)$ and u is even Lipschitz continuous. It follows that if I is finite then there is no v satisfying (C)′ such that (H2) holds for $H(x,p)=-g(x)\,|p|^{\alpha}$. However, if I is infinite, then

$$v(x)=\int_1^{|x|} \frac{1}{g(s)^{1/\alpha}}\,ds \qquad \text{on} \quad |x|>1$$

satisfies (C)′ and (H2) holds. In this class of examples, (H2)′ is necessary and sufficient for uniqueness in BUC.

We continue dissecting the role of (H2). As mentioned above, it played a dual role in first obtaining bounds on $d\leqslant 1$ and then again in getting refined estimates in $d\leqslant 1$. We can split the assumption into pieces designed to handle these tasks separately. To obtain the preliminary bounds it is enough to know that there is a function v satisfying (C) (in particular, (1.4)) such that

$(\mathrm{H2})_{\mathrm{a}}$ For each $R>0$ there is a constant C_R such that

$$H(x, t, r, p) - H(x, t, r, p+\lambda Dv) \leqslant C_R$$

for $(x, t, r) \in V \times [0, T] \times \mathbf{R}$, $|p| \leqslant R$ and $0 \leqslant \lambda \leqslant R$ holds.

In order to continue to the second stage, one only needs that there is a μ which satisfies the requirements of (C)′ (in particular, (1.4)′ in place of (1.4)) in place of v such that

$(\mathrm{H2})_{\mathrm{b}}$ There is a local modulus σ_H such that

$$H(x, t, r, p) - H(x, t, r, p+\lambda D\mu(x)) \leqslant \sigma_H(\lambda, |p|)$$

whenever $0 \leqslant \lambda \leqslant 1$, $(x, r, t, p) \in \Omega \times [0, T] \times \mathbf{R} \times V^*$.

Then the existence and uniqueness results remain valid with (H2) replaced by $(\mathrm{H2})_{\mathrm{a}}$ and $(\mathrm{H2})_{\mathrm{b}}$. Examples show that this is an interesting generality. Indeed, if H is bounded on $V \times [0, T] \times \mathbf{R} \times B_R^*$ for each R, then $(\mathrm{H2})_{\mathrm{a}}$ is automatically satisfied. Hence $H(x, p) = \max((\min(-(x, p), |p|), -|p|)$ satisfies $(\mathrm{H2})_{\mathrm{a}}$. It also satisfies $(\mathrm{H2})_{\mathrm{b}}$ with $\mu(x) = \log(1+|x|^2)$ when V is Hilbert. Another example, with the same μ, is given by $H(x, p) = \cos((x, p))$. Neither satisfies (H2).

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