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Metric sparsification and operator norm localization

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Abstract

We study an operator norm localization property and its applications to the coarse Novikov conjecture in operator K-theory. In particular, we introduce a sufficient geometric condition (called metric sparsification) for the operator norm localization property. This is used to give many examples of finitely generated groups with infinite asymptotic dimension and the operator norm localization property. We also show that a sequence of expanding graphs does not possess the operator norm localization property.

Keywords: Operator norm localization; Metric sparsification; Operator K-theory; Novikov conjecture

1. Introduction

Operator norm is a global invariant and is often difficult to estimate. In this paper, we study a localization property which allows us to estimate the operator norm locally relative to a metric space. This property is motivated by the coarse Novikov conjecture in operator K-theory. More precisely, a metric space X is said to have operator norm localization property if there exists $0 < c \leqslant 1$ such that for every $r > 0$, there is $R > 0$ for which, if ν is a positive locally finite Borel measure on X, H is a separable infinite-dimensional Hilbert space and T is a bounded linear operator acting on $L^2(X, \nu) \otimes H$ with propagation r, then there exists an unit vector $\xi \in L^2(X, \nu) \otimes H$ satisfying the $\mathrm{Diam}(\mathrm{Supp}(\xi)) \leqslant R$ and $\|T\xi\| \geqslant c\|T\|$. If X has finite asymptotic

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dimension, then X has operator norm localization property [7]. We introduce a natural coarse geometric property on metric spaces, called metric sparsification, to study the operator norm localization property. Roughly speaking, this property says that there exists a constant $0 < c \leqslant 1$ such that, for every positive finite Borel measure μ on X, there exists a subset E, which is a union of "well separated" subsets of "controlled" diameters such that $\mu(E) \geqslant c\mu(X)$. We show that c can be chosen arbitrarily close to 1.

We show that the metric sparsification property implies the operator norm localization property, but is more flexible than the latter. We prove for instance that any solvable locally compact group equipped with a proper, locally finite left-invariant metric has metric sparsification property. This provides the first examples of finitely generated groups (as metric spaces with word metric) with infinite asymptotic dimension satisfying operator norm localization property, as for instance $asdim(\mathbf{Z} \wr \mathbf{Z}) = \infty$ ($\mathbf{Z} \wr \mathbf{Z}$ is the wreath product of $\mathbf{Z}$ with $\mathbf{Z}$). This also implies that connected Lie groups and their discrete subgroups have metric sparsification property. We obtain several permanence properties for the operator norm localization property. We also show that a sequence of expanding graphs does not possess operator norm localization property. Finally in the last section of this paper, we apply the operator norm localization property to prove the coarse Novikov conjecture for certain sequences of expanders.

2. Operator norm localization

In this section, we introduce an operator norm localization property for a metric space. We show that this property is invariant under coarse geometric equivalence.

Recall that a Borel measure on a metric space is said to be locally finite if every bounded Borel subset has finite measure.

Definition 2.1. (See Roe [6].) Let X be a metric space with a positive locally finite Borel measure ν, let H be a separable and infinite-dimensional Hilbert space. A bounded operator $T : L^2(X,\nu) \otimes H \to L^2(X,\nu) \otimes H$, is said to have propagation at most r if for all $\varphi, \psi \in L^2(X,\nu) \otimes H$ such that $d(\mathrm{Supp}(\varphi), \mathrm{Supp}(\psi)) > r$,

$$\langle A\varphi, \psi \rangle = 0.$$

Note that if X is discrete, then we can write

$$L^2(X,\nu) \otimes H = \bigoplus_{x \in X} (\delta_x \otimes H),$$

where δ_x is the Dirac function at x. Every bounded operator acting on $L^2(X,\nu) \otimes H$ has a corresponding matrix representation

$$T = (T_{x,y})_{x,y \in X},$$

where $T_{x,y} : \delta_y \otimes H \to \delta_x \otimes H$ is a bounded operator. For T to have propagation r, it is equivalent to saying that the matrix coefficient $T_{x,y}$ of T vanishes when $d(x,y) > r$. The space of operators acting on $L^2(X,\nu) \otimes H$ with propagation at most r will be denoted by $\mathcal{A}_r(X,\nu)$.

Let $\|T\|$ denote the operator norm of a bounded linear operator T.

Definition 2.2. Let (X, ν) be a metric space equipped with a positive locally finite Borel measure ν. Let f be a (non-decreasing) function $\mathbf{N} \to \mathbf{N}$. We say that (X, ν) has operator norm localization property relative to f with constant $0 < c \leqslant 1$ if, for all $k \in \mathbf{N}$, and every $T \in \mathcal{A}_k(X, \nu)$, there exists non-zero $\varphi \in L^2(X, \nu) \otimes H$ satisfying

(i) $\mathrm{Diam}(\mathrm{Supp}(\varphi)) \leqslant f(k)$,
(ii) $\|T\varphi\| \geqslant c\|T\|\|\varphi\|$.

Definition 2.3. A metric space X is said to have operator norm localization property if there exists a constant $0 < c \leqslant 1$ and a (non-decreasing) function $f : \mathbf{N} \to \mathbf{N}$ such that, for every positive locally finite Borel measure ν on X, (X, ν) has operator norm localization property relative to f with constant c.

We point out that a locally compact metric space X has operator norm localization property if (X, ν_0) has operator norm localization property for some positive locally finite Borel measure ν_0 such that there exists $r_0 > 0$ for which every closed ball with radius r_0 has positive measure. This can be seen as follows. We can decompose X into a countable disjoint union of uniformly bounded Borel subsets $\{X_i\}_{i\in I}$ such that every bounded subset of X is contained in a union of finitely many members of $\{X_i\}_{i\in I}$ and $\nu_0(X_i) > 0$. We decompose

$$L^2(X, \nu_0) \otimes H = \bigoplus_{i\in I} \big(L^2(X_i, \nu_0) \otimes H\big).$$

For every other positive locally finite Borel measure ν, we have a similar decomposition:

$$L^2(X, \nu) \otimes H = \bigoplus_{i\in I} \big(L^2(X_i, \nu) \otimes H\big).$$

Let $W : L^2(X, \nu) \otimes H \to L^2(X, \nu_0) \otimes H$, be an isometry such that

$$W\big(L^2(X_i, \nu) \otimes H\big) \subseteq L^2(X_i, \nu_0) \otimes H$$

for every $i \in I$. If T is a bounded operator acting on $L^2(X, \nu) \otimes H$ with propagation $r > 0$, then WTW^* is a bounded operator acting on $L^2(X, \nu_0) \otimes H$ with propagation $r + 2D$ and $\|WTW^*\| = \|T\|$, where $D = \sup\{\mathrm{Diam}(X_i): i \in I\}$. It follows that if (X, ν_0) has operator norm localization property relative to f with constant $0 < c \leqslant 1$, then (X, ν) has operator norm localization property relative to $f + D$ with constant c.

Let F be a Borel map from a metric space X to another metric space Y. Recall that F is said to be coarse if: (1) for every $r > 0$, there exists $R > 0$ such that $d(F(x), F(y)) < R$ for every pair of points x and y in X satisfying $d(x, y) < r$; (2) the inverse image $F^{-1}(B)$ for every bounded subset B of Y is bounded. We say that X is coarsely equivalent to Y if there exist coarse maps $F : X \to Y$ and $G : Y \to X$, such that there exist a constant C satisfying $d(G(F(x)), x) < C$ for all $x \in X$, and $d(F(G(y)), y) < C$ for all $y \in Y$.

The following proposition was indicated to us by the referee.

Proposition 2.4. *If a metric space has operator norm localization property, then it has the property with constant c for all* $0 < c < 1$.

Proof. Let (X, ν) be a metric space equipped with a positive locally finite Borel measure ν. Let f be a (non-decreasing) function $\mathbf{N} \to \mathbf{N}$ such that (X, ν) has operator norm localization property relative to f with constant $0 < c_0 \leqslant 1$ for some $c_0 > 0$. For each $T \in \mathcal{A}_k(X, \nu)$, we have $T^n \in \mathcal{A}_{nk}(X, \nu)$. Without loss of generality, we can assume $\|T\| = 1$. Hence, by the definition of the operator norm localization property, there exists $\varphi \in L^2(X, \nu) \otimes H$ satisfying

(i) $\mathrm{Diam}(\mathrm{Supp}(\varphi)) \leqslant f(nk)$,
(ii) $\|T^n \varphi\| \geqslant c_0 \|\varphi\|$.

This implies that $\|T(T^j \varphi)\| \geqslant c_0^{1/n} \|T^j \varphi\|$ for some $0 \leqslant j \leqslant n-1$. The support of $T^j \varphi$ has at most diameter $g(k) = (n-1)k + f(nk)$. Hence (X, ν) has operator norm localization property relative to g with constant $0 < c_0^{1/n} \leqslant 1$. $\square$

Proposition 2.5. *The operator norm localization property is invariant under coarse equivalence.*

Proof. Let X and Y be two coarsely equivalent metric spaces. Let $F : X \to Y$ and $G : Y \to X$ be two coarse maps as in the definition of coarse equivalence. There exist two increasing functions $\rho_1, \rho_2 : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{t \to \infty} \rho_1(t) = \infty$ and

$$\rho_1\big(d(x, x')\big) \leqslant d\big(F(x), F(y)\big) \leqslant \rho_2\big(d(x, x')\big).$$

We shall prove that if Y has operator norm localization property, then so does X. Let ν be a positive locally finite measure on X and let $\nu' = F(\nu)$. It is not difficult to see that there exists an isometry

$$W : L^2(X, \nu) \otimes H \to L^2(Y, \nu') \otimes H$$

satisfying

$$\mathrm{Supp}(W\varphi) \subseteq \big\{y \in Y:\ d\big(y, F\big(\mathrm{Supp}(\varphi)\big)\big) \leqslant 1\big\}.$$

For every $T \in \mathcal{A}_k(X, \nu)$, we have that $\|T\| = \|WTW^*\|$ and $WTW^* \in \mathcal{A}_{k+1}(Y, \nu')$. These properties of W imply that X has operator norm localization property. $\square$

Recall that a metric space X is said to coarsely embed into Y if X is coarsely equivalent to a subset of Y (with the metric induced from Y). The proof of Proposition 2.4 shows the following:

Proposition 2.6. *If a metric space X coarsely embeds into another metric space Y, then X has operator norm localization property if Y does.*

It is an open question to find a geometric condition equivalent to the operator norm localization property.

3. Metric sparsification property

In this section, we introduce a coarse geometric invariant for metric spaces called the metric sparsification property. We prove in particular that any locally compact solvable group has metric sparsification property. As a consequence, every connected Lie group satisfies this property.

Definition 3.1. Let X be a metric space. We say that X has metric sparsification property with constant $0 < c \leqslant 1$ (for short we say that X has $\mathrm{MS}(c)$), if there exists a (non-decreasing) function $f : \mathbf{N} \to \mathbf{N}$ such that for all $m \in \mathbf{N}$, and every finite positive Borel measure μ on X, there is a Borel subset $\Omega = \bigsqcup_{i \in I} \Omega_i$ such that

(i) $d(\Omega_i, \Omega_j) \geqslant m$ for all $i \neq j \in I$,
(ii) $\mathrm{Diam}(\Omega_i) \leqslant f(m)$ for all $i \in I$,
(iii) $\mu(\Omega) \geqslant c\mu(X)$.

When we need to be more explicit, we will say that X has $\mathrm{MS}(c)$ with function f. If m and μ are given, and if we want to say that a subset Ω satisfies the conditions of Definition 3.1, we will simply write $\Omega = \Omega(\mu, f, m, c)$.

Remark 3.2. If the asymptotic dimension of a metric space X is n, then one checks easily that X has metric sparsification property with constant $\frac{1}{n+1}$. In particular, Gromov-hyperbolic groups and groups acting geometrically on CAT(0) cubical complexes have metric sparsification property.

Actually, similar to the operator norm localization property, the constant c can be chosen as close as we want to 1.

Proposition 3.3. *If X has metric sparsification property, then it has the property with constant c for all* $0 < c < 1$.

Proof. Assume that X has $\mathrm{MS}(1-\varepsilon)$ with function f for some $0 < \varepsilon < 1$. It is enough to prove that it has $\mathrm{MS}(1-\varepsilon^2)$ with function $\tilde{f}(m) = f(2f(m) + 3m) + f(m) + m$.

First, take $\bigsqcup_{i \in I} \Omega_i = \Omega(\mu, f, m, 1-\varepsilon)$, then look at the finite measure μ' obtained by taking the restriction of μ to the complement of Ω. Applying the property to μ' for $m' = 2f(m) + 3m$, we get a subset $\Omega' = \bigsqcup_{i \in I'} \Omega'_i = \Omega'(\mu', f, m', 1-\varepsilon)$. Now, one can checks that $\tilde{\Omega} = \Omega' \cup \Omega = \tilde{\Omega}(\mu, \tilde{f}, m, 1-\varepsilon^2)$, where $\tilde{f}(m) = f(2f(m) + 3m) + f(m) + m$. Indeed, let us partition I into two subsets I_1 and I_2, I_1 corresponding to subsets Ω_i which are at distance at least m from Ω'. Now, define $\tilde{I} = I' \sqcup I_1$. For $i \in I'$, $\tilde{\Omega}_i$ is the union of Ω'_i with all Ω_j which are at distance $< m$ from Ω'_i; and for $i \in I_1$, $\tilde{\Omega}_i = \Omega_i$. Clearly, if $i \in I_1$, $j \in \tilde{I}$, then $d(\tilde{\Omega}_i, \tilde{\Omega}_j) \geqslant m$. If $i, j \in I_1$, $i \neq j$, then

$$d(\tilde{\Omega}_i, \tilde{\Omega}_j) \geqslant d(\Omega'_i, \Omega'_j) - 2f(m) - 2m \geqslant m' - 2f(m) - 2m \geqslant m.$$

On the other hand, we have

$$\mathrm{Diam}(\tilde{\Omega}_i) \leqslant \mathrm{Diam}(\tilde{\Omega}_i) + f(m) + m \leqslant f\big(2f(m) + 3m\big) + f(m) + m. \qquad \square$$

Definition 3.4. We say that a family of metric spaces has uniform $\mathrm{MS}(c)$ if there is an f that works for all the elements of the family.

Proposition 3.5. *Let X and Y be two metric spaces. If $F : X \to Y$ is a coarse embedding, and if Y has metric sparsification property, then so does X.*

Proof. As F is a coarse embedding, there exist two increasing functions $\rho_1, \rho_2 : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{t\to\infty} \rho_1(t) = \infty$ and

$$\rho_1\big(d(x,x')\big) \leqslant d\big(F(x), F(y)\big) \leqslant \rho_2\big(d(x,x')\big).$$

Assume that Y has $\mathrm{MS}(c)$. Let μ be a finite measure on X and let $m \in \mathbf{N}$. Let $\mu' = F(\mu)$, and let $\Omega' = (\mu', f, m, c)$ (for some f). Let $\Omega = F^{-1}(\Omega')$. Then one immediately checks that $\Omega = \Omega(\mu, \rho_1^{-1} \circ f \circ \rho_2, \rho_2^{-1}(m), c)$. □

Corollary 3.6. *The metric sparsification property is invariant under coarse equivalence.*

Corollary 3.7. *Let X' be a metric space and let $X \subset X'$ be a metric subspace of X', i.e. a Borel subset of X' equipped with the induced distance. Then X has metric sparsification property if X does.*

Let us prove an easy but useful lemma.

Lemma 3.8. *Let X be a metric space and assume that for every $m \in \mathbf{N}$, there is an m-disjoint family of metric subspaces $(X_j)_{j\in I}$ with uniform* $\mathrm{MS}(c)$*. Then there is a function f such that for every $m \in \mathbf{N}$, and every finite measure μ on X supported on $\bigcup_j X_j$, there exists $\Omega = \Omega(\mu, f, m, c)$ (included in $\bigcup_j X_j$).*

Proof. Let μ_j be the restriction of μ to X_j. As (X_j) has uniform $\mathrm{MS}(c)$, there is a function f_m such that for every $j \in J$, there exists $\Omega_j = \Omega_j(\mu_j, f_m, m, c)$. Now take $\Omega = \bigcup_j \Omega_j$. Clearly, $\Omega = \Omega(\mu, f, m, c)$ for $f(m) = f_m(m)$. □

Proposition 3.9. *Let G be a locally compact group equipped with some proper, locally bounded left-invariant metric d. Let G_n be a non-decreasing, exhaustive sequence of open subgroups of G. If the G_n have* $\mathrm{MS}(c)$ *for the same constant $c > 0$, then so does G.*

Proof. Say that for all $n \in \mathbf{N}$, G_n has $\mathrm{MS}(c)$ with function f_n. Fix $m \in \mathbf{N}$ and take a finite measure μ on G. As G is locally compact and d is proper, there exits $n = n(m)$ such that $B(1, m) \subset G_n$. Hence the set of left cosets of G modulo G_n is an m-separated partition of G. Let μ_n be the restriction of μ to G_n. Let $\Omega_n = \Omega_n(\mu_n, f_n, m, c)$. Then $\Omega = \Omega_{n(m)} = \Omega(\mu, f, m, c)$ where $f(m) = f_{n(m)}(m)$. □

Proposition 3.10. *The metric sparsification property is stable under extension. More precisely, let G be a locally compact compactly generated group and let N be a closed normal subgroup of G. Assume that N has* $\mathrm{MS}(c)$ *for the induced metric, and that G/N has* $\mathrm{MS}(c')$*. Then G has* $\mathrm{MS}(cc')$.

Proof. To fix the ideas, we equip G with the word metric associated to a compact generating subset S, and G/N with the word metric associated to $\pi(S)$, where π is the projection $\pi : G \to G/N$.

Fix $m \in \mathbf{N}$ and take some finite measure μ on G. Let $\overline{\mu} = \pi(\mu)$, and let $\overline{\Omega} = \overline{\Omega}(\overline{\mu}, \overline{f}, m, c')$. Hence $\overline{\Omega} = \bigsqcup_{i \in I} \overline{\Omega}_i$, where $\mathrm{Diam}(\overline{\Omega}_i) \leqslant \overline{f}(m)$ and $d(\overline{\Omega}_i, \overline{\Omega}_j) \geqslant m$ for all $i \neq j$. Write $X_i = X_i(m) = \pi^{-1}(\overline{\Omega}_i)$. As π is 1-Lipschitz, we have that for all $i \neq j$,

$$d\big(\pi^{-1}(X_i), \pi^{-1}(X_j)\big) \geqslant m.$$

For every $i \in I$, let $x_i \in \overline{\Omega}_i$ and let $g_i \in G$ such that $x_i = \pi(g_i)$. As N is normal and $X_i \subset B_Q(x_i, m) = x_i B_Q(1, m)$, we have

$$\begin{aligned} g_i N \subset \pi^{-1}(X_i) &\subset g_i B_G\big(1, \overline{f}(m)\big) N \\ &= g_i N B_G\big(1, \overline{f}(m)\big) \\ &= \big\{g \in G,\ d(g, x_i N) \leqslant \overline{f}(m)\big\}. \end{aligned}$$

Hence, the obvious injection $N \to X_i$ where 1 is sent to x_i is a coarse equivalence with ρ_1 and ρ_2 depending only on m, hence uniform in i. Hence the family X_i has uniform MS(c) (see the proof of Proposition 3.5). Let μ_m be the restriction of μ to $\bigcup_i X_i$. Note that $\mu_m(G) = \overline{\mu}(\overline{\Omega}) \geqslant c'\overline{\mu}(G/N) = c'\mu(G)$. By Lemma 3.8, there exists $\Omega = \Omega(\mu_m, f, m, c)$ (for some f). Hence, together with the previous remark, it yields $\Omega = \Omega(\mu, f, m, cc')$. □

Theorem 3.11. *Any solvable locally compact group equipped with a proper locally finite left invariant metric has metric sparsification property.*

As a consequence, the wreath product $\mathbf{Z} \wr \mathbf{Z}$ has metric sparsification property. This implies that $\mathbf{Z} \wr \mathbf{Z}$ has operator norm localization property despite the fact it has infinite asymptotic dimension.

Proof. By Proposition 3.9, we can assume that G is compactly generated. By Proposition 3.10, we can assume that G is abelian, and then again Proposition 3.9 reduces the problem to $G = \mathbf{Z}$ or $\mathbf{R}$, which have asymptotic dimension 1. □

Corollary 3.12. *Every Borel subset of a connected Lie group, or of an algebraic group over $\mathbf{Q}_p$ has metric sparsification property.*

Proof. This follows from the fact that these groups have co-compact solvable closed subgroups, which are therefore coarsely equivalent to them (we can also deduce it from the fact that they have finite asymptotic dimension). □

4. Link to operator norm localization property

In this section, we show that the metric sparsification property implies the operator norm localization property.

Proposition 4.1. *The metric sparsification property implies the operator norm localization property.*

Proof. Let ν be a positive locally finite Borel measure on X. For any measurable subset U of X, let P_U be the orthogonal projector on the space of functions of $L^2(X) \otimes H$ supported on U.

Clearly, $A \in \mathcal{A}_k$ means that for any subsets $U, V \subset X$ such that $d(U, V) > k$, $P_U A P_V = 0$. We deduce that if $\psi \in L^2(X) \otimes H$ is supported in U, then $A\psi$ is supported in $[U]_k := \{x \in X, d(x, U) \leqslant k\}$. As a result, we have

Lemma 4.2. *If $\psi \in L^2(X, \nu) \otimes H$ is a sum of non-zero $\psi_i \in L^2(X, \nu) \otimes H$ whose supports are piecewise at distance larger than $m > 2k$, then*

$$d\big(\operatorname{Supp}(A\psi_i), \operatorname{Supp}(A\psi_j)\big) \geqslant m - 2k > 0$$

for all $i \neq j$.

Consequently,

$$\frac{\|A\psi\|}{\|\psi\|} \leqslant \sup_{i \in I} \frac{\|A\psi_i\|}{\|\psi_i\|}.$$

Now let $k \in \mathbf{N}$ and $A \in \mathcal{A}_k(X, \nu)$. Let $\varphi \in L^2(X, \nu) \otimes H$. Consider the finite measure $d\mu = \|A\varphi\|_H^2 \, d\nu$ and some $m > 2k$. Let $\Omega = \bigcup_{i \in I} \Omega_i = \Omega(\mu, f, 3m, c)$ for some $c < a(X)$, where f is as in Definition 3.1. Let P_Ω be the orthogonal projector on $L^2(\Omega) \otimes H$. Therefore, $P_\Omega \varphi$ is a sum of $\varphi_i = P_{\Omega_i} \varphi \in L^2(X, \nu) \otimes H$ (which we can assume to be non-zero) whose supports are piecewise at distance larger than $3m$ and have diameter at most $f(3m)$. Let $[\Omega]_m = \{x \in X, \ d(x, \Omega) \leqslant m\}$. Note that $\Omega' = [\Omega]_m = \bigcup_{i \in I} [\Omega_i]_m$, and that $\Omega' = \Omega'(\mu, f', m, c)$ with $f'(m) = f(3m)$.

Lemma 4.3. *For all $\psi \in L^2(X, \nu) \otimes H$, $\|A P_{[U]_m} \psi\| \geqslant \|P_U A \psi\|$.*

Proof. As $A \in \mathcal{A}_k(X, \nu)$ and $m > 2k$, we have $P_U A P_{X \setminus [U]_m} = 0$. Hence $P_U A P_{[U]_m} = P_U A$. So the lemma follows. □

Using the first part of Lemmas 4.2 and 4.3, we obtain

$$\begin{aligned} \|A P_{\Omega'} \varphi\|^2 &= \sum_i \|A P_{[\Omega_i]_m} \varphi\|^2 \\ &\geqslant \sum_i \|P_{\Omega_i} A \varphi\|^2 \\ &= \|P_\Omega A \varphi\|^2. \end{aligned}$$

Hence,

$$\frac{\|A P_{\Omega'} \varphi\|^2}{\|P_{\Omega'} \varphi\|^2} \geqslant \frac{\|A P_{\Omega'} \varphi\|^2}{\|\varphi\|^2} \geqslant \frac{\|P_\Omega A \varphi\|^2}{\|\varphi\|^2} = \frac{\mu(\Omega)}{\|\varphi\|^2} \geqslant c \frac{\mu(X)}{\|\varphi\|^2} = c \frac{\|A\varphi\|^2}{\|\varphi\|^2}.$$

Applying this inequality to some $\varphi \in L^2(X, \nu) \otimes H$ such that $\|A\varphi\| / \|\varphi\| \geqslant (1 - \varepsilon)\|A\|$, and applying the second part of Lemma 4.2 to $\psi = P_{\Omega'} \varphi$, we get that X has operator norm localization property with constant $\sqrt{c}(1 - \varepsilon)$, for arbitrary small $\varepsilon > 0$. □

5. Permanence properties for operator norm localization

In this section, we prove several permanence properties for the operator norm localization property.

Let Γ be a group acting on a metric space X. For every $k \geqslant 0$, the k-stabilizer $W_k(x_0)$ of a point $x_0 \in X$ is defined to be the set of all $g \in \Gamma$ with $gx_0 \in B(x_0, k)$, where $B(x_0, k)$ is the closed ball with center x_0 and radius k. The concept of k-stabilizer is introduced by Bell and Dranishnikov in their work on permanence properties of asymptotic dimension [2].

Proposition 5.1. *Let Γ be a finitely generated group acting isometrically on a metric space X. If X has metric sparsification property with a constant $0 < c \leqslant 1$ and there exist $0 < c' \leqslant 1$ and $x_0 \in X$ such that $W_k(x_0)$ has operator norm localization property with constant c' for each $k > 0$, then Γ has operator norm localization property with constant $\sqrt{c}c'$ as a metric space with a word metric.*

Proof. We define a map $\pi : \Gamma \to X$ by: $\pi(g) = gx_0$ for all $g \in \Gamma$. Let S be the finite generating set in the definition of the word metric for Γ and let $\lambda = \max\{d(\gamma x_0, x_0),\ \gamma \in S\}$. It is easy to see that π is λ-Lipschitz, i.e. $d(\pi(x), \pi(y)) \leqslant \lambda d(x, y)$ for all x and y in Γ.

Let ν be a positive locally finite measure on Γ and H be a separable infinite-dimensional Hilbert space. Let $T : \ell^2(\Gamma, \nu) \otimes H \to \ell^2(\Gamma, \nu) \otimes H$, be a bounded linear operator with propagation r for some $r > 0$. For each vector $v \in \ell^2(\Gamma, \nu) \otimes H$, we define a finite measure μ on Γx_0 by

$$\mu\big(\{x\}\big) = \| P_{\pi^{-1}(x)} T v \|^2_{\ell^2(\Gamma,\nu)\otimes H}$$

for every $x \in \Gamma x_0$, where $P_{\pi^{-1}(x)}$ is the projection from $\ell^2(\Gamma, \nu) \otimes H$ to its subspace $\ell^2(\pi^{-1}(x), \nu) \otimes H$.

By the definition of metric sparsification property, there exists a subset $\Omega = \bigsqcup_{i\in I} \Omega_i$ of Γx_0 such that

(i) $d(\Omega_i, \Omega_j) \geqslant (\lambda + 10)(r + 10)$ for all $i \neq j \in I$,
(ii) $\mathrm{Diam}(\Omega_i) \leqslant D$ for some $D > 0$ and all $i \in I$, where D is independent of ν and v,
(iii) $\mu(\Omega) \geqslant c\mu(\Gamma x_0)$.

Notice that there exists $k > 0$ such that $\pi^{-1}(\Omega_i)$ is coarsely equivalent to a subset of $W_k(x_0)$ for all $i \in I$ with a uniform ρ_1 and ρ_2, where ρ_1 and ρ_2 are control functions as in the proof of Proposition 2.5. By Proposition 2.5, $\pi^{-1}(\Omega_i)$ has uniform operator norm localization property with constant c' for all $i \in I$ in the sense that each Ω_i has operator norm localization property with constant c' and the function f in Definition 2.3 is independent of $i \in I$. This, together with the above properties of Ω_i and the fact that T has propagation r, implies that

$$\| P_{\pi^{-1}(\Omega)} T v \|^2 \geqslant c \| T v \|^2$$

and $P_{\pi^{-1}(\Omega)} T$ decomposes

$$P_{\pi^{-1}(\Omega)} T = \bigoplus_{i\in I} T_i,$$

where each T_i is an operator acting on $\ell^2(\{g \in \Gamma\colon d(g, \pi^{-1}(\Omega_i)) \leqslant r\}) \otimes H$ with propagation r. Note that $\{g \in \Gamma\colon d(g, \pi^{-1}(\Omega_i)) \leqslant r\}$ is uniformly coarsely equivalent to $\pi^{-1}(\Omega_i)$ and hence has uniform operator norm localization property with constant c' for all $i \in I$. It follows that Γ has operator norm localization property with constant $\sqrt{cc'}$. □

Next we shall prove the following countable union result for operator norm localization property. We say that a family of metric spaces $\{X_i\}_{i \in I}$ has uniform operator norm localization property with constant $0 < c \leqslant 1$ if X_i has operator norm localization property with constant c for each i and the function f in Definition 2.3 is independent of $i \in I$.

Proposition 5.2. *Let X be a metric space and $X = \bigcup_{i \in I} X_i$, where each X_i is a Borel subset of X. If $\{X_i\}_{i \in I}$ has uniform operator norm localization property and, for each $r > 0$, there exists a Borel subset $Y_r \subseteq X$ having operator norm localization property with constant c such that $\{X_i - Y_r\}_{i \in I}$ is r-disjoint, then X has operator norm localization property.*

Proof. Let ν be a positive locally finite Borel measure on X. Let T be a bounded linear operator acting on $L^2(X, \nu) \otimes H$ with propagation $r > 0$. For every $1 > \delta > 0$, there exists a unit vector $\xi \in L^2(X, \nu) \otimes H$ satisfying $\|T\xi\| \geqslant (1 - \delta)\|T\|$.

Let

$$Z_k = \bigl\{x \in X\colon\ 10(k-1) \leqslant d(x, Y_{10r}) < 10(k+1)r\bigr\}$$

for each $k \in \mathbb{N}$. Let $\xi_k \in L^2(X, \nu) \otimes H$ be defined by: $\xi_k(x) = \xi(x)$ for all $x \in Z_k$ and $\xi_k(x) = 0$ for all $x \in X - Z_k$. We have $\|\xi\|^2 = \sum_k \|\xi_k\|^2$. Hence for each large $N \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ satisfying $\|\xi_{k_0}\| < \frac{1}{N}$.

Let $U_1 = \bigcup_{k<k_0} Z_k$ and $U_2 = \bigcup_{k>k_0} Z_k$. Notice that U_1 and U_2 are $10r$-disjoint if both U_1 and U_2 are non-empty. By our assumptions and Proposition 2.5, the r-neighborhood of U_1 has operator norm localization property with constant c and the r-neighborhood of U_2 is the union of pairwise $5r$-disjoint subsets having uniform operator norm localization property with constant c. Let P_i be the projection from $L^2(X, \nu) \otimes H$ onto $L^2(U_i, \nu) \otimes H$ for $i = 1, 2$. By the choice of k_0 and the fact T has propagation r, we have

$$\max\bigl\{\|TP_1\|, \|TP_2\|\bigr\} \leqslant \|T\| \leqslant \left(\frac{1}{1-\delta} + \frac{1}{N}\right)\max\bigl\{\|TP_1\|, \|TP_2\|\bigr\}.$$

The above inequality, together with our assumptions and the fact that TP_i has propagation r and is supported on the r-neighborhood of U_i, implies our result. □

Corollary 5.3. *Let A and B be two finitely generated groups with a common subgroup C. The amalgamated product $A *_C B$ has operator norm localization property if and only A and B have operator norm localization property.*

Proof. It is enough to prove the "if" part. We follow the strategy in Bell and Dranishnikov [2]. Bell and Dranishnikov constructed a tree on which $A *_C B$ acts isometrically [2]. Recall that a tree has asymptotic dimension 1 and hence has operator norm localization property. Proposition 5.2, together with the argument in the proofs of Theorem 5 and Proposition 4 in [2], shows that the

k-stabilizer of this action has operator norm localization property for each $k > 0$. Our corollary now follows from Proposition 5.1. □

By using Propositions 5.1, 5.2 of this paper and constructions in Section 5 of [2], we can prove the following permanence result for operator norm localization property in the case of HNN extensions.

Corollary 5.4. *Let G be a finitely generated group with a word metric. Let $\phi : A \to G$, be a monomorphism of a subgroup A of G, let G' be the HNN extension of G. If G has operator norm localization property, then G' has operator norm localization property.*

We should point out that similar permanence results for finite asymptotic dimension was obtained by Bell and Dranishnikov in [2].

6. Expanding graphs and operator norm localization property

In this section, we show that any expanding sequence of graphs does not have operator norm localization property. In particular, this implies that any expanding sequence of graphs does not have the metric sparsification property defined in this paper.

For convenience of readers, we briefly recall the concept of expanding graphs [5].

Definition 6.1. Let $X = X(V, E)$ be a finite graph with V as its vertex set and E as its edge set. Define the Cheeger constant of X by

$$h(X) = \inf_{A,B \subseteq V} \frac{|E(A, B)|}{\min(|A|, |B|)},$$

where the infimum is taken over all disjoint partition $V = A \cup B$ and $E(A, B)$ is the set of all edges connecting vertices in A to vertices in B.

Definition 6.2. An infinite sequence of graphs $\{X_n(V_n, E_n)\}_{n=1}^{\infty}$ of bounded degree is said to be a sequence of expanding graphs if there exists $h > 0$ such that $h(X_n) \geqslant h$ for all n and the number of elements in V_n goes to ∞ as $n \to \infty$.

In the sense of probability, most sequences of graphs are expanding [5].

Definition 6.3. The Laplacian Δ of the graph $X = X(V, E)$ is the operator on $l^2(V)$ defined by

$$\Delta f(x) = \sum_{y \in V} \delta_{xy}\big(f(x) - f(y)\big)$$

for every $f \in \ell^2(V)$, where $\delta_{x,y}$ is the number of edges between x and y.

It is not difficult to show that Δ is self-adjoint and positive. Let $\lambda_1(X)$ be the smallest positive eigenvalue.

The following result is well known [5].

Proposition 6.4. *$\{X_n\}_n$ is an expanding sequence of graphs if and only if there exists $\lambda > 0$ such that $\lambda_1(X_n) \geqslant \lambda$ for all n.*

Let $\{X_n\}_{n=1}^{\infty}$ be an infinite sequence of graphs. We define a metric on the disjoint union $\bigcup_n X_n$ such that the restriction of the metric on each connected component of X_n is the natural path metric and $d(X_i, X_j) > i + j$ if $i \neq j$. Let $V = \bigcup_n V_n \subseteq \bigcup_n X_n$ be given its subspace metric.

Theorem 6.5. *If $\{X_n\}_n$ is an infinite expanding sequence of graphs, then the metric space V defined as above does not have operator norm localization property.*

Proof. Let Δ_n be the Laplacian of the graph X_n. Let p_n be the projection from $\ell^2(V_n)$ to the one-dimensional subspace of constant functions. By abuse of notation, we denote the operator $p_n \otimes I$ acting on $\ell^2(V_n) \otimes H$ by p_n and the operator $\Delta_n \otimes I$ acting on $\ell^2(V_n) \otimes H$ by Δ_n. Let $p = \bigoplus_n p_n$ and $\Delta = \bigoplus_n \Delta_n$. We have

$$p = \lim_{t \to +\infty} \exp(-t\Delta),$$

where the limit is taken in operator norm (as operators acting on the Hilbert space $\ell^2(V) \otimes H$). It follows that, for any $\epsilon > 0$, there exist an operator T and $r > 0$ in $B(\ell^2(V) \otimes H)$ such that $\|T - p\| < \epsilon$ and T has propagation r. The fact that T has finite propagation implies that there exists some large N such that $\ell^2(V_n) \otimes H$ is invariant under T and T^* if $n > N$. We denote the restriction of T to $\ell^2(V_n) \otimes H$ by T_n if $n > N$. Consider

$$S_k = \bigoplus_{n \leqslant k} 0 \bigoplus_{n > k} T_n \in B\big(\ell^2(V) \otimes H\big)$$

if $k \geqslant N$ and

$$Q_k = \bigoplus_{n \leqslant k} 0 \bigoplus_{n > k} p_n \in B\big(\ell^2(V) \otimes H\big)$$

if $k \geqslant N$. We observe that S_k has propagation r and $\|S_k - Q_k\| < \epsilon$ if $k \geqslant N$.

Now we assume by contradiction that V has operator norm localization property. By assumption, there exist $C > 0$ (independent of ϵ and r) and R (dependent on r) and a unit vector $v_k \in \ell^2(V) \otimes H$ such that $\|S_k\| \leqslant C\|S_k v_k\|$ and $\mathrm{Diam}(\mathrm{Supp}(v_k)) < R$ for all $k > N$. We have

$$\|S_k v_k\| \leqslant \|Q_k v_k\| + \epsilon$$

for all $k > N$. By the definition of Q_k and the support condition of v_k, we know $\|Q_k v_k\| \to 0$ as $k \to \infty$. Consequently we have

$$\|S_k\| \leqslant (1 + C)\epsilon$$

if k is large enough. However, by the definition of S_k and the fact that Q_k has norm 1, we have

$$\|S_k\| > 1 - \epsilon$$

if $k > N$. This is a contradiction if we choose ϵ small enough and k large enough. □

7. Applications to K-theory

In this section, we discuss applications of the operator norm localization property to the coarse Novikov conjecture.

Let Γ be a finitely generated residually finite group. We can assume that there is a sequence of normal subgroups of finite index

$$\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_i \supseteq \cdots$$

such that

$$\bigcap_{i=1}^{\infty} \Gamma_i = \{e\}.$$

Endow Γ/Γ_i with the quotient metric, that is,

$$d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2)\colon \gamma_1, \gamma_2 \in \Gamma_i\}.$$

Let $X(\Gamma) = \bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$ be the disjoint union of Γ/Γ_i. We give a metric on $X(\Gamma)$ such that its restriction to each Γ/Γ_i is the quotient metric defined above and

$$\lim_{n+m\to\infty,\ n\neq m} d(\Gamma/\Gamma_n, \Gamma/\Gamma_m) = \infty.$$

The metric space $X(\Gamma)$ is called the box metric space [7].

Recall that the strong Novikov conjecture states that the Baum–Connes map $\mu_r : K_*^{\Gamma}(E\Gamma) \to K_*(C_r^*(\Gamma))$, is injective [4,1], where $E\Gamma$ is the universal space for free and proper Γ actions and $C_r^*(\Gamma)$ is the reduced group C^*-algebra.

If X is a discrete metric space with bounded geometry, the coarse geometric Novikov conjecture states that the Baum–Connes map $\mu : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(C^*(X))$, is injective, where $P_d(X)$ is the Rips complex and $C^*(X)$ is the Roe algebra associated to X [9]. If X does not have bounded geometry, then there is a counter-example to the coarse geometric Novikov conjecture [8].

Theorem 7.1. *If Γ has operator norm localization property and the classifying space $E\Gamma/\Gamma$ for free Γ-actions has homotopy type of a compact CW complex, then the strong Novikov conjecture for Γ and all subgroups Γ_n $(n = 1, 2, 3, \ldots)$ implies the coarse geometric Novikov conjecture for $X(\Gamma)$.*

Recall that if Γ is an infinite property T group, then $X(\Gamma)$ is a sequence of expanders [5]. Hence Theorem 7.1 implies the coarse Novikov conjecture for many interesting examples of sequences of expanders. A similar result at the level of maximal C^*-algebra is proved in [3] without the operator norm localization property.

Definition 7.2. (See Roe [6].) Let X be a discrete metric space and let T be an operator acting on $\ell^2(X) \otimes H$ with finite propagation. T is called locally compact if $T_{x,y}$ is compact for every x and y in X, where $T = (T_{x,y})_{x,y\in X}$ is the matrix representation of T with respect to the Hilbert space decomposition $\ell^2(X) \otimes H = \bigoplus_{x\in X}(\delta_x \otimes H)$. We denote by $\mathbb{C}[X]$ the algebra of all locally compact operators acting on $\ell^2(X) \otimes H$ with finite propagation. The Roe algebra $C^*(X)$ is the operator norm closure of $\mathbb{C}[X]$.

If Γ is a finitely generated group with a word metric, we denote by $C^*(|\Gamma|)$ the Roe algebra for Γ as a metric space with a word metric. If Γ' is a subgroup of Γ, we denote by $\mathbb{C}[|\Gamma|]^{\Gamma'}$ the fixed point subalgebra of $\mathbb{C}[|\Gamma|]$, i.e. $\mathbb{C}[|\Gamma|]^{\Gamma'}$ consists of all operators T in $\mathbb{C}[|\Gamma|]$ satisfying $T_{gx,gy} = T_{x,y}$ for all $g \in \Gamma'$ and $x, y \in \Gamma$. We denote by $C^*_{r,\Gamma'}(|\Gamma|)$ the operator norm closure of $\mathbb{C}[|\Gamma|]^{\Gamma'}$.

Let $T \in \mathbb{C}[X(\Gamma)]$. Suppose that T has finite propagation l. Let n be the smallest positive integer such that $d(\gamma, e) > 2l$ for all $\gamma \in \Gamma_n$ and $d_{X(\Gamma)}(\Gamma/\Gamma_i, \Gamma/\Gamma_j) > 2l$ if $i \neq j$ and $i \geqslant n$ and $j \geqslant n$, where e is the identity element in Γ. Let

$$Z = \bigsqcup_{i=1}^{n-1} \Gamma/\Gamma_i, \qquad Y = \bigsqcup_{i=n}^{\infty} \Gamma/\Gamma_i.$$

T decomposes as follows

$$T = T^0 \bigoplus_{i \geqslant n} T_i,$$

where T^0 acts on $\ell^2(Z) \otimes H$ and T_i acts on $\ell^2(\Gamma/\Gamma_i) \otimes H$ for each $i \geqslant n$. Let S_i be the operator acting on $\ell^2(\Gamma) \otimes H$ defined by

$$S_{i;x,y} = \begin{cases} T_{i;[x],[y]}, & \text{if } d(x, y) \leqslant l, \\ 0, & \text{otherwise,} \end{cases}$$

where, for $x, y \in \Gamma$, $S_{i;x,y}$ denotes the (x, y)-entry of the matrix representation of S_i and, for $[x], [y] \in \Gamma/\Gamma_i$, the operator $T_{i;[x],[y]}$ is the $([x], [y])$-entry in the matrix representation of T_i.

We define a map

$$\phi : \mathbb{C}\big[X(\Gamma)\big] \to \prod_{i=1}^{\infty} \mathbb{C}\big[|\Gamma|\big]^{\Gamma_i} \Big/ \bigoplus_{i=1}^{\infty} \mathbb{C}\big[|\Gamma|\big]^{\Gamma_i}$$

by

$$\phi(T) = \Big(\bigoplus_{i<n} 0\Big) \oplus \prod_{i \geqslant n}^{\infty} S_i.$$

It is not difficult to verify that ϕ is a homomorphism.

We should note that this homomorphism is used by N. Higson in his unpublished work on counter-examples to the coarse Baum–Connes conjecture.

Lemma 7.3. *If Γ has operator norm localization property, then ϕ extends to a bounded homomorphism*

$$\phi : C^*\big(X(\Gamma)\big) \to \prod_{i=1}^{\infty} C^*_{r,\Gamma_i}\big(|\Gamma|\big) \Big/ \bigoplus_{i=1}^{\infty} C^*_{r,\Gamma_i}\big(|\Gamma|\big).$$

The proof of this lemma follows from the definition of the operator norm localization property and is therefore omitted. Now the proof of Theorem 7.1 follows from our lemma and the argument in the proof of part III of Theorem 5.2 in [3].

8. Remarks and questions

In this section, we list a few open questions and make several remarks about the operator norm localization and metric sparsification property.

Question 1. Is the metric sparsification property equivalent to the operator norm localization property?

Question 2. Does every finitely generated linear group equipped with a word metric have metric sparsification property or operator norm localization property?

Question 3. Does every CAT(0) group have metric sparsification or operator norm localization property?

Quantitative versions of the metric sparsification or operator norm localization property. In Definitions 3.1 and 2.2, we can consider, for a given constant c, the infimum of all functions f. We will call these functions respectively the MS-profile and the OL-profile of X. Recall that the asymptotic behavior of an increasing function is its class modulo the relation $f \approx g$ if there exists $C > 0$ such that $C^{-1}g(C^{-1}t) - C \leqslant f(t) \leqslant Cg(Ct) + C$. Here are a few facts which result from our proofs:

- The asymptotic behavior of the OL-profile is invariant under quasi-isometries (but obviously not under coarse equivalence) and does not depend on c.
- The asymptotic behavior of the MS-profile is invariant under quasi-isometries for a given c but we do not know if it depends on c. Proposition 3.3 however implies that having polynomial growth (resp. linear growth) for the MS-profile does not depend on c.
- The OL-profile grows asymptotically faster than the MS-profile.
- The best situation (at least for an infinite graph) is when the OL-profile (resp. the MS-profile) grows linearly, which happens in particular for spaces with finite asymptotic dimension of linear type (e.g. Gromov hyperbolic groups, discrete subgroups of connected Lie groups, groups acting properly and cocompactly on CAT(0) cubical complexes …). Is the converse also true?

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