The hyperspace of the regions below of continuous maps is homeomorphic to $c_0$

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Abstract

For a compact metric space $(X, d)$, we use $\downarrow USC(X)$ and $\downarrow C(X)$ to denote the families of the regions below of all upper semi-continuous maps and the regions below of all continuous maps from $X$ to $I = [0, 1]$, respectively. In this paper, we consider the two spaces topologized as subspaces of the hyperspace $Cld(X \times I)$ consisting of all non-empty closed sets in $X \times I$ endowed with the Vietoris topology. We shall show that $\downarrow C(X)$ is Baire if and only if the set of isolated points is dense in $X$, but $\downarrow C(X)$ is not a $G_{\delta\sigma}$-set in $\downarrow USC(X)$ unless $X$ is finite. As the main result, we shall prove that if $X$ is an infinite locally connected compact metric space then $(\downarrow USC(X), \downarrow C(X)) \approx (Q, c_0)$, where $Q = [-1, 1]^{\omega}$ is the Hilbert cube and $c_0 = \{(x_n) \in Q: \lim_{n \to \infty} x_n = 0\}$.

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1. Introduction

For a metric space $X$, the hyperspace $Cld(X)$ is the set consisting of all non-empty closed subsets in $X$ endowed with the Vietoris topology which is generated by $\{U^-, U^+: U \subset X$ is open\} as a subbase, where

$$U^- = \{A \in Cl(X) | A \cap U \neq \emptyset\} \quad \text{and} \quad U^+ = \{A \in Cl(X) | A \subset U\}.$$ 

It is well known that $Cld(X)$ with this topology is metrizable if and only if $X$ is a compactum (i.e. a compact metric space) [11, Theorem I.3.4]. For a compactum $X = (X, d)$, the Vietoris topology of $Cld(X)$ is induced by the Hausdorff metric $d_H$ defined as follows:

$$d_H(A, B) = \max_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b).$$

The Curtis–Schori–West Hyperspace Theorem is a celebrated theorem in infinite-dimensional topology which states that $Cld(X)$ is homeomorphic to $(\approx)$ the Hilbert cube $Q = [-1, 1]^{\omega}$ if and only if $X$ is a non-degenerate Peano

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As a subspace of \( Cld_X \) unless \( X \) is an infinite non-locally-connected compactum. And also, in [21], it was proved that, although \( X \) is an infinite non-locally-connected compactum, the hyperspace \( \text{Cld}_f(Y) \) endowed with the so-called Fell topology and its some natural subspaces, for example, the subspace consisting of all compact subsets. Particularly, they proved a result similar to the Curtis–Schori–West Hyperspace Theorem.

For a compactum \((X,d)\), we consider the sets \( C(X) \) and \( \text{USC}(X) \) which consist of all continuous maps and all upper semi-continuous maps from \( X \) to \( I = [0,1] \), respectively. In [20], a map \( f : X \to I \) is carried by a bijection to its region below \( \downarrow f = \{(x,\lambda) \in X \times I: \lambda \leq f(x)\} \) in the space \( X \times I \). For a family \( A \subseteq \text{USC}(X) \), let \( \downarrow A = \{ \downarrow f: f \in A \} \). As a subspace of \( \text{Cld}(X \times I) \), \( \downarrow A \) is a metric space. Let \( \text{USC}(X,\{0,1\}) \) be the set of all upper semi-continuous from \( X \) to \( \{0,1\} \). When we identify a subset of \( X \) with its characteristic function, we may think \( \text{USC}(X,\{0,1\}) = \text{Cld}(X) \cup \{ \emptyset \} \).

Thus the Curtis–Schori–West Hyperspace Theorem can be rewritten as \( \downarrow \text{USC}(X,\{0,1\}) \setminus \{ \emptyset \} \approx Q \) if and only if \( X \) is a non-degenerate Peano continuum, where, such as \( \downarrow \text{USC}(X), \downarrow \text{USC}(X,\{0,1\}) \) is topologized as a subspace of \( \text{Cld}(X \times I) \). In [20], the author proved that \( \downarrow \text{USC}(X) \approx Q \) and \( \downarrow C(X) \) is homotopy dense in \( \downarrow \text{USC}(X) \) when \( X \) is an infinite locally connected compactum (see Section 4 in the present paper).

Let \( X \) and \( Y \) be two spaces. For two subspaces \( A \) and \( B \) of \( X \) and \( Y \), respectively, if \( X \) has a homeomorphism \( h : X \to Y \) such that \( h(A) = B \), then \( (X,A) \) and \( (Y,B) \) are called pair-homeomorphic and denoted by \( (X,A) \approx (Y,B) \). In the present paper, we shall prove the following theorems:

**Theorem 1.** If \( X \) is an infinite locally connected compact metric space then \( \downarrow \text{USC}(X), \downarrow C(X) \approx (Q,c_0) \), where \( Q = [-1,1]^\omega \) is the Hilbert cube and \( c_0 = \{ (x_n) \in Q: \lim_{n \to \infty} x_n = 0 \} \).

Noting that \( c_0 \) is not a Baire space, the following Theorem 2 shows that Theorem 1 need not be true for any non-locally-connected compactum. And also, in [21], it was proved that, although \( \downarrow \text{USC}(X) \approx Q \), \( \downarrow C(X) \) is homeomorphic to neither \( c_0 \) nor \( R^\omega \), where \( R \) is the set of all real numbers with the usual topology, even though \( X \) is a convergent sequence with its limit, which implies that the structure of \( \downarrow C(X) \) would be a little complicated when \( X \) is an infinite non-locally-connected compactum.

**Theorem 2.** For a compactum \( X \), \( \downarrow C(X) \) is Baire if and only if the set of isolated points is dense in \( X \).

Let \( C_p(X) \) and \( C_u(X) \) be the spaces of \( C(X) \) topologized by the pointwise convergence topology and the uniformly convergence topology, respectively. In [7] (cf. [15, Theorem 6.3.8]), it was proved that \( C_p(X) \) is not a \( G_{\delta_0} \)-set of \( I^X \) unless \( X \) is discrete. In [20], we proved that if \( X \) is an infinite compactum then \( \downarrow C(X) \) is not a \( G_{\delta} \)-set of \( \downarrow \text{USC}(X) \). Here, we have

**Theorem 3.** No \( \downarrow C(X) \) is a \( G_{\delta_0} \)-set of \( \downarrow \text{USC}(X) \) for any infinite compactum \( X \).

Trivially, \( f \mapsto \downarrow f \) gives a bijection from the set \( C(X) \) onto \( \downarrow C(X) \). Notice that \( d(f,g) = \sup\{|f(x) - g(x)|: x \in X\} \) is an admissible metric on \( C_u(X) \) if \( X \) is a compactum. It is trivial that \( d_H(\downarrow f, \downarrow g) \leq d(f,g) \) for all \( f,g \in C(X) \). Thus \( f \mapsto \downarrow f \) is also continuous from the space \( C_u(X) \) onto \( \downarrow C(X) \) for every compactum \( X \). And it is also a homeomorphism from \( C_p(X) = C_u(X) \) onto \( \downarrow C(X) \) if \( X \) is finite, but we shall prove the following corollary of Theorems 2 and 3:

**Corollary 1.** For any infinite compactum \( X \), no pair among \( C_p(X), \downarrow C(X) \) and \( C_u(X) \) is homeomorphic.

In Section 2 we shall recall some necessary fundamental concepts and facts on hyperspaces and the uniqueness on absorbers which are the main instrument to prove the pair-homeomorphism. In Section 3, we shall prove Theorem 2, 3 and Corollary 1. And the last section contains proof of Theorem 1.

**2. Preliminaries**

All spaces under discussion here are assumed to be separable and metrizable.

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1 A Peano continuum is a connected and locally connected compact metrizable space.
In this section we recall some necessary fundamental concepts and facts. For more information on them, please refer to [14, 15].

A closed subset $A$ of a space $X$ is said to be a Z-set if for every continuous map $\varepsilon : X \to (0, \infty)$ there is a continuous map $f : X \to X \setminus A$ with $d(f(x), x) < \varepsilon(x)$ for every $x \in X$. It is trivial that every Z-set is closed nowhere dense but the converse is not necessarily true. If $X$ is compact then obviously $\varepsilon$ map can be replaced by an arbitrary positive real number $\varepsilon$. A $Z_\sigma$-set in a space is a countable union of Z-sets. A space is called a $Z_\sigma$-space if it is a $Z_\sigma$-set in itself. A Z-embedding is an embedding with a Z-set image. We say that a subspace $Y$ of $X$ is homotopy dense in $X$ provided that there exists a homotopy $h : X \times I \to X$ such that $h_0 = \text{id}_X$ and $h_t(X) \subset Y$ for every $t > 0$.

Let $\mathcal{M}_0$ denote the class of compacta, and for a topological class $C$ let $(\mathcal{M}_0, C)$ denote the class of the pairs $(Z, C)$ such that $Z \in \mathcal{M}_0$ and $C \in C$ with $C \subset Z$.

**Definition 1.** Let $X$ be a copy of Hilbert cube $Q$. We say that a subspace $Y$ of $X$ is strongly $C$-universal in $X$ provided that for every $(M, C) \in (\mathcal{M}_0, C)$, every continuous map $f : M \to X$, every closed subset $K$ of $M$ such that $f|K : K \to X$ is a Z-embedding and every $\varepsilon > 0$ there is a Z-embedding $g : M \to X$ such that $g|K = f|K$, $g^{-1}(Y) \setminus K = C \setminus K$ and $d(g(m), f(m)) \leq \varepsilon$ for $m \in M$.

**Definition 2.** We say that a subset $Y$ is a $C$-absorber in $X$ if

(a) $Y \in C$,
(b) $Y$ is contained in a $Z_\sigma$-set of $X$, and
(c) $Y$ is strongly $C$-universal in $X$.

**Lemma 1.** [2, Theorem 8.2] (cf. [3]) If $X$ and $Y$ are $C$-absorbers in a copy $M$ of $Q$, then $(M, X) \approx (M, Y)$.

In this paper we are concerned with the class $F_{\sigma\delta}$ of absolute $F_{\sigma\delta}$ spaces. It was proved that $c_0 = \{(x_n) \in Q: \lim_{n \to \infty} x_n = 0\}$ is a $F_{\sigma\delta}$-absorber in $Q$ in [8]. The proof of the fact that $c_0 = \{(x_n) \in Q: \lim_{n \to \infty} x_n = 0\}$ is strongly $F_{\sigma\delta}$-universal in $Q$ in [9] (cf. [15, Theorem 6.12.15]) can easily be modified to show the following lemma.

**Lemma 2.** Let $Q_u = [0, 1]^\omega$ and $c_1 = \{(x_n) \in Q_u: \lim_{n \to \infty} x_n = 1\}$. Then $c_1$ is strongly $F_{\sigma\delta}$-universal in $Q_u \approx Q$.

In the following, $\mathbb{N}$ denotes the set of all natural numbers and, as stated before, $\mathbb{R}$ denotes the set of all real numbers and $I = [0, 1]$. We assume that $(X, d)$ is a compactum. Then, $d((x, \lambda), (y, \mu)) = \max\{d(x, y), |\lambda - \mu|\}$ is an admissible metric on $X \times I$ and so $d_H$ is on the hyperspace $\text{Cld}(X \times I)$. Let $\overline{A}$ and $\text{Int}(A)$ denote the closure and the interior of a set $A$ in a space, respectively and let $B(a, \varepsilon) = \{x \in X: d(a, x) < \varepsilon\}$. Let $\phi : A \to B$ be a map from a set $A$ to a set $B$. If $A \subseteq \text{USC}(X)$ and/or $B \subseteq \text{USC}(Y)$ for spaces $X$ and $Y$. We may define a corresponding map $\downarrow \phi: \downarrow A \rightarrow \downarrow B$ or $\downarrow \phi: A \rightarrow \downarrow B$ or $\downarrow \phi: \downarrow A \rightarrow B$ as $\downarrow \phi(\downarrow f) = \downarrow (\phi(f))$ or $\downarrow \phi(f) = \downarrow (\phi(f))$ or $\downarrow \phi(\downarrow f) = \phi(f)$, respectively.

3. Proofs of Theorems 2 and 3

If $X$ is finite, then Theorem 2 is trivial and Theorem 3 does not apply. Hence, in this section, we assume that $X$ is an infinite compactum and $X_0$ denotes the set of all isolated points of $X$.

**Lemma 3.** If $X_0 = X$ then $\downarrow C(X)$ is a Baire space.

**Proof.** Let $\mathcal{A} = \{A \subseteq X_0: A$ is finite$\}$. For $A \in \mathcal{A}$ and $n \in \mathbb{N}$, let

$$U_{A,n} = \left\{ \downarrow f \in \downarrow \text{USC}(X): f(x) < \frac{1}{n} \text{ for all } x \in X \setminus A \right\}.$$
Then \(U_{A,n} = ((A \times I) \cup ((X \setminus A) \times [0, \frac{1}{n}]))^+ \cap \downarrow \text{USC}(X)\) is open in \(\downarrow \text{USC}(X)\). Let \(U_n = \bigcup \{U_{A,n} : A \in \mathcal{A}\}\). Then \(U_n\) is a dense open set in \(\downarrow \text{USC}(X)\). We only need to verify that \(U_n\) is dense in \(\downarrow \text{USC}(X)\). For \(f \in C(X)\) and \(\varepsilon > 0\), since \(\overline{X_0} = X\) there exists \(A \in \mathcal{A}\) such that \(d_H(\{(a, f(a)) : a \in A\}, G(f)) < \varepsilon\), where \(G(f) = \{(x, f(x)) : x \in X\}\). Let
\[
g(x) = \begin{cases} f(x) & x \in A, \\
0 & x \in X \setminus A.
\end{cases}
\]
Then \(\downarrow g \in U_n\) and \(d_H(\downarrow g, \downarrow f) < \varepsilon\). It follows from the density of \(\downarrow \mathcal{C}(X)\) in \(\downarrow \text{USC}(X)\) that \(U_n\) is dense in \(\downarrow \text{USC}(X)\).

For later use, we shall show a stronger result than the converse of Lemma 3. We first show two technical lemmas.

**Lemma 4.** For every continuous map \(\varepsilon : \downarrow \mathcal{C}(X) \rightarrow (0, 1)\) and for every \(a \in X \setminus X_0\) there exists a map \(\varphi : \mathcal{C}(X) \rightarrow \mathcal{C}(X)\) such that the map \(\downarrow \varphi : \downarrow \mathcal{C}(X) \rightarrow \downarrow \mathcal{C}(X)\) is continuous and also for every \(f \in \mathcal{C}(X)\),

(a) \(d_H(\downarrow f, \downarrow \varphi(\downarrow f)) < \varepsilon(\downarrow f)\);

(b) \(\varphi(f)(a) = 0\).

**Proof.** For \(f \in \mathcal{C}(X)\), define
\[
\delta(\downarrow f) = \sup \{\eta > 0 : d_H(\downarrow f, \downarrow f \cap ((X \setminus B(a, \eta)) \times I)) < \varepsilon(\downarrow f)\}.
\]
Then \(\delta(\downarrow f) > 0\) since \(a \in X \setminus X_0\) and \(f\) is continuous.

We shall prove that \(\delta : \downarrow \mathcal{C}(X) \rightarrow \mathbb{R}\) is lower semi-continuous and then there exists a continuous function \(\zeta : \downarrow \mathcal{C}(X) \rightarrow \mathbb{R}\) such that \(0 < \zeta(\downarrow f) < \delta(\downarrow f)\) for each \(f \in \mathcal{C}(X)\) [10,13] (cf. [15, Corollary A.7.6]). Define \(\varphi : \mathcal{C}(X) \rightarrow \mathcal{C}(X)\) by
\[
\varphi(f)(x) = \min \left\{1, \frac{d(x, a)}{\zeta(\downarrow f)}\right\} f(x),
\]
for \(f \in \mathcal{C}(X)\) and \(x \in X\). It is easy to verify that \(\varphi\) is as required.

It remains to show that \(\delta\) is lower semi-continuous. For every fixed \(f \in \mathcal{C}(X)\) and \(\eta \in (0, \delta(\downarrow f))\), since \(\delta(\downarrow f) - \frac{\eta}{2} < \delta(\downarrow f)\),
\[
d_H(\downarrow f, \downarrow f \cap \left((X \setminus B(a, \delta(\downarrow f) - \frac{\eta}{2})) \times I\right)) < \varepsilon(\downarrow f),
\]
and hence there exists \(n \in \mathbb{N}\) such that
\[
d_H(\downarrow f, \downarrow f \cap \left((X \setminus B(a, \delta(\downarrow f) - \frac{\eta}{2})) \times I\right)) < \frac{n - 1}{n} \varepsilon(\downarrow f).
\]
Choose \(\xi \in (0, \min\left\{\frac{1}{2n} \varepsilon(\downarrow f), \frac{\eta}{2}\right\})\) such that \(|\varepsilon(\downarrow g) - \varepsilon(\downarrow f)| < \frac{1}{2n} \varepsilon(\downarrow f)\) for every \(\downarrow g \in B(\downarrow f, \xi) \cap \downarrow \mathcal{C}(X)\). We shall prove that \(\delta(\downarrow g) \geq \delta(\downarrow f) - \eta\) holds for every \(\downarrow g \in B(\downarrow f, \xi) \cap \downarrow \mathcal{C}(X)\). In fact, for every \((x_1, \lambda_1) \in (B(a, \delta(\downarrow f) - \eta)) \times I\) \(\cap \downarrow g\), there exists \((x_2, \lambda_2) \in \downarrow f\) such that \(d_H((x_1, \lambda_1), (x_2, \lambda_2)) < \xi\) since \(d_H(\downarrow f, \downarrow g) < \xi\). Then \((x_2, \lambda_2) \in (B(a, \delta(\downarrow f) - \frac{\eta}{2})) \times I\) \(\cap \downarrow f\) and hence there exists \((x_3, \lambda_3) \in ((X \setminus B(a, \delta(\downarrow f) - \frac{\eta}{2})) \times I) \cap \downarrow f\) such that \(d((x_2, \lambda_2), (x_3, \lambda_3)) < \frac{n - 1}{n} \varepsilon(\downarrow f)\). Using \(d_H(\downarrow f, \downarrow g) < \xi\) again there exists \((x_4, \lambda_4) \in \downarrow g\) such that \(d((x_3, \lambda_3), (x_4, \lambda_4)) < \xi\) which implies that \((x_4, \lambda_4) \in ((X \setminus B(a, \delta(\downarrow f) - \eta)) \times I) \cap \downarrow g\). It is not hard to verify that
\[
d((x_1, \lambda_1), (x_4, \lambda_4)) \leq \varepsilon(\downarrow g).
\]
By the definition of \(\delta(\downarrow g)\) we have \(\delta(\downarrow g) \geq \delta(\downarrow f) - \eta\). This shows that \(\downarrow \delta\) is lower semi-continuous. \(\Box\)
Lemma 5. Let $F = E \cup Z \subset \downarrow C(X)$ be closed. If $Z$ is a $Z$-set in $\downarrow C(X)$ and for every $\downarrow f \in E$ there exists $a \in X$ such that $f(a) = 0$, then $F$ is a $Z$-set in $\downarrow C(X)$.

Proof. For every continuous map $\varepsilon : \downarrow C(X) \to (0, 1)$, choose a map $\psi : C(X) \to C(X)$ such that $\downarrow \psi : \downarrow C(X) \to \downarrow C(X)$ is continuous, $\downarrow \psi(\downarrow C(X)) \cap Z = \emptyset$ and $d_H(\downarrow \psi(\downarrow f), \downarrow f) < \frac{1}{2} \varepsilon(\downarrow f)$ for all $f \in C(X)$. Define $\varphi : C(X) \to C(X)$ by

$$\varphi(f)(x) = \max \left\{ \psi(f)(x), \min \left\{ \frac{1}{2} \varepsilon(\downarrow f), \frac{1}{2} d(\downarrow \psi(\downarrow f), Z) \right\} \right\}$$

for $f \in C(X)$ and $x \in X$. Then $\downarrow \varphi : \downarrow C(X) \to \downarrow C(X)$ is continuous, $d(\downarrow \varphi, \text{id}) < \varepsilon(\downarrow f)$ and $\downarrow \varphi(\downarrow C(X)) \cap F = \emptyset$. We are done. □

Using the above two lemmas, we shall prove the following lemma:

Lemma 6. If $\overline{X}_0 \neq X$ then $\downarrow C(X)$ is a $Z_\sigma$-space and hence it is not a Baire space.

Proof. Choose a countable set $D = \{d_1, d_2, \ldots \}$ such that $\overline{D} = X \setminus X_0$. For $n, m \in \mathbb{N}$, let

$$F_{n,m} = \left\{ \downarrow f \in \downarrow C(X) : f(d_n) \geq \frac{1}{m} \right\}.$$ 

Then $F_{n,m}$ is a $Z$-set in $\downarrow C(X)$. In fact, it is trivial that $F_{n,m}$ is closed. Moreover, for every $n \in \mathbb{N}$ and every continuous function $\varepsilon : \downarrow C(X) \to (0, 1)$, by Lemma 4, there exists a continuous function $\downarrow \varphi_n : \downarrow C(X) \to \downarrow C(X)$ such that

(a) $d_H(\downarrow f, \downarrow \varphi_n(\downarrow f)) < \varepsilon(\downarrow f)$;
(b) $\varphi_n(f)(d_n) = 0$

for $f \in C(X)$. Hence $\downarrow \varphi_n(\downarrow C(X)) \cap F_{n,m} = \emptyset$ for all $n, m \in \mathbb{N}$.

Now let $F = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (\downarrow C(X) \setminus F_{n,m})$. Then $F$ is also a $Z$-set in $\downarrow C(X)$. To this end, by Lemma 5, it suffices to verify that $f(a) = 0$ for all $\downarrow f \in F$ and all $a \in X \setminus \overline{X}_0$ since $X \setminus \overline{X}_0 \neq \emptyset$. Note that $g(a) = 0$ for $a \in X \setminus \overline{X}_0$ and $\downarrow g \in \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (\downarrow C(X) \setminus F_{n,m})$. Let $\downarrow f \in F$ and $a \in X \setminus \overline{X}_0$. Then there exists $\delta > 0$ such that $B(a, \delta) \subset X \setminus \overline{X}_0$. For every $\varepsilon \in (0, \delta)$, choose $\downarrow g \in \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (\downarrow C(X) \setminus F_{n,m})$ such that $d_H(\downarrow f, \downarrow g) < \varepsilon$. Then there exists $(x, \lambda) \in \downarrow g$ such that $d((x, \lambda), (a, f(a))) < \varepsilon$. It follows from $\lambda \leq g(x) = 0$ that $f(a) < \varepsilon$. Hence $f(a) = 0$.

Finally, note that $\downarrow C(X) = F \cup \bigcup_{n,m=1}^{\infty} F_{n,m}$. We finish the proof. □

Proof of Theorem 2. It follows from Lemmas 3 and 6. □

Remark. Notice that we in fact proved the stronger statement that $\downarrow C(X)$ is a $Z_\sigma$-space if and only if it is not Baire if and only if $\overline{X}_0 \neq X$.

Now we turn to prove Theorem 3. We first give two lemmas.

Lemma 7. Let $F_n \subset \downarrow \text{USC}(X)$ be closed for each $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} F_n \supset \downarrow \text{USC}(X) \setminus \downarrow C(X)$. Then there exists $n_0 \in \mathbb{N}$ such that $\{ f \in C(X) : \downarrow f \in F_{n_0} \}$ has a non-empty interior in $C_u(X)$.

Proof. For convenience, let $F_0 = \emptyset$. For $n \in \mathbb{N}$, put $E_n = \{ f \in C(X) : \downarrow f \in F_n \}$. Assume that $\text{Int} E_n = \emptyset$ in $C_u(X)$ for every $n \in \mathbb{N}$. We shall derive a contradiction.

Pick an arbitrary element $x_\infty \in X \setminus X_0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in $X \setminus \{ x_\infty \}$ converging to $x_\infty$ and choose a sequence of positive numbers $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $B(x_n, \varepsilon_n) \cap (B(x_m, \varepsilon_m) \cup \{ x_\infty \}) = \emptyset$ for all $n \neq m$.

We shall by induction on $n$ define sequences of continuous functions $(f_n : n \in \mathbb{N})$ and $(g_n : n \in \mathbb{N} \cup \{ 0 \})$, a sequence of positive real numbers $(\delta_n : n \in \mathbb{N})$ and a sequence of integers $1 \leq i(1) < i(2) < \cdots$ such that the following conditions are satisfied:

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(a) \( g_0 \equiv 1 \);
(b) \( \delta_0 = \frac{2}{3} \) and \( \delta_n \leq \frac{\delta_n - 1}{2} \);
(c) \( f_n(x_i(n)) = 0 \);
(d) \( f_n|X \times B(x_i(n), \epsilon_i(n)) = g_{n-1}|X \times B(x_i(n), \epsilon_i(n)) \);
(e) \( g_n(x) \geq 1 - \frac{1}{3} \sum_{m=1}^{n} \frac{1}{2^m} \) for \( x \in X \times \bigcup_{m=1}^{n} B(x_i(m), \epsilon_i(m)) \);
(f) \( g_n(x_i(m)) \leq \frac{1}{3} \sum_{i=1}^{n} \frac{1}{2^i} \) for \( m \leq n \);
(g) \( d(g_n, f_n) < \frac{\delta_n - 1}{4} \);
(h) \( d_H(\downarrow f_n, \downarrow g_{n-1}) < \frac{\delta_n - 1}{4} \);
(i) \( \downarrow g_n \notin F_n \);
(j) \( B(\downarrow g_{n-1}, \delta_n) \cap F_{n-1} = \emptyset \),

for \( n \geq 1 \).

Let \( g_0 \equiv 1 \) and \( \delta_0 = \frac{2}{3} \). Then, for every \( n \), there exists \( h_n \in C(X) \) such that \( h_n \leq g_0 \) and

\[
    h_n(x) = \begin{cases} 
0 & x = x_n, \\
 g_0(x) & x \in X \setminus B(x_n, \epsilon_n).
\end{cases}
\]

Then \( \downarrow h_n \to \downarrow g_0 \). It follows that there exists \( i(1) \in \mathbb{N} \) such that \( d_H(\downarrow h_{i(1)}, \downarrow g_0) < \frac{\delta_0}{4} \). Let \( f_1 = h_{i(1)} \). Then \( f_1 \notin \text{Int}E_1 \) and hence there exists \( g_1 \in C(X) \) such that \( d(f_1, g_1) < \frac{\delta_0}{4} \) and \( g_1 \notin E_1 \). It is clear that \( f_1, g_1, i(1) \) and \( \delta_0 \) satisfy the above conditions.

Assume that \( f_n, g_n, i(n) \) and \( \delta_n-1 \) have been determined. Since \( F_n \) is closed in \( \downarrow \text{USC}(X) \) and \( \downarrow g_n \notin F_n \) there exists \( \delta_n \in (0, \frac{\delta_n - 1}{4}) \) such that \( B(\downarrow g_n, \delta_n) \cap F_n = \emptyset \). As above, for every \( m > i(n) \), there exists \( k'_m \in C(X) \) such that

\[
    k'_m(x) = \begin{cases} 
0 & x = x_m, \\
 g_n(x) & x \in X \setminus B(x_m, \epsilon_m).
\end{cases}
\]

Let \( k_m(x) = \min\{k'_m(x), g_n(x)\} \) for each \( x \in X \). Then \( k_m \in C(X) \) and \( \lim_{m \to \infty} \downarrow k_m = \downarrow g_n \). It follows that there exist \( i(n+1) > i(n) \) and \( f_{n+1} = k_{i(n+1)} \in C(X) \) such that \( d(\downarrow g_n, \downarrow f_{n+1}) < \frac{\delta_n}{4} \).

Then

\[
    f_{n+1}(x) = \begin{cases} 
0 & x = x_{i(n+1)}, \\
 g_n(x) & x \in X \setminus B(x_{i(n+1)}, \epsilon_{i(n+1)}).
\end{cases}
\]

By the assumption \( \text{Int}E_{n+1} = \emptyset \) and \( f_{n+1} \notin \text{Int}E_{n+1} \), there exists \( g_{n+1} \in C(X) \) such that \( d(f_{n+1}, g_{n+1}) < \frac{\delta_n}{4} \) and \( g_{n+1} \notin E_{n+1} \). By the definitions of \( \delta_n, E_{n+1}, f_{n+1}, g_{n+1}, (b)(c)(d)(g)(h)(i)(j) \) hold for \( n+1 \). It remains to check (e) and (f). Since \( d(f_{n+1}, g_{n+1}) < \frac{\delta_n}{4} \), \( g_n(x) \geq 1 - \frac{1}{3} \sum_{m=1}^{n} \frac{1}{2^m} \) for every \( x \in X \setminus \bigcup_{m=1}^{n} B(x_i(m), \epsilon_i(m)) \), and

\[
    f_{n+1}|X \setminus B(x_{i(n+1)}, \epsilon_{i(n+1)}) = g_{n+1}|X \setminus B(x_{i(n+1)}, \epsilon_{i(n+1)}),
\]

we have

\[
    g_{n+1}(x) \geq 1 - \frac{1}{3} \sum_{m=1}^{n} \frac{1}{2^m} - \frac{\delta_n}{4} \geq 1 - \frac{1}{3} \sum_{m=1}^{n+1} \frac{1}{2^m}
\]

for every \( x \in X \setminus \bigcup_{m=1}^{n+1} B(x_i(m), \epsilon_i(m)) \). Hence (e) holds. From

\[
    g_{n+1}(x_i(m)) \leq \frac{1}{3} \sum_{i=1}^{n} \frac{1}{2^i}, \quad \text{for } m \leq n,
\]

\[
    f_{n+1}|X \setminus B(x_{i(n+1)}, \epsilon_{i(n+1)}) = g_{n+1}|X \setminus B(x_{i(n+1)}, \epsilon_{i(n+1)}),
\]

we obtain that

\[
    g_{n+1}(x_i(m)) \leq f_{n+1}(x_i(m)) + \frac{\delta_n}{4} = g_n(x_i(m)) + \frac{\delta_n}{4} \leq \frac{1}{3} \sum_{i=1}^{n} \frac{1}{2^i} + \frac{\delta_n}{4} \leq \frac{1}{3} \sum_{i=1}^{n+1} \frac{1}{2^i}.
\]
for every \( m \leq n \), and
\[
\begin{align*}
\varepsilon_{n+1}(x_{i(n+1)}) & \leq \frac{\delta_n}{4} \leq 1 \sum_{i=1}^{n+1} \frac{1}{2^i}.
\end{align*}
\]

Thus (f) holds.

By (g), we have \( d_H(\downarrow g_n, \downarrow f_n) < \frac{\delta_n}{4} \) for every \( n \in \mathbb{N} \). It follows from (h) and (b) that \( d_H(\downarrow g_n, \downarrow g_{n-1}) \leq d_H(\downarrow g_n, \downarrow f_n) + d_H(\downarrow f_n, \downarrow g_{n-1}) \leq \frac{\delta_n}{4} + \frac{\delta_{n-1}}{4} \leq \frac{\delta_n}{2} \) for every \( n \in \mathbb{N} \). Thus, \( (\downarrow g_n)_{n \in \mathbb{N}} \)

is a Cauchy sequence in \( \downarrow \text{USC}(X) \) and hence \( \lim_{n \to \infty} \downarrow g_n = G \) exists in \( \downarrow \text{USC}(X) \). We shall show that \( G \in \downarrow \text{USC}(X) \setminus \downarrow C(X) \), but \( G \notin \bigcup_{n=1}^{\infty} F_n \), which contradicts \( \bigcup_{n=1}^{\infty} F_n \supset \downarrow \text{USC}(X) \setminus \downarrow C(X) \).

Let \( G = \downarrow g \), where \( g \in \text{USC}(X) \). To show \( G \in \downarrow \text{USC}(X) \setminus \downarrow C(X) \), it is enough to verify that \( g(x_{i(n)}) \leq \frac{1}{2} \)

for all \( n \) but \( g(x_{\infty}) \geq \frac{2}{3} \). Trivially, \( g(x_{\infty}) \geq \frac{2}{3} \) since \( g(x_{\infty}) \geq \lim_{n \to \infty} g_n(x_{i(n+1)}) \geq \frac{2}{3} \). Fix \( m \in \mathbb{N} \). From (b), (d), (g), we have that \( (g_n|B(x_{i(m)}, \varepsilon_{i(m)}))_{n \geq m} \)

is a Cauchy sequence in \( (C(B(x_{i(m)}, \varepsilon_{i(m)})), d) \) and hence \( k = \lim_{n \to \infty} g_n|B(x_{i(m)}, \varepsilon_{i(m)}) \) exists in \( C(B(x_{i(m)}, \varepsilon_{i(m)})) \). It follows from (f) that \( k(x_{i(m)}) \leq \frac{1}{4} \). Thus there exist \( \delta \in (0, \varepsilon_{i(m)}) \) and \( N \in \mathbb{N} \) such that \( g_n(x) < \frac{1}{2} \)

for every \( x \in B(x_{i(m)}, \delta) \) and \( n > N \). Hence \( g_n \notin (B(x_{i(m)}, \delta) \times (\frac{1}{2}, 1]^\circ) \)

for \( n > N \). It follows from \( \lim_{n \to \infty} \downarrow g_n = \downarrow g \) that \( \downarrow g \notin (B(x_{i(m)}, \delta) \times (\frac{1}{2}, 1]^\circ) \). We have \( g(x_{i(m)}) \leq \frac{1}{2} \).

For every \( n \in \mathbb{N} \), we consider
\[
\begin{align*}
d(\downarrow g_n, G) \leq d(\downarrow g_n, \downarrow g_{n+1}) + d(\downarrow g_{n+1}, \downarrow g_{n+2}) + \cdots + \frac{\delta_n}{2^n} + \frac{\delta_{n+1}}{2^{n+1}} + \cdots < \sum_{i=1}^{\infty} \frac{1}{2^i} = \delta_n.
\end{align*}
\]

Then by (j), \( G \notin F_n \). It follows that \( G \notin \bigcup_{n=1}^{\infty} F_n \). We are done. \( \square \)

**Corollary 2.** For every \( G_\delta \)-set \( G \) of \( \downarrow \text{USC}(X) \), if \( G \subset \downarrow C(X) \) then \( \{ f \in C(X) : \downarrow f \in G \} \) is not dense in \( C_a(X) \).

For each \( f \in C(X) \) and each \( a \in (0, \frac{1}{2}) \), if \( 1 - a \geq f(x) \geq a \)

for all \( x \in X \) then for every \( \varepsilon \in (0, a] \), let
\[
[f - \varepsilon, f + \varepsilon] = \{ \downarrow g \in \downarrow \text{USC}(X) : \| f(x) - g(x) \| \leq \varepsilon \text{ for all } x \in X \}.
\]

**Lemma 8.** For each \( f \in C(X) \) and each \( a \in (0, \frac{1}{2}) \), if \( 1 - a \geq f(x) \geq a \)

for all \( x \in X \) then for every \( \varepsilon \in (0, a] \), there exists a bijection \( \Phi : [f - \varepsilon, f + \varepsilon] \to \downarrow \text{USC}(X) \) such that \( \Phi \) is an order-preserving pair-homeomorphism from \( ([f - \varepsilon, f + \varepsilon], \downarrow C(X) \cap [f - \varepsilon, f + \varepsilon]) \)

onto \( (\downarrow \text{USC}(X), \downarrow C(X)) \).

**Proof.** Let \( Y = \{ (x, t) : x \in X, f(x) - \varepsilon \leq t \leq f(x) + \varepsilon \} \) and define \( \varphi : Y \to X \times I \) as follows
\[
\varphi(x, t) = \left( x, \frac{1}{2 \varepsilon} (t - f(x) + \varepsilon) \right),
\]

for \( (x, t) \in Y \). It is trivial to verify that \( \varphi \) is a continuous bijection and hence a homeomorphism by the compactness of \( Y \). Moreover, obviously, for every \( (x, t_1), (x, t_2) \in Y \), if \( t_1 \leq t_2 \), then \( \varphi(x, t_1) \leq \varphi(x, t_2) \).

Define \( \Phi : [f - \varepsilon, f + \varepsilon] \to \downarrow \text{USC}(X) \) by,
\[
\Phi(\downarrow g) = \{ \varphi(x, t) \in X \times I : (x, t) \in \downarrow g \cap Y \},
\]

for every \( \downarrow g \in [f - \varepsilon, f + \varepsilon] \). Then \( \Phi(\downarrow g) \in \downarrow \text{USC}(X) \). It is easy to see that \( d_H(\Phi(\downarrow g_1), \Phi(\downarrow g_2)) \leq \frac{1}{2} d_H(\downarrow g_1, \downarrow g_2) \)

for \( \downarrow g_1, \downarrow g_2 \in [f - \varepsilon, f + \varepsilon] \). Then \( \Phi \) is a homeomorphism since \( \varphi \) is. Moreover, for every \( \downarrow g \in \downarrow \text{USC}(X) \cap [f - \varepsilon, f + \varepsilon] \), obviously, \( \varphi(\downarrow g) \in \downarrow C(X) \) if and only if \( g \in C(X) \). Hence \( \Phi : ([f - \varepsilon, f + \varepsilon], \downarrow C(X) \cap [f - \varepsilon, f + \varepsilon]) \to (\downarrow \text{USC}(X), \downarrow C(X)) \)

is a pair-homeomorphism. At last, it follows from the definition of \( \Psi \) that, for all \( \downarrow g_1, \downarrow g_2 \in [f - \varepsilon, f + \varepsilon], \downarrow g_1 \subset \downarrow g_2 \) implies that \( \Phi(\downarrow g_1) \subset \Phi(\downarrow g_1), \) that is, \( \Phi \) is order-preserving. We complete the proof. \( \square \)

Now we give a proof of Theorem 3. It is similar to the corresponding result in [7] (cf. [15, Theorem 6.3.8]).

**Proof of Theorem 3.** Otherwise, put \( \downarrow C(X) = \bigcup_{n=1}^{\infty} G_n \), where \( G_n \) is a \( G_\delta \)-set in \( \downarrow \text{USC}(X) \) and \( G_1 = \emptyset \). We shall derive a contradiction.
For $n \in \mathbb{N}$, let $E_n = \{ f \in C(X) : \downarrow f \in G_n \}$. We shall by induction on $n$ define a sequence of \{ $f_n : n \in \mathbb{N}$ \} in $C(X)$ and a decreasing sequence of positive real numbers \{ $\epsilon_n : n \in \mathbb{N}$ \} such that

(a) $f_1 \equiv \frac{1}{2}$ and $\epsilon_1 = \frac{1}{2}$;
(b) $\epsilon_{n+1} < \frac{\epsilon_n}{2}$;
(c) $[f_{n+1} - \epsilon_{n+1}, f_{n+1} + \epsilon_{n+1}] \subset [f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4}]$; and
(d) $B(f_n, \epsilon_n) \cap E_n = \emptyset$,

for every $n \in \mathbb{N}$. Assume that $f_i$ and $\epsilon_i$ have been determined for $i \leq n$. By Lemma 8 we obtain that

$$(\left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right] \cup C(X) \cap \left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right]) \approx \left( \downarrow \text{USC}(X), \downarrow C(X) \right).$$

And since

$$G_{n+1} \cap \left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right] \subset \downarrow C(X) \cap \left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right]$$

and $G_{n+1} \cap \left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right]$ is a $G_\delta$-set in $\left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right]$, it follows from Corollary 2 that $\{ f \in C(X) : \downarrow f \in G_{n+1} \cap \left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right] \}$ is not dense in $\{ f \in C(X) : \downarrow f \in [f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4}] \}$, that is, there exist $f_{n+1}$ and $\epsilon_{n+1} \in (0, \frac{\epsilon_n}{4})$ such that

(1) $\downarrow f_{n+1} + \left[ f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4} \right] \cap \downarrow C(X);
(2) $[f_{n+1} - \epsilon_{n+1}, f_{n+1} + \epsilon_{n+1}] \subset [f_n - \frac{\epsilon_n}{4}, f_n + \frac{\epsilon_n}{4}]$;
(3) $B(f_{n+1}, \epsilon_{n+1}) \cap E_{n+1} = \emptyset$.

Thus $f_{n+1}$ and $\epsilon_{n+1}$ satisfy the above conditions. We completed the inductive definition.

It follows from (b) and (c) that the sequence $(f_n)_n$ converges uniformly to a continuous function $f : X \to I$. For $x \in X$, $n \in \mathbb{N}$, we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{n+1}(x)| + |f_{n+1}(x) - f_{n+2}(x)| + \cdots$$
$$\leq \frac{1}{2}(\epsilon_n + \epsilon_{n+1} + \cdots)$$
$$= \frac{\epsilon_n}{2} + \frac{1}{2}(\epsilon_{n+1} + \epsilon_{n+2} + \cdots)$$
$$\leq \frac{\epsilon_n}{2} + \sum_{j=1}^{\infty} \frac{1}{2^j} \epsilon_{n+1},$$

which implies that

$$\sup |f_n(x) - f(x)| \leq \frac{\epsilon_n}{2} + \epsilon_{n+1} < \epsilon_n$$

for $n \in \mathbb{N}$. By (d) we have

$$f \in \bigcap_{n=1}^{\infty} B(f_n, \epsilon_n) \subset C(X) \setminus \bigcup_{n=1}^{\infty} E_n = \emptyset,$$

which is impossible. The proof is finished. \qed

**Proof of Corollary 1.** The well-known Kadec–Anderson Theorem states that when $X$ is an infinite compactum the space $C_u(X, R) \approx R^\omega \approx I^2$, where $C_u(X, R)$ is the set $C(X, R)$ consisting of all continuous maps from $X$ to $R$ endowed with the uniform convergence topology $\{12,1\}$. As its closed subspace, $C_u(X)$ is topologically complete. The Dobrowolski–Marciszewski–Mogilski Theorem proved that $C_p(X) \approx c_0$ if $X$ is a non-discrete countable metrizable space [9] (cf. [15, Theorem 6.12.15]). Moreover, $C_p(X)$ is not metrizable if $X$ is not countable. As a conclusion, if $X$ is countable, then we have that $C_u(X)$ is topologically complete, $\downarrow C(X)$ is Baire but not topologically complete by Theorems 2 and 3, and $C_p(X) \approx c_0$ is not Baire; if $X$ is not countable, then we have that $C_u(X)$ is topologically complete, $\downarrow C(X)$ is metrizable but not topologically complete, and $C_p(X)$ is not metrizable. \qed
4. Proof of Theorem 1

Let $X$ be an infinite locally connected compactum. Then $X = \bigoplus_{i=1}^{n} X_i$ for a finite family of $\{X_i: i = 1, 2, \ldots, n\}$ of Peano continua and there exists at least one $X_i$ which is non-degenerate. Note that $(Q \times Q, c_0 \times c_0) \approx (Q, c_0)$ and $(I \times Q, I \times c_0) \approx (Q, c_0)$. If we may prove Theorem 1 holds for all non-degenerate Peano continua, then $(\downarrow \text{USC}(X), \downarrow C(X)) \approx (\prod_{i=1}^{n} \text{USC}(X_i), \prod_{i=1}^{n} \downarrow C(X_i)) \approx (Q, c_0)$. This shows that Theorem 1 holds for all infinite locally connected compacta. For every Peano continuum $X$, there exists an admissible convex metric $d$ (see [5,16]), whence each two points $x, x' \in X$ can be joined by an arc in $X$ isometric to the segment $[0, d(x, x')]$ in $\mathbb{R}$. Therefore, in the remainder of this section, we assume that $X$ is a non-degenerate Peano continuum with an admissible convex metric $d$. In [20], we proved the following three lemmas.

**Lemma 9.** $\downarrow \text{USC}(X) \approx Q$.

**Lemma 10.** $\downarrow C(X)$ is homotopy dense in $\downarrow \text{USC}(X)$.

**Lemma 11.** $\downarrow C(X)$ is a $F_{\sigma\delta}$-set in $\downarrow \text{USC}(X)$. Therefore, $\downarrow C(X)$ belongs to $\mathcal{F}_{\sigma\delta}$.

We need two corollaries of them.

**Corollary 3.** $\downarrow C(X)$ is an AR.

**Proof.** As well known [17], if $Y$ is a homotopy dense subspace in $X$ then $Y$ is an ANR(AR) if and only if $X$ is. It follows directly from Lemmas 9 and 10 that $\downarrow C(X)$ is an AR. \hfill $\Box$

**Corollary 4.** There exists a homotopy $H : \downarrow \text{USC}(X) \times I \to \downarrow \text{USC}(X)$ such that $H_0 = \text{id}_{\downarrow \text{USC}(X)}$, $H_t(\downarrow \text{USC}(X)) \subset \downarrow C(X)$ for each $t > 0$ and $d_H(H(\downarrow f, t), \downarrow f) \leq t$ for each $f \in \text{USC}(X)$ and each $t \in I$.

**Proof.** It is a combination of [15, Proposition 4.1.7] and Lemma 10. \hfill $\Box$

**Lemma 12.** $\downarrow C(X)$ is contained in a $Z_{\sigma}$-set of $\downarrow \text{USC}(X)$.

**Proof.** It follows directly from Lemma 6 and Corollary 4. \hfill $\Box$

To show that $\downarrow C(X)$ is $\mathcal{F}_{\sigma\delta}$-universal in $\downarrow \text{USC}(X)$, we need some technical lemmas. Let us recall $c_1 = \{(x_n) \in Q_0 : \lim_{n \to \infty} x_n = 1\}$ and $c_1$ is strongly $\mathcal{F}_{\sigma\delta}$-universal in $Q_0$.

**Lemma 13.** For every $x = (x_n) \in Q_0$, define a map $\varphi(x)$ from $I$ to $I$ whose graph is the broken line through points $(1, 1), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2^2}, x_1\right), \left(\frac{1}{2^3}, x_2\right), \ldots, \left(1, x'_0\right)$. Then $\varphi(x) \in \text{USC}(I)$, and $\varphi(x) \in C(I)$ if and only if $x \in c_1$. Moreover, $\downarrow \varphi : Q_0 \to \downarrow \text{USC}(I)$ is an embedding, and for each $a \in (0, 1]$,

$$|\varphi(x)(t) - \varphi(x')(t)| \leq \max\{|x_n - x'_n| : n \leq -\log_2 a + 1\},$$

for all $x, x' \in Q_0$ and all $t \in [a, 1]$.

**Proof.** The proof is trivial. \hfill $\Box$

Now we fix a point $x_0 \in X$. Without loss of generality, we may assume that $\sup\{d(x, x_0) : x \in X\} = 1$.

**Lemma 14.** The map $M : \downarrow C(X) \times (0, 1) \to I$ defined by

$$M(\downarrow f, t) = \max\{f(x) : d(x, x_0) \leq t\}$$

is continuous.
Lemma 13. By Lemma 2, there exists an embedding \( \phi \) such that \( \phi : X \rightarrow M(\downarrow f_k, t_k) \). Then \( \lim_{k \rightarrow \infty} M(\downarrow f_k, t_k) = \lim_{k \rightarrow \infty} \phi(x_k) \leq f(x) \) and \( x_k \rightarrow x \) for some \( x \in X \) and some subsequence \( (x_k) \) of \( (x) \), since \( \downarrow f_k \rightarrow \downarrow f \) and \( X \) is compact. It follows from \( t_k \rightarrow t \) that \( d(x, x_0) \leq t \). Thus \( \lim_{k \rightarrow \infty} M(\downarrow f_k, t_k) \leq M(\downarrow f, t) \).

Fact 2. For every \( \varepsilon > 0 \), there exists \( K \in \mathbb{N} \) such that \( M(\downarrow f_k, t_k) \geq M(\downarrow f, t) - \varepsilon \) for \( k > K \).

In fact, choose \( x \in X \) such that \( d(x, x_0) \leq t \) and \( f(x) = M(\downarrow f, t) \). In the case that \( d(x, x_0) < t \), from \( \downarrow f_k \rightarrow \downarrow f \) it follows that there exist \( x_k \in X \) and \( s_k \leq f(x_k) \) such that \( \lim_{k \rightarrow \infty} (x_k, s_k) = (x, f(x)) \). Then \( d(x_k, x_0) \leq t \) for large enough \( k \). Thus, there exists \( K \in \mathbb{N} \) such that \( M(\downarrow f_k, t_k) \geq f(x_k) \geq s_k \geq f(x) - \varepsilon = M(\downarrow f, t) - \varepsilon \) for \( k > K \). In the case that \( d(x, x_0) = t \), from the continuity of \( f \) and the convexity of \( d \) it follows that there exists \( y \in B(x_0, t) \) such that \( f(y) \geq f(x) - \frac{\varepsilon}{t} \). By the above case, there exist \( x_k \in X \) and \( s_k \leq f(x_k) \) such that \( \lim_{k \rightarrow \infty} (x_k, s_k) = (y, f(y)) \). Thus there exists \( K \in \mathbb{N} \) such that \( M(\downarrow f_k, t_k) \geq f(y) - \frac{\varepsilon}{t} \geq f(x) - \varepsilon = M(\downarrow f, t) - \varepsilon \) for \( k > K \).}

Remark. It is not hard to give examples to show that neither \( M : \downarrow \text{USC}(X) \times (0, 1] \rightarrow I \) nor \( M : \downarrow \text{C}(X) \times I \rightarrow I \) is necessarily continuous when we define \( M \) by using the formula in the above lemma.

Now we are in a position to show our key lemma in the section.

Lemma 15. \( \downarrow \text{C}(X) \) is strongly \( \mathcal{F}_{\alpha \delta} \)-universal in \( \downarrow \text{USC}(X) \approx Q \).

Proof. Let \( C \) and \( K \) be a \( \mathcal{F}_{\alpha \delta} \)-subset and a compact set of a compactum \( Y \), respectively. Let \( \Phi : Y \rightarrow \text{USC}(X) \) be a map such that \( \downarrow \Phi : Y \rightarrow \downarrow \text{USC}(X) \) is continuous and \( \downarrow \Phi|K : K \rightarrow \downarrow \text{USC}(X) \) is a \( Z \)-embedding. By [4, Lemma 1.1] and Corollary 3, without loss of generality, we may assume that \( \downarrow \Phi(K) \cap \downarrow \Phi(Y \setminus K) = \emptyset \). For every \( \varepsilon \in (0, 1) \), let \( \delta : Y \rightarrow [0, 1) \) be a map defined by

\[
d(\Phi(y), \Phi(K)) \leq \frac{1}{4} \min\{\varepsilon, d(\Phi(y), \Phi(K))\}.
\]

Then \( \delta \) is continuous and \( \delta(y) = 0 \) if and only if \( y \in K \).

Let \( M : \downarrow \text{C}(X) \times (0, 1] \rightarrow I \) be the map defined in the above lemma and \( H : \downarrow \text{USC}(X) \times I \rightarrow \downarrow \text{USC}(X) \) be a continuous map satisfying the conditions in Corollary 4. For \( y \in Y \setminus K \), let

\[
k(y) = H(\downarrow \Phi(y), \delta(y)), \quad \text{and} \quad m(y) = M(k(y), \delta(y)).
\]

Then \( k : Y \setminus K \rightarrow \downarrow \text{C}(X) \) and \( m : Y \setminus K \rightarrow I \) are continuous. Suppose \( \varphi : Q \rightarrow \text{USC}(I) \) is the map defined in Lemma 13. By Lemma 2, there exists an embedding \( j : Y \rightarrow Q_u \) such that \( j^{-1}(c_1) = C \). Let \( \phi = \varphi \circ j \). Then \( \phi^{-1}(C(k)) = C \) and \( \downarrow \phi : Y \rightarrow \downarrow \text{USC}(I) \) is an embedding. Now we define a map \( \Psi : Y \rightarrow \text{USC}(X) \) as follows: for \( y \in K \), let \( \Psi(y) = \Phi(y) \) and for \( y \in Y \setminus K \), let

\[
\Psi(y)(x) = \begin{cases} 
\frac{k(y)(x)}{2(\delta(y) - d(x,x_0) - \delta(y))} m(y) + \frac{d(x,x_0) - 2\delta(y)}{2\delta(y)} k(y)(x), & d(x,x_0) \geq \delta(y), \\
3\delta(y) - 6d(x,x_0) + \frac{6d(x,x_0) - 2\delta(y)}{2\delta(y)} m(y), & \frac{\delta(y)}{2} \leq d(x,x_0) \leq \delta(y), \\
\delta(y)(x) / \frac{2\delta(y)}{\delta(y)}, & 0 \leq d(x,x_0) \leq \frac{\delta(y)}{2}. 
\end{cases}
\]

To complete the proof of the lemma, we need the following facts:

Fact 1. \( \Psi(y) \) is well-defined and belongs to \( \text{USC}(X) \) for each \( y \in Y \), and \( \Psi(y) \) is continuous in the set \( X \setminus \{x_0\} \) for each \( y \in Y \setminus K \). This is trivial.
Fact 2. $\downarrow \Psi : Y \to \downarrow \text{USC}(X)$ is continuous\(^2\) at every point $y \in Y$. In the case that $y \in K$, it follows from the following Fact 3. Here we only consider the case that $y \in Y \setminus K$.

Claim A. For every $x \in X$ and every $\gamma > 0$, there exists $\beta > 0$ such that
\[ \downarrow \Psi (c) \cap B((x, \Psi (y)(x)), \gamma) \neq \emptyset \] (1)
for each $c \in B(y, \beta)$.

We consider the following cases:

Case a: $d(x, x_0) > \delta(y)$. Then $\Psi (y)(x) = k(y)(x)$. By the continuity of $\delta$, there exists $\beta_1 > 0$ such that $d(x, x_0) > \delta(c) > 0$ for each $c \in B(y, \beta_1)$. Thus, $\Psi (c)(x) = k(c)(x)$ for $c \in B(y, \beta_1)$. Since $\downarrow \Psi : Y \setminus K \to \downarrow \text{USC}(X)$ is continuous and $\downarrow k(y) \cap B((x, k(y)(x)), \gamma) \neq \emptyset$, there exists $\beta \in (0, \beta_1)$ such that $\downarrow k(c) \cap B((x, k(c)(x)), \gamma) \neq \emptyset$ for all $c \in B(y, \beta)$. Hence, (1) holds.

Case b: $\frac{\delta(y)}{2} < d(x, x_0) < \delta(y)$. As the same as in case a, there exists $\beta_1 > 0$ such that $\frac{\delta(c)}{2} < d(x, x_0) < \delta(c)$ for all $c \in B(y, \beta_1)$. Trivially, there exists $\alpha > 0$ such that, for all real numbers $a, b, c, d$, if $|d(x, x_0) - a| < \alpha$, $|\delta(y) - b| < \alpha$, $|k(y)(x) - c| < \alpha$ and $|m(y) - d| < \alpha$ then
\[ \frac{2(b - a)}{b} d + \frac{2a - b}{b} c - \Psi (y)(x) < \gamma. \]

Since $m : Y \setminus K \to I$ and $\delta : Y \setminus K \to (0, 1)$ are continuous, there exists $\beta_2 \in (0, \beta_1)$ such that $|m(y) - m(c)| < \alpha$ and $|\delta(y) - \delta(c)| < \alpha$ if $d(y, c) < \beta_2$. Since $\downarrow \Psi : Y \setminus K \to \downarrow \text{C}(X)$ is continuous, there exists $\beta \in (0, \beta_2)$ such that, for all $c \in B(y, \beta)$, there exist $x(c) \in X$ and $s(c) \leq k(c)(x(c))$ satisfying:

(a) $\frac{\delta(c)}{2} < d(x(c), x_0) < \delta(c)$;
(b) $d(x(c), x) < \gamma$;
(c) $|s(c) - k(y)(x)| < \alpha$;
(d) $|d(x(c), x_0) - d(x, x_0)| < \alpha$.

Let
\[ t(c) = \frac{2(\delta(c) - d(x(c), x_0))}{\delta(c)} m(c) + \frac{2d(x(c), x_0) - \delta(c)}{\delta(c)} s(c). \]

Then $t(c) \leq \Psi (c)(x(c))$ and $|t(c) - \Psi (y)(x)| < \gamma$. Thus $(x(c), t(c)) \in \downarrow \Psi (c) \cap B((x, \Psi (y)(x)), \gamma)$ for all $c \in B(y, \beta)$. We have that (1) holds.

Case c: $\frac{\delta(y)}{3} < d(x, x_0) < \frac{\delta(y)}{2}$. As the same as the above cases, there exists $\beta_1 > 0$ such that $\frac{\delta(c)}{3} < d(x, x_0) < \frac{\delta(c)}{2}$ for all $c \in B(y, \beta_1)$. By the continuity of $\delta$ and $m$, there exists $\beta \in (0, \beta_1)$ such that $|\Psi (y)(x) - \Psi (c)(x)| < \gamma$ for each $c \in B(y, \beta)$. Thus (1) holds for $c \in B(y, \beta)$.

Case d: $0 < d(x, x_0) < \frac{\delta(y)}{3}$. Choose $\beta_1 > 0$ such that $d(x, x_0) < \frac{\delta(c)}{3}$ and $\delta(c) \leq \frac{2}{3} \delta(y)$ for all $c \in B(y, \beta_1)$. Then
\[ \left| \Psi (c)(x) - \Psi (y)(x) \right| = \left| \delta(c) \phi(c) \left( \frac{3d(x, x_0)}{\delta(c)} \right) - \delta(y) \phi(y) \left( \frac{3d(x, x_0)}{\delta(y)} \right) \right| \leq \left| \delta(c) - \delta(y) \right| + \left| \phi(c) \left( \frac{3d(x, x_0)}{\delta(c)} \right) - \phi(y) \left( \frac{3d(x, x_0)}{\delta(y)} \right) \right| \leq \left| \delta(c) - \delta(y) \right| + \left| \phi(c) \left( \frac{3d(x, x_0)}{\delta(c)} \right) - \phi(y) \left( \frac{3d(x, x_0)}{\delta(c)} \right) \right| + \left| \phi(y) \left( \frac{3d(x, x_0)}{\delta(c)} \right) - \phi(y) \left( \frac{3d(x, x_0)}{\delta(y)} \right) \right|. \]

Let $a = \frac{3d(x, x_0)}{\delta(y)}$. Choose $\beta \in (0, \beta_1)$ such that $|\delta(c) - \delta(y)| < \frac{1}{3} \gamma$, $|\phi(y) - \frac{3d(x, x_0)}{\delta(c)}| < \frac{1}{3} \gamma$ and $|j(c)n - j(y)n| < \frac{1}{3} \gamma$ for all $n \leq -\log_2 a + 1$ and $c \in B(y, \beta)$. By Lemma 13, we have

\(^2\) More careful verification of the continuity of $\downarrow \Psi$ is needed when we note that the map $\downarrow \tau : \downarrow \text{C}(\mathbb{I}) \times \downarrow \text{C}(\mathbb{I}) \to \downarrow \text{C}(\mathbb{I})$ defined by $\tau(f, g) = \frac{1}{2} f + \frac{1}{2} g$ is not continuous. In fact, let $f_n = \frac{1}{2}(1 + \sin nx)$ and $g_n = 1 - f_n$, then $\lim_{n \to \infty} f_n = \lim_{n \to \infty} g_n = 1$ but $\lim_{n \to \infty} \downarrow (\frac{1}{2} f_n + \frac{1}{2} g_n) = \downarrow \frac{1}{2}. $
\[ \phi(c) \left( \frac{3d(x, x_0)}{\delta(c)} \right) - \phi(y) \left( \frac{3d(x, x_0)}{\delta(c)} \right) = \varphi(j(c)) \left( \frac{3d(x, x_0)}{\delta(c)} \right) - \varphi(j(y)) \left( \frac{3d(x, x_0)}{\delta(c)} \right) \leq \max \{|j(c)_n - j(y)_n|; n \leq -\log_2 a + 1\} \leq \frac{1}{3} \gamma, \]

since \( \frac{3d(x, x_0)}{\delta(c)} \geq a \) when \( c \in B(y, \beta) \). It follows that (1) holds for \( c \in B(y, \beta) \).

Case c: \( x = x_0 \). Then \( \Psi(y)(x) = \delta(y) \). It follows from the continuity of \( \delta \) that there exists \( \beta > 0 \) such that \( |\delta(c) - \delta(y)| < \gamma \) for all \( c \in B(y, \beta) \). Then \( (x, \Psi(c)(x)) \in B((x, \Psi(y)(x), \gamma) \).

Thus, for every \( x \in A \), we have \( (x, \Psi(y)(x), \gamma) \) such that \( \delta(c) \neq 0 \) implies that \( \Psi(y) \) is continuous in the set \( X \setminus \{x_0\} \). Since \( d \) is a convex metric, \( A \) is a nowhere dense closed set of \( X \). Moreover, \( \delta(y) > 0 \) implies that \( \Psi(y) \) is continuous in the set \( X \setminus \{x_0\} \). Thus, for every \( x \in A \) and every \( y \in X \), there exist \( x' \in X \setminus (A \cup \{x_0\}) \) and \( y' > 0 \) such that \( B((x', \Psi(y)(x')), \gamma') \subset B((x, \Psi(y)(x)), \gamma) \). By Cases (a)-(d), there exists \( \beta > 0 \) such that \( \cup \Psi(c) \cap B((x', \Psi(y)(x')), \gamma') \neq \emptyset \) for all \( c \in B(y, \beta) \). It follows that (1) holds for every \( c \in B(y, \beta) \).

Claim B. For each convergent sequence \( (y_k) \) in \( Y \setminus K \) with the limit \( y \in Y \setminus K \) and each convergent sequence \( (x_k, t_k) \) in \( X \times I \) such that \( t_k \leq \Psi(y_k)(x_k) \) and \( \lim_{k \to \infty} (x_k, t_k) = (x, t) \), we have \( t \leq \Psi(y)(x) \).

Case a: \( d(x_k, x_0) \geq \delta(y_k) \) and \( d(x_k, x_0) \geq \delta(y_k) \) for each \( k \). Then \( \Psi(y_k)(x_k) = k(y_k)(x_k) \) and \( \Psi(y_k)(x_k) = k(y_k)(x_k) \) for all \( k \). It follows from the continuity of \( \delta \) and \( (x_k, t_k) \in \downarrow \Psi(y_k) \) that \( (x_k, t_k) \in \downarrow \Psi(y_k) \). Hence \( (x, t) \in \downarrow \Psi(y_k) \) implies that \( t \leq \delta(k(y_k)(x_k)) = \Psi(y_k)(x_k) \).

Case b: \( \frac{\delta(y_k)}{2} \leq d(x_k, x_0) \leq \delta(y_k) \) and \( \frac{\delta(y_k)}{2} \leq d(x_k, x_0) \leq \delta(y_k) \) for all \( k \). Then
\[ \Psi(y)(x) = \frac{2\delta(y) - 2d(x, x_0)}{\delta(y)} m(y) + \frac{2d(x, x_0) - \delta(y)}{\delta(y)} k(y)(x) \]
and
\[ \Psi(y_k)(x_k) = \frac{2\delta(y_k) - 2d(x_k, x_0)}{\delta(y_k)} m(y_k) + \frac{2d(x_k, x_0) - \delta(y_k)}{\delta(y_k)} k(y_k)(x_k). \]

Since \( \frac{2d(x_k, x_0) - \delta(y_k)}{\delta(y_k)} > 0 \) and \( t_k \leq \Psi(y_k)(x_k) \) for all \( k \), there exist \( s_k \leq k(y_k)(x_k) \) such that
\[ t_k = \frac{2\delta(y_k) - 2d(x_k, x_0)}{\delta(y_k)} m(y_k) + \frac{2d(x_k, x_0) - \delta(y_k)}{\delta(y_k)} s_k \]
for all \( k \). By the compactness of \( I \), we assume that \( \lim_{k \to \infty} (x_k, s_k) = (x, s) \) exists. It follows from the continuity of \( \downarrow \Psi: Y \setminus K \to \downarrow C(X) \) that \( s \leq k(y)(x) \). Thus we have
\[ t = \lim_{k \to \infty} t_k \leq \Psi(y)(x). \]

Case c: \( \frac{\delta(y)}{2} \leq d(x, x_0) \leq \frac{\delta(y)}{2} \) and \( \frac{\delta(y_k)}{2} \leq d(x_k, x_0) \leq \frac{\delta(y_k)}{2} \) for each \( k \). Then
\[ \Psi(y)(x) = 3\delta(y) - 6d(x, x_0) + \frac{6d(x, x_0) - 2\delta(y)}{\delta(y)} m(y) \]
and
\[ \Psi(y_k)(x_k) = 3\delta(y_k) - 6d(x_k, x_0) + \frac{6d(x_k, x_0) - 2\delta(y_k)}{\delta(y_k)} m(y_k). \]

Note that
\[ (y, x) \mapsto 3\delta(y) - 6d(x, x_0) + \frac{6d(x, x_0) - 2\delta(y)}{\delta(y)} m(y) \]
is continuous map from \((Y \setminus K) \times X \) to \( I \). It follows that \( t = \lim_{k \to \infty} t_k \leq \Psi(y)(x). \)

Case d: \( 0 < d(x, x_0) \leq \frac{\delta(y)}{2} \) and \( 0 < d(x_k, x_0) \leq \frac{\delta(y_k)}{2} \) for all \( k \). Then
\[ \Psi(y)(x) = \delta(y) \phi(y) \left( \frac{3d(x, x_0)}{\delta(y)} \right) \quad \text{and} \quad \Psi(y_k)(x_k) = \delta(y_k) \phi(y_k) \left( \frac{3d(x_k, x_0)}{\delta(y_k)} \right). \]
Note $| \Psi (y_k)(x_k) - \Psi (y)(x)| \leq | \Psi (y_k)(x_k) - \Psi (y)(x_k)| + | \Psi (y)(x_k) - \Psi (y)(x)|$. Since $\frac{3d(x,x_0)}{\delta(y)} > 0$, we have that there exists $a > 0$ such that $\frac{3d(x,x_0)}{\delta(y_k)} \geq a$ for large enough $k$. Using the same method as in Proof of case d in Claim A, we may show that $\lim_{k \to \infty} | \Psi (y_k)(x_k) - \Psi (y)(x)| = 0$. Since $\Psi (y)$ is continuous at $x$, $\lim_{k \to \infty} | \Psi (y_k)(x_k) - \Psi (y)(x)| = 0$. Thus, we have $\lim_{k \to \infty} | \Psi (y_k)(x_k) - \Psi (y)(x)| = 0$, that is, $\lim_{k \to \infty} \Psi (y_k)(x_k) = \Psi (y)(x)$. It follows that $t = \lim_{k \to \infty} k \leq \lim_{k \to \infty} \Psi (y_k)(x_k) = \Psi (y)(x)$.

Case e: $x = x_0$. Since $\delta(y) > 0$ we may without loss of generality assume that $0 \leq d(x_k, x_0) < \frac{\delta(y)}{3}$. Then $t_k \leq \Psi (y_k)(x_k) \leq \delta(y)$. Thus $t \leq \delta(y) = \Psi (y)(x_0)$.

Case f: Otherwise, there exists a subsequence $(k_i)$ of $(k)$ such that $x, y$, and the sequences $(x_{k_i}), (y_{k_i})$ satisfy the conditions of one of the above cases. Thus, we have that $t \leq \Psi (y)(x)$ holds.

Now we show that $\downarrow \Psi : Y \setminus K \to \downarrow \text{USC}(X)$ is continuous. To this end, it suffices to verify that $(\downarrow \Psi |(Y \setminus K))^{-1}(U^-)$ and $(\downarrow \Psi |(Y \setminus K))^{-1}(U^+)$ are open in $Y$ for all open sets $U$ in $X \times I$. Suppose that $U$ is an open set of $X \times I$.

For every $y \in (\downarrow \Psi |(Y \setminus K))^{-1}(U^-)$, we have $\downarrow \Psi (y) \cap U \neq \emptyset$. Then there exist $(x, t) \in X \times I$ and $\gamma > 0$ such that $(x, t) \in \downarrow \Psi (y) \cap U$ and $B((x, t), \gamma) \subset U$. By Claim A, there exists $\beta > 0$ such that $\downarrow \Psi (c) \cap B((x, \Psi (y)(x)), \gamma) \neq \emptyset$ for all $c \in B(y, \beta)$. It is easy to verify that $\downarrow \Psi (c) \cap U \neq \emptyset$ if $c \in B(y, \beta)$, that is, $c \in \downarrow \Psi |(Y \setminus K))^{-1}(U^-)$. Hence, $(\downarrow \Psi |(Y \setminus K))^{-1}(U^-)$ is open.

If $(\downarrow \Psi |(Y \setminus K))^{-1}(U^+)$ is not open in $Y \setminus K$, then there exists $y \in (\downarrow \Psi |(Y \setminus K))^{-1}(U^+)$ and a sequence $(y_k)$ in $Y \setminus K$ such that $\lim_{k \to \infty} y_k = y$ but no $k$ satisfies $y_k \in (\downarrow \Psi |(Y \setminus K))^{-1}(U^+)$. Thus, $\downarrow \Psi (y) \subseteq U$ but $\downarrow \Psi (y_k) \not\subseteq U$. For every $k$, choose $(x_k, t_k) \in \downarrow \Psi (y_k) \cap U$. Then $t_k \leq \Psi (y_k)(x_k)$ for each $k$. Without loss of generality, we assume that $\lim_{k \to \infty} (x_k, t_k) = (x, t)$ exists. Then $(x, t) \in U$. But, by Claim B, we have $t \leq \Psi (y)(x)$, that is, $(x, t) \in \downarrow \Psi (y)$. A contradiction occurs.

**Fact 3.** $d_H(\downarrow \Psi (y), \downarrow \Phi (y)) \leq 3\delta(y)$ for each $y \in Y$. Therefore, $d_H(\downarrow \Psi, \downarrow \Phi) < \varepsilon$ and $\Phi (Y \setminus K) \cap \Psi (K) = \Psi (Y \setminus K) \cap \Phi (K) = \emptyset$.

Note that $m(y) = M(\downarrow k(y), \delta(y))$ for every $y \in Y \setminus K$. Thus,

$$k(y)(x) \leq m(y) = \max \{ k(y)(z): d(z, x_0) \leq \delta(y) \}$$

for every point $x \in X$ with $d(x, x_0) \leq \delta(y)$. Hence there exists $x_1 \in X$ such that $d(x_0, x_1) \leq \delta(y)$ and $k(y)(x_1) = m(y)$. Let $A = \downarrow k(y) \cap ((X \setminus B(x_0, \delta(y))) \times I)$. Then

$$A \cup \{ x_1 \times [0, m(y)] \} \subset \downarrow k(y) \subset A \cup \left( B(x_0, \delta(y)) \times [0, m(y)] \right).$$

(2)

From $\{ x \in X: d(x, x_0) = \frac{\delta(y)}{3} \neq \emptyset$ and the definition of $\Psi (y)(x)$, it follows that $\Psi (y)(x_2) = m(y)$ for some $x_2 \in X$ with $d(x_0, x_2) = \frac{\delta(y)}{3}$. Hence

$$m(y) \leq \max \{ \Psi (y)(x): d(x, x_0) \leq \delta(y) \}.$$

On the other hand, $\Psi (y)(x)$ is a weighted average value of either $k(y)(x)$ and $m(y)$ or $m(y)$ and $\delta(y)$ if $\frac{\delta(y)}{3} \leq d(x, x_0) \leq \delta(y)$. And $\Psi (y)(x) \leq \delta(y)$ if $d(x, x_0) \leq \frac{\delta(y)}{3}$. It follows that

$$\max \{ \Psi (y)(x): d(x, x_0) \leq \delta(y) \} \leq \max \{ \delta(y), m(y) \}.$$

Since $\Psi (y)(x) = k(y)(x)$ for all $x \in X$ with $d(x, x_0) \geq \delta(y)$, we have that

$$A \cup \{ x_2 \times [0, \max \{ \delta(y), m(y) \}] \} \subset \downarrow \Psi (y) \subset A \cup \left( B(x_0, \delta(y)) \times [0, \max \{ \delta(y), m(y) \}] \right).$$

(3)

It is not hard to check that $d_H(B, C) \leq 2\delta(y)$ for every pair of $B$ and $C$ in the family which consists of the first and the last terms of both (2) and (3). It follows that $d_H(\downarrow k(y), \downarrow \Psi (y)) \leq 2\delta(y)$.

By the definition of $k$ and the properties of $H$, we have that

$$d_H(\downarrow \Psi (y), \downarrow \Phi (y)) \leq d_H(\downarrow \Psi (y), \downarrow k(y)) + d_H(\downarrow k(y), \downarrow \Phi (y)) \leq 3\delta(y).$$
Fact 4. \(\downarrow \Psi : Y \rightarrow \downarrow \text{USC}(X)\) is a Z-embedding.

Let \(y_1, y_2 \in Y\) with \(y_1 \neq y_2\). We verify that \(\Psi(y_1) \neq \Psi(y_2)\). From Fact 3 we assume that \(y_1, y_2 \notin K\). If \(\delta(y_1) \neq \delta(y_2)\), then \(\Psi(y_1)(x_0) = \delta(y_1) \neq \delta(y_2) = \Psi(y_2)(x_0)\). If \(\delta(y_1) = \delta(y_2) = \delta > 0\) and there exists \(t \in I\) such that \(\Phi(y_1)(t) \neq \Phi(y_2)(t)\). It follows from the convexity of \(d\) that there exists \(x \in X\) such that \(\frac{3d(x, x_0)}{\delta} = t\). Thus, \(\Psi(y_1)(x) = \delta \Phi(y_1)(\frac{3d(x, x_0)}{\delta}) \neq \delta \Phi(y_2)(\frac{3d(x, x_0)}{\delta}) = \Psi(y_2)(x)\). This shows that \(\downarrow \Psi : Y \rightarrow \downarrow \text{USC}(X)\) is an injection. Moreover, note that, for each \(y \in Y \setminus K\), there exists \(x \in X\) such that \(\frac{3d(x, x_0)}{\delta(y)} = \frac{1}{2}\) and hence \(\Psi(y)(x) = 0\). It follows from Lemma 5 that \(\downarrow \Psi(Y)\) is a Z-set in \(\downarrow \text{USC}(X)\).

Fact 5. \(\Psi^{-1}(C(X)) \setminus K = C \setminus K\).

Note that, for each \(y \in Y \setminus K\), \(\Psi(y)\) is continuous in \(X \setminus \{x_0\}\) and it is continuous at \(x_0\) if and only if \(\phi(y) \in C(I)\) if and only if \(y \in C\).

It follows from the above five Facts that \(\downarrow \Psi : Y \rightarrow \downarrow \text{USC}(X)\) is a Z-embedding and satisfies \(\downarrow \Psi|_K = \downarrow \Phi|_K\), \(\downarrow \Psi^{-1}((\downarrow C(X)) \setminus K = C \setminus K\) and \(d_H(\downarrow \Psi(y), \downarrow \Phi(y)) < \varepsilon\) for each \(y \in Y\). This shows that \(\downarrow C(X)\) is strongly \(\mathcal{F}_{\sigma\delta}\)-universal in \(\downarrow \text{USC}(X)\).

Proof of Theorem 1. It follows from Lemmas 1, 11, 12 and 15.

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