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Book Review

DAVID HILBERT, *Theory of Algebraic Invariants*, translated by Reinhard C. Laubenbacher and edited by Bernd Sturmfels, Cambridge Univ. Press, Cambridge, 1993, xiv + 191pp.

Hilbert wrote, at the end of his 1893 paper [5] proving the existence of a finite set of covariants generating all the covariants of a form with an arbitrary number of variables, "Thus I believe the most important general goals of the theory of function fields generated by invariants have been obtained. [Hiermit sind, glaube ich, die wichtigsten allgemeinen Ziele einer Theorie der durch die Invarianten gebildeten Funktionenkörper erreicht.]" This is the starting point of the story, perpetuated more or less uncritically to present day, that Hilbert "killed" invariant theory. It is therefore ironic that in 1993, exactly 100 years later, there is sufficient interest in invariant theory for a translation of a set of lecture notes by Hilbert on that very subject to be published.

The book under review is a translation of notes from Hilbert's lectures in the summer of 1897 prepared by Sophus Marxsen, who was Hilbert's assistant at that time. The book has two parts. The first part is a short introduction to the classical core of invariant theory, with topics skillfully chosen so that a comprehensive yet self-contained account can be rendered within the given number of hours. The treatment is standard and might well have appeared in the more elementary portions of the books of Gordan [4] or Salmon [10]. Incidentally, Hilbert thought that the latter book and the book of Faà di Bruno [3] are "the best introduction to invariant theory." The topics covered in the first part are: definitions of forms, invariants and covariants, decomposition of a linear transformation into elementary linear transformations, differential operators on polynomials, characterizations of covariants by differential equations, recursions and generating functions for the dimensions of vector spaces of covariants having given degrees and weights, Hermite's reciprocity theorem, transvectants, the representation of the product of a covariant times a suitable power of the base form as a polynomial of transvectants, the covariants of the quadratic, cubic and quartic, solutions of the cubic and quartic equations by radicals using covariants and their syzygies, irrational invariants, simultaneous covariants of a family of base forms, construction of new covariants from old by transvection, polars, and the Aronhold process, representations of covariants by roots, and a brief account of the symbolic

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method. The final remark to Part I give a summary of how Hilbert viewed classical invariant theory.

The second part is mostly concerned with Hilbert's own contributions to invariant theory. It begins with two proofs of the theorem that there exists a finite set of covariants generating all the covariants of a form. The "finiteness principle" used in the second proof is the Hilbert basis theorem, which assures the existence of a finite set of generators for any ideal in a ring of polynomials in finitely many variables over a field. An essential and by no means easy ingredient needed to complete this proof is the Cayley omega operator and Cayley's theorem that the omega operator applied to certain polynomials a suitable number of times yields covariants. Unlike some modern treatment, where this point is hastily dismissed because of its symbol-manipulative nature, a careful proof for the case of binary forms is given in these notes. The finiteness theorem is used to study "null forms," that is, forms for which all the *in*variants vanish. The fundamental result here is the theorem: If $I_1, I_2, ..., I_\mu$ are invariants such that $I_1 = 0$, $I_2 = 0, ..., I_\mu = 0$ imply that every invariant is zero, then every invariant can be expressed as an integral algebraic function of $I_1, I_2, ..., I_\mu$. This theorem was the inspiration for the *Nullstellensatz*; unfortunately, the *Nullstellensatz* is only stated and no proof is given. Next, syzygies, or relations between invariants, for binary forms of degree 3 and 4 and for systems of two and three quadratics are computed. Using syzygies, characterizations of the binary null forms in terms of multiple linear factors and ternary null forms using Newton polygons are given, in addition to the Hilbert syzygy theorem, which asserts the existence of a finite free resolution. The book ends with a geometric application, the inflexion point problem for plane curves of order 3, and a glimpse of Plücker and other coordinates.

Compared with other books in which Hilbert had a direct hand—the earnest axiomatic style of "Foundations of Geometry" and the sheer indigestibility of his books on logic and foundations come to mind—this book attempts much less and is aimed at the beginning student. It is in fact a somewhat conventional textbook. There is perhaps a biographical explanation for this. According to Constance Reid [9, Chap. 7], Hilbert left invariant theory in 1893 and spent much of the time until April 1897 writing the *Zahlbericht*. In the winter of 1898, he lectured on the foundations of geometry. Thus, the lectures on invariant theory was an interlude. If only we had his lecture notes from Königsberg.

As the editor pointed out, this set of lecture notes has potential uses as a historical source. Therefore, it is not inappropriate to point out one or two significant details in the second part. Part II starts with two proofs of the finite basis theorem by Hilbert. The first, which applies only to binary forms, is built from two main ideas: the representation of the invariants as polynomials in the differences of roots and a finiteness principle— Lemma 2 on page 117—stating that "a system of linear homogeneous diophantine equations has a finite number of nonnegative solutions such that every other nonnegative solution can be obtained from these linearly and homogeneously with nonnegative integer coefficients." Lemma 2 has many similarities with the better known Hilbert basis theorem. Both are proved inductively, by induction on the number of variables. Neither proof is explicitly constructive, but both can be made constructive. When both theorems are set side by side, it seems clear that Lemma 2 is the source for the Hilbert basis theorem. The notes gives the impression that every component of the first proof is due to Hilbert. The introductory paragraphs (pp. 115–116) contain the following sentences: "Later, Gordan proved this theorem ... using the symbolic method. But this proof is very cumbersome. (...) But then, Hilbert gave a very simple and relatively short proof of the same fact." The same impression is given in later biographies of Hilbert (see [11, 9]). In addition, the editor seemed to have attributed Lemma 2 to Hilbert, since he added a footnote calling a minimal generating set of nonnegative solutions a "Hilbert basis."

The curious point about all this, assuming that these notes are an accurate transcription of Hilbert's lectures, is that Lemma 2 *is in fact due to Gordan*, who proved the finiteness theorem for binary forms in 1868 using Lemma 2 and taking transvectants. See page 196 of the first volume of Gordan's book [4]; an account in English can be found in [5]. Hilbert referred to Lemma 2 and more generally acknowledged his debt to earlier work in [6], where he wrote "In the following, a different proof for (the finiteness) theorem will be given. (...) This proof has close analogies with the original method of P. Gordan and (...) runs parallel to the train of thought of the proof given by F. Mertens. [Im folgenden wird für diesen fundamentalen Satz ein anderer Beweis erbracht, welcher mit dem ursprünglichen Verfahren von P. Gordan nahe Analogien aufweist, während andererseits der Gedankengang dem von F. Mertens gegebenen Beweise parallel läuft.]"

Another historical surprise concerns Hilbert's views on constructibility in 1897. In Lecture 37, we read

With each mathematical theorem, three things are to be distinguished. First one needs to settle the basic question of whether the theorem is valid; one has to prove its existence, so to speak. Second, one can ask whether there is any way to determine how many operations are needed at the most to carry out the assertions of the theorem. (...) Third, it has to be actually carried out; this is the least interesting question.

Following this approach, Hilbert considered in later lectures how his proof can be made more constructive. Thus, it seems that in invariant theory, constructibility was not a burning issue in 1897. One should be careful not to project the bitter foundational disputes of the 1920s—that battle between the frogs and the mice, in the words of Einstein—back to the 1890s. It is also implausible that Gordan disbelieved either of Hilbert's proofs, since the first used Gordan's own lemma and the second had the same degree of nonconstructiveness as Gordan's own. The famous remark (see [11]), that the proof "is theology," if he had made it at all, might well have been made in the sense of Erdős, that the proof came from God's book.

It is perhaps time that we rehabilitate Gordan's reputation. Far from a mathematician who left "a curiously negative mark upon the history of mathematics" [9, p. 29], Gordan's work included one of the first existence proofs in algebra. Another major result of Gordan, equally far ahead of his time, can be found in [1]. In equal measure, we should also reassess the traditional image—derived from German romanticism mixed with more than a tinge of the *Führerprinzip*—of Hilbert as a lonely misunderstood hero. The strength of Hilbert's work, it seems to me, lies more in the clarity with which he saw through the ideas of his contemporaries than in sheer originality. A scholarly study of invariant theory in the second half of the nineteenth century, supplementing Fisher's pioneering sociological study [2], should be one of the top priorities in the history of mathematics.

It remains to say that the editor, the translator, and the publisher have done an excellent job. This attractive yet affordable little book is a very readable introduction to invariant theory; indeed, it is the only one giving equal emphasis to both the classical and modern approaches. This is a book not only for invariant theorists, but also for algebraists who are interested in the origins of their subject.

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