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Poisson Equations and Morrey Spaces

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INTRODUCTION

In this paper we study the Dirichlet problem for equation

$$Lu = -(a_{ij} u_{x_i})_{x_j} = f, \qquad (*)$$

where L is a second-order uniformly elliptic operator in a bounded domain Ω and f is taken in the Morrey space $L^{1,\lambda}$, $0<\lambda<n$. (For the precise statement see Section 1).

The solution we consider is a very weak one introduced in [LSW] because in general the Dirichlet problem for Eq. $(*)$ does not have a weak (variational) solution under our assumption on f.

The purpose of our work is to study the regularity properties of the solution as the parameter λ increases from 0 to n. In fact we prove L^p regularity for $0<\lambda<n-2$, and that the solution belongs (locally) to the space BMO if $\lambda=n-2$. For $\lambda>n-2$ we prove the local Hölder continuity of the solution.

Furthermore, we consider the cases in which f belongs to some spaces related to the so-called Stummel–Kato classes which fall in between $L^{1,n-2}$ and $L^{1,\lambda}(\lambda>n-2)$ obtaining boundedness and continuity of the solution. The study of the Stumel and $L^{1,\lambda}(\lambda>n-2)$ cases is much along the lines of [CFG] and our previous work [D].

We also show that if f is taken in the Stummel class S (in fact in a slightly larger one) the solution has its gradient in $L^2(\Omega)$ and then it is a variational solution. The same is obviously true in the case $L^{1,\lambda}$ $(\lambda>n-2)$ because of the inclusion $L^{1,\lambda} \subseteq S$ $(\lambda>n-2)$.

Finally we give some regularity results for the gradient of the solution for $\lambda>n-2$ under some regularity hypotheses on Ω and $(a_{ij}(x))$.

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1. Preliminary Results

Let Ω be an open bounded subset of $\mathbb{R}^n$ ($n \geqslant 3$). In what follows some regularity on $\partial\Omega$ is needed. Precise assumptions are made in the statement of the theorems. However, we require at least the following condition to be satisfied in

$$\exists A \in]0, 1[: |\Omega_r(x)| \geqslant A \; |B_r(x)| \qquad 0 < r < \operatorname{diam} \Omega, \tag{1.1}$$

where $B_r(x)$ is the closed ball centered at x with radius r and $\Omega_r(x) := B_r(x) \cap \Omega$.

Here and in the following we set $|E|$ for the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^n$.

For $0 < \lambda < n$ and $f \in L^1(\Omega)$ we let $\|f\|_{1,\lambda} := \sup_{0 < r < \operatorname{diam} \Omega,\, x \in \Omega} r^{-\lambda} \int_{\Omega_r(x)} |f(y)| \, dy$. $L^{1,\lambda}(\Omega) := \{f \in L^1(\Omega) : \|f\|_{1,\lambda} < +\infty\}$ is the classical Morrey space. We also use some other function spaces.

For $f \in L^1_{\mathrm{loc}}(\Omega)$ we let $\|f\|_{\mathrm{BMO}} := \sup ⨍_B |f - f_B| \, dx$, where the sup is taken on the closed balls $B \subseteq \Omega$. Also the symbol $⨍$ stands for the average and $f_B = ⨍_B f(x) \, dx$. $\mathrm{BMO} := \{f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{\mathrm{BMO}} < +\infty\}$. We also set

$$\tilde{S}(\Omega) := \left\{ f \in L^1(\Omega) : \sup_{\substack{x \in \Omega \\ r > 0}} \int_{\Omega_r(x)} |f(y)| \, |x - y|^{2-n} \, dy < +\infty \right\}$$

and

$$S(\Omega) := \left\{ f \in L^1(\Omega) : \sup_{\substack{x \in \Omega \\ r > 0}} \int_{\Omega_r(x)} |f(y)| \, |x - y|^{2-n} \, dy \leqslant \eta(r) \right\}$$

for some increasing function η: $]0, \operatorname{diam} \Omega[\to \mathbb{R}^+$ that $\lim_{r \to 0} \eta(r) = 0$.

Obviously $S \subseteq \tilde{S}$. S and $\tilde{S}$ are variants of the classical Stummel–Kato classes.

It is easy to show that (see [D])

$$L^{1,\lambda}(\Omega) \subseteq S(\Omega) \subseteq \tilde{S}(\Omega) \subseteq L^{1,\mu}(\Omega) \qquad \text{for} \quad 0 < \mu \leqslant n - 2 < \lambda < n.$$

We also use the Sobolev space $H_0^{1,p}(\Omega)$ $1 \leqslant p \leqslant +\infty$ and their duals $H^{-1,q}(\Omega)$ $(1/p + 1/q = 1)$. Also $H_0^{1,2}(\Omega) \equiv H_0^1(\Omega)$ and $H^{-1,2}(\Omega) \equiv H^{-1}(\Omega)$. We consider in Ω the Dirichlet problem for the divergence form equation

$$Lu \equiv -(a_{ij}(x) u_{x_i})_{x_j} = f, \tag{1.2}$$

where we assume

$$\begin{gathered} a_{ij}(x) \in L^\infty(\Omega), \qquad a_{ij}(x) = a_{ji}(x) \qquad \text{for} \quad i, j = 1, \ldots, n \\ \exists \nu \in \mathbb{R}^+ : \nu^{-1} \, |\xi|^2 \leqslant a_{ij}(x) \xi_i \xi_j \leqslant \nu \, |\xi|^2 \; \forall \xi \in \mathbb{R}^n, \end{gathered} \tag{1.3}$$

a.e. x in Ω.

We make various assumptions on the known term f which, for the time being, we suppose to be an $L^1(\Omega)$ function.

Under this assumption there is, in general, no hope for the existence of a variational solution to the Dirichlet problem for (1.2).

Then, following [LSW], we give the

DEFINITION 1. Let μ be a bounded variation measure in Ω. We say that $u \in L^1(\Omega)$ is a very weak soution to the Dirichlet problem in Ω for equation $Lu = \mu$ iff

$$\int_\Omega uL\varphi \, dx = \int_\Omega \varphi \, d\mu \qquad \forall \varphi \in H_0^1(\Omega) \cap C^0(\bar{\Omega}) \text{ such that } L\varphi \in C^0(\bar{\Omega}). \tag{1.4}$$

It is well known (see [ST]) that under our assumptions (1.1), (1.3) there is a unique very weak solution. In the case $\mu = \delta_y$, $y \in \Omega$, the corresponding very weak solution $g(x, y)$ is called the *Green function* for L and we have (see [ST]) the representation formula

$$u(x) = \int_\Omega g(x, y) \, d\mu \qquad \text{a.e. in } \Omega.$$

It is also clear, from the definitions, that a weak solution to the Dirichlet problem, i.e.,

$$u \in H_0^1(\Omega) : \int_\Omega a_{ij} u_{x_i} \psi_{x_j} \, dx = \int_\Omega \psi \, d\mu \qquad \forall \psi \in H_0^1(\Omega),$$

whenever it exists, is the very weak solution in the sense of the above definition.

It is also known [LSW, p. 67] that $g(x, y)$ satisfies the estimate

$$C_1 |x-y|^{2-n} \leqslant g(x, y) \leqslant C_2 |x-y|^{2-n}, \tag{1.5}$$

where C_1 and C_2 are positive constants depending only on n and ν and the right-hand side estimate is true under the assumption $|x-y| \leqslant 1/4 \, d(y, \partial\Omega)$.

In their paper [GW] Grüter and Widman give a different definition of Green function establishing many useful estimates which we need in the following.

Precisely Grüter and Widman define, in their theorem (1.1), as a Green function $\tilde{g}(x, y)$ for the operator L the unique nonnegative function such that

$$\forall y \in \Omega \quad \tilde{g}(\cdot, y) \in H^1(\Omega \setminus B_r(y)) \cap H_0^{1,1}(\Omega) \qquad \forall r \in \,]0, \operatorname{dist}(y, \partial\Omega)[$$

and satisfying

$$\int_\Omega a_{ij}(x)\, g_{x_i}(x, y) \varphi_{x_j}(x)\, dx = \varphi(y) \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{1.6}$$

LEMMA 1.1. *Assume* (1.1), (1.3), *and suppose* $a_{ij}(x)$ *are so smooth that for* $\forall \eta \in H_0^{1,\infty}(\Omega)$ *the weak solution* φ *to the Dirichlet problem*

$$\begin{aligned} L\varphi &= \eta \\ \varphi &\in H_0^1(\Omega) \end{aligned} \qquad \textit{is in } H_0^{1,p} \textit{ for some } p > n.^1 \tag{1.7}$$

Then

$$g(x, y) = \tilde{g}(x, y).$$

Proof. Let φ be the solution of (1.7). Then $\int_\Omega a_{ij}(x) \varphi_{x_i}(x) \psi_{x_j}(x)\, dx = \int_\Omega \eta(x) \psi_{x_j}(x)\, dx$ for any $\psi \in H_0^1(\Omega)$ and also, by density, for all $\psi \in H^{1,p'}(\Omega)$ $(1/p + 1/p' = 1,\ p' < n/n-1)$. On the other hand by [GW, Theorem 1.1, (1.7)] we know that $\tilde{g}(\cdot, y)$ is in $H_0^{1,s}(\Omega)$ for any $s \in [1, n/n-1[$. Hence

$$\int_\Omega a_{ij}(x) \varphi_{x_i}(x)\, \tilde{g}_{x_j}(x, y)\, dx = \int_\Omega \eta(x)\, \tilde{g}(x, y)\, dx.$$

By definition (1.6) the solution $\varphi \in H_0^{1,p}(\Omega)$ $(p > n)$ of (1.7) satisfies

$$\int_\Omega a_{ij}(x) \varphi_{x_i}(x)\, \tilde{g}_{x_j}(x, y)\, dx = \varphi(y)$$

and then

$$\int_\Omega \eta(x)\, \tilde{g}(x, y)\, dx = \varphi(y). \tag{1.8}$$

By the [LSW] definition of $g(x, y)$ using $L\varphi = \eta$ we also have

$$\int_\Omega \eta(x)\, g(x, y)\, dx = \varphi(y). \tag{1.9}$$

Finally, by (1.8) and (1.9),

$$\int_\Omega \eta(x)\, g(x, y)\, dx = \int_\Omega \eta(x)\, \tilde{g}(x, y)\, dx \qquad \forall \eta \in H_0^{1,\infty}(\Omega)$$

and this in turn implies $g(x, y) = \tilde{g}(x, y)$.

[1] $a_{ij}(x) \in C^0(\bar{\Omega})$ will suffice.

We now quote an estimate from [GW] which is useful in the next section.

LEMMA 1.2 [GW, Theorem 3.3]. *Suppose Ω satisfies a uniform exterior sphere condition. Assume* (1.3) *are satisfied and furthermore suppose a_{ij} are Dini-continuous. Then*

$$|g_x(x, y)| \leq K|x-y|^{1-n} \qquad \forall x, y \in \Omega, \tag{1.10}$$

where K is a positive constant depending only on n, v, Ω and the modulus of continuity of the coefficients.

Observe that in the above lemma we denoted by g the Green function using Lemma 1. Lemma 1.2 enables us to prove

LEMMA 1.3. *Consider the very weak solution u to the Dirichlet problem for $Lu=f$, where we suppose the assumption of Lemma* 1.2 *to be satisfied by Ω and a_{ij} and f is any $L^1(\Omega)$ function. Then*

$$u_{x_i}(x) = \int_\Omega g_{x_i}(x, y) f(y)\, dy.$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$ and consider

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial x_i} \int_\Omega g(x, y) f(y)\, dy, \varphi(x) \right\rangle \\
&\quad = -\int_\Omega \left(\int_\Omega g(x, y) f(y)\, dy \right) \frac{\partial \varphi}{\partial x_i}\, dx \\
&\quad = -\int_\Omega f(y) \left(\int_\Omega g(x, y) \frac{\partial \varphi}{\partial x_i}\, dx \right) dy \\
&\quad = \int_\Omega f(y) \left(\int_\Omega g_{x_i}(x, y) \varphi(x)\, dx \right) dy \\
&\quad = \int_\Omega \left(\int_\Omega g_{x_i}(x, y) f(y)\, dy \right) \varphi(x)\, dx \\
&\quad = \left\langle \int_\Omega g_{x_i}(x, y) f(y)\, dy, \varphi(x) \right\rangle \ \forall \varphi \in C_0^\infty(\Omega).
\end{aligned}$$

In the above calculation we exchanged twice the order of integrations.

This is possible because both $g(x, y) f(y) \varphi_{x_i}(x)$ and $g_{x_i}(x, y) f(y) \varphi(x)$ are in $L^1(\Omega \times \Omega)$. Indeed, using (1.5)

$$\iint_{\Omega \times \Omega} |g(x, y) f(y) \varphi_{x_i}(x)| \, dy \, dx$$

$$\leqslant C \max_{\Omega} |\varphi_{x_i}(x)| \iint_{\Omega \times \Omega} |f(y)| \, |x-y|^{2-n} \, dy \, dx < +\infty.$$

Similarly, using (1.10)

$$\iint_{\Omega \times \Omega} |g_{x_i}(x, y) f(y) \varphi(y)| \, dy \, dx$$

$$\leqslant C \max_{\Omega} |\varphi(x)| \iint_{\Omega \times \Omega} |f(y)| \, |x-y|^{1-n} \, dy \, dx < +\infty.$$

We conclude this section with a simple result:

LEMMA 1.4. ([2]) *$\tilde{S}(\Omega) \subseteq H^{-1}(\Omega)$. In particular $S(\Omega)$ and $L^{1,\lambda}$ $(\lambda > n-2)$ are also contained in $H^{-1}(\Omega)$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$, $f \in \tilde{S}(\Omega)$.

$$|\langle f, \varphi \rangle| \leqslant C \int_\Omega |f(x)| \left(\int_\Omega |\nabla \varphi(y)| \, |x-y|^{1-n} \, dy \right) dx$$

$$= C \int_\Omega |\nabla \varphi(y)| \left(\int_\Omega |f(x)| \, |x-y|^{1-n} \, dx \right) dy$$

$$\leqslant C \, \| \, |\nabla \varphi| \, \|_{L^2(\Omega)} \, \|I_1(f)\|_{L^2(\Omega)},$$

where we set $I_1(f)(x) = \int_\Omega |f(x)| \, |x-y|^{1-n} \, dx$.

But

$$\|I_1(f)\|_{L^2(\Omega)} = \left[\int_\Omega \left(\int_\Omega |f(x)| \, |x-y|^{1-n} \, dx \right) \times \left(\int_\Omega |f(z)| \, |z-y|^{1-n} \, dz \right) dy \right]^{1/2}$$

$$= \left[\int_\Omega |f(z)| \left[\int_\Omega |f(x)| \times \left(\int_\Omega |x-y|^{1-n} \, |z-y|^{1-n} \, dy \right) dx \right] dz \right]^{1/2}.$$

[2] We thank Prof. C. Simader for pointing out the proof of the lemma.

Using well-known properties of the Riesz kernels (see, e.g., [LA, p. 45 (1.1.3)]) we estimate the inner integral by a multiple of $|x-z|^{2-n}$ obtaining

$$\| I_1 |f| \|_{L^2(\Omega)} \leqslant C \left[\int_\Omega |f(z)| \left(\int_\Omega |f(x)| \, |x-z|^{2-n} \, dx \right) dz \right]^{1/2}. \quad (1.11)$$

By the assumption $f \in \tilde{S}$ the inner integral is uniformly bounded in Ω and the conclusion follows. The quantity on the right-hand side of (1.11) is the energy of the measure $|f(x)|$ (see, e.g., [LA, p. 77 (1.4.1)]).

2. Regularity Results

In this section we consider the very weak solution (see (1.4)) to the Dirichlet problem

$$\begin{aligned} Lu &= f \\ u_{\partial\Omega} &= 0. \end{aligned} \quad (2.1)$$

Here and in the following L is defined by (1.2) and (1.1), (1.3) are satisfied. f belongs to some $L^{1,\lambda}(\Omega)$, $0 < \lambda < n$.

Theorem 2.1. *Let $\lambda \in \,]0, n-2[$, $f \in L^{1,\lambda}(\Omega)$. Then the solution u to* (2.1) *is in the weak $L^{p_\lambda}(\Omega)$ space, where $1/p_\lambda = 1 - 2/(n-\lambda)$. In particular $u \in L^p(\Omega)$ for all $p < p_\lambda$.*

Proof. We have

$$|u(x)| \leqslant \int_\Omega g(x, y) \, |f(y)| \, dy \leqslant C \int_\Omega |f(y)| \, |x-y|^{2-n} \, dy$$

and the conclusion follows by known properties of the Newtonian potential (see, e.g., [A] or [CF]).

Theorem 2.2. *Let $\lambda = n-2$. Then the solution u to* (2.1) *is locally in $BMO(\Omega)$, i.e., $\forall \Omega' \Subset \Omega$, $d := \operatorname{dist}(\Omega', \partial\Omega)$, there is a positive constant $C = C(n, \nu, d)$ such that for any $B_r(x)$, with $x \in \Omega'$ and $0 < r < d/2$*

$$⨍_{B_r(x)} |u(x) - u_{B_r(x)}| \, dx \leqslant C.$$

Proof. Let $B \equiv B_r(x_0)$ one of the balls in the conclusion of the theorem. Set

$$f_1 := f_{\chi_{B^*}}, \qquad f_2 := f(1-\chi_{B^*}), \qquad \text{where} \quad B^* \equiv B_{2r}(x_0).$$

Then we have $u = u_1 + u_2$, where $u_1(x) = \int_\Omega g(x, y) f_1(y)\, dy$ and $u_2(x) = \int_\Omega g(x, y) f_2(y)\, dy$ are the very weak solutions of

$$\begin{matrix} Lu = f_1 \\ u_{1_{\partial\Omega}} = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} Lu_2 = f_2 \\ u_{2_{\partial\Omega}} = 0 \end{matrix} \quad \text{respectively.}$$

We now estimate the BMO norm of u_1 and u_2.

As for u_1 we have

$$\fint_B |u_1(x) - u_{1_B}|\, dx \leqslant 2\, |u_1|_B \leqslant 2C \int_{B^*} |f(y)| \fint_B |x-y|^{2-n}\, dx\, dy$$

$$\leqslant C r^{2-n} \int_{B^*} |f(y)|\, dy \leqslant C \|f\|_{L^{1,n-2}(\Omega)},$$

where we used estimate (1.5).

The estimate for u_2 is slightly more delicate.

$$\fint_B |u_2(x) - u_{2_B}|\, dx$$

$$= \fint_B \left| \int_{\Omega \setminus B^*} g(x, y) f(y)\, dy - \fint_B \int_{\Omega \setminus B^*} g(z, y) f(y)\, dy\, dz \right| dx$$

$$= \fint_B \left| \int_{\Omega \setminus B^*} \left[g(x, y) - \fint_B g(z, y)\, dz \right] f(y)\, dy \right| dx$$

$$\leqslant \fint_B \int_{\Omega \setminus B^*} \left| g(x, y) - \fint_B g(z, y)\, dz \right| |f(y)|\, dy\, dx$$

$$= \int_{\Omega \setminus B^*} |f(y)| \fint_B \left| g(x, y) - \fint_B g(z, y)\, dz \right| dx\, dy = \int_{\Omega_d} \cdots + \int_{\Omega^d} \cdots,$$

where we set $\Omega_d = \{y : |x_0 - y| \leqslant d\} \cap (\Omega \setminus B^*)$ and $\Omega^d = \{y : |x_0 - y| > d\} \cap (\Omega \setminus B^*)$. To estimate the first integral we use the fact that, when restricted to B, $g(\cdot, y)$ is a weak solution to the equation $Lu = 0$, and then we can apply De Giorgi–Nash's theorem to obtain

$$\fint_B \left| g(x, y) - \fint_B g(z, y)\, dz \right| dx$$

$$\leqslant C(n, \nu) \left(\fint_{B^{(x_0)}_{|x_0 - y|/2}} g^2(x, y)\, dx \right)^{1/2} (2r/|x_0 - y|)^\alpha.$$

Now, using Harnack's inequality together with estimate (1.5), we get

$$\fint_B \left| g(x, y) - \fint_B g(z, y)\, dz \right| dx \leqslant C(n, v) r^\alpha\, |x_0 - y|^{2-n-\alpha}.$$

Then, for the first integral, we have

$$\int_{\Omega_d} |f(y)| \fint_B \left| g(x, y) - \fint_B g(z, y)\, dz \right| dx\, dy$$

$$\leqslant C r^\alpha \int_{\Omega_d} |f(y)|\, |x_0 - y|^{2-n-\alpha}\, dy$$

and, setting $\Omega_{d,k} := \{ y \in \Omega_d : 2^k r \leqslant |x_0 - y| \leqslant 2^{k+1} r,\ k \in \mathbb{N} \}$,

$$\int_{\Omega_d} |f(y)|\, |x_0 - y|^{2-n-\alpha}\, dy$$

$$= \sum_{k=1}^{+\infty} \int_{\Omega_{d,k}} |f(y)|\, |x_0 - y|^{2-n-\alpha}\, dy \leqslant C r^{-\alpha}\, \|f\|_{L^{1,n-2}(\Omega)}.$$

To bound the second term we observe that for any $x \in B$ $|x - y| \geqslant |x_0 - y| - r > d/2$.

Hence, once more using (1.5),

$$\int_{\Omega^d} |f(y)| \left(\fint_B \left| g(x, y) - \fint_B g(z, y)\, dz \right| dx \right) dy$$

$$\leqslant C d^{2-n} \int_{\Omega^d} |f(y)|\, dy \leqslant C\, \|f\|_{L^{1,n-2}(\Omega)}$$

and this completes the proof.

We now turn to the study of the regularity properties of the solution u if f is in $\tilde{S}$ and in its subspaces S and $L^{1,\lambda}(\Omega)$, $\lambda > n-2$. Let us remark that, because of Lemma 1.4, $|\nabla u| \in L^2(\Omega)$ and the very weak solution is in fact the weak (variational) solution to the Dirichlet problem (2.1)

THEOREM 2.3. *Let $f \in \tilde{S}$. Then the solution u to* (2.1) *is bounded in Ω.*

Proof. We have

$$|u(x)| \leqslant \int_\Omega g(x, y)\, |f(y)|\, dy \leqslant C \int_\Omega |f(y)|\, |x - y|^{2-n}\, dy$$

$$\leqslant C \sup_{\substack{x \in \Omega \\ r > 0}} \int_{\Omega_r(x)} |f(y)|\, |x - y|^{2-n}\, dy.$$

THEOREM 2.4. *Let $f \in S$. Then the solution u to* (2.1) *is continuous in Ω.*

THEOREM 2.5. *Let $f \in L^{1,\lambda}$ $(n-2<\lambda<n)$. Then the solution u to* (2.1) *is locally Hölder-continuous in Ω.*

Proof of Theorems 2.4, 2.5. The proof very closely follows the proof of Theorem 3.1 in [CFG] and Theorem 3.1 in [D].

In both the cases, obviously, we must substitute f in place of Vu in the equations considered.

We conclude this section with some results on the regularity of the gradient of the solution u to (2.1).

THEOREM 2.6. *Suppose Ω satisfies a uniformly exterior sphere condition. Assume* (1.3) *are satisfied and a_{ij} are Dini-continuous in Ω. Let $f \in L^{1,\lambda}$ $\lambda \in]n-2, n-1[$.*

Then the gradient of the solution u to (2.1) *is in the weak L^{p_λ} space where $p_\lambda = (n-\lambda)/(n-\lambda-1)$.*

Proof. We have, by Lemmas 1.2 and 1.3,

$$|u_{x_i}| \leq \int_\Omega |g_{x_i}(x, y)| \, |f(y)| \, dy \leq K \int_\Omega |f(y)| \, |x-y|^{1-n} \, dy$$

and the right-hand side is in the weak L^{p_λ} space of the conclusion by a theorem of Adams (see [A] or [CF)].

THEOREM 2.7. *Suppose Ω satisfies a uniform exterior sphere condition. Assume* (1.3) *are satisfied with a_{ij} Hölder-continuous. Let $f \in L^{1,n-1}$. Then the gradient of the solution u to* (2.1) *is locally in BMO* (*see Theorem* 2.2. *above*).

Indeed, take f_1 and f_2 as in Theorem 2.2. By the linearity and Lemma 1.3 we have

$$u_{x_i} = (u_1)_{x_i} + (u_2)_{x_i} = \int_\Omega g_{x_i}(x, y) f_1(y) \, dy + \int_\Omega g_{x_i}(x, y) f_2(y) \, dy.$$

Now the proof goes on in much the same way as Theorem 2.2 using the bound for the gradient of the Green function given by Lemma 1.2 and the Hölder estimate in [GW] Theorem 3.5].

Similarly we have

THEOREM 2.8. *Under the same assumptions on Ω and a_{ij} of the previous theorem, if $f \in L^{1,\lambda}$, $n-1<\lambda<n$, we may conclude that $|\nabla u|$ is locally in $C^{0,\alpha}(\Omega)$.*

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