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Nearest interval, triangular and trapezoidal approximation of a fuzzy number preserving ambiguity

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ABSTRACT

The ambiguity was introduced to simplify the task of representing and handling of fuzzy numbers. We find the nearest real interval, nearest triangular (symmetric) fuzzy number, nearest trapezoidal (symmetric) fuzzy number of a fuzzy number, with respect to average Euclidean distance, preserving the ambiguity. A simpler and elementary method, to avoid the Karush–Kuhn–Tucker theorem and the laborious calculus associated with it and to prove the continuity is used. We give algorithms for calculus and several examples. The approximations are discussed in relation to data aggregation.

1. Introduction

The fuzzy numbers are often used to represent the uncertain and incomplete information in decision making, linguistic controllers, expert systems, data mining, pattern recognition, etc. Because complicated fuzzy numbers may cause important difficulties in data processing, different kinds of approximations (interval, triangular, trapezoidal, parametric) were proposed in many recent papers [1–4,6–10,18,27–29,37,43–48]. The aim of the present paper is to continue the development of the topic.

To capture the relevant information, to simplify the task of representing and handling of fuzzy numbers, the value and the ambiguity of a fuzzy number were introduced in [24]. The authors also discussed how to approximate a given fuzzy number by a suitable trapezoidal one with the same value and ambiguity. Because it is not possible to uniquely determine a trapezoidal fuzzy number, which is characterized by four numbers, from two conditions, some additional conditions must be introduced. An idea is to determine the trapezoidal fuzzy number by minimizing the distance between the initial fuzzy number and its approximation. In the present paper we completely solve the problem to find the nearest trapezoidal approximation of a fuzzy number, with respect to a well-known metric, preserving the ambiguity. The triangular approximation and interval approximation are determined too. They can be useful for practitioners which prefer to work with more simplified data. In applications is sometimes better to consider symmetric data such as we compute the nearest symmetric triangular and the nearest symmetric trapezoidal fuzzy number with respect to average Euclidean distance, under the preserving of ambiguity. We propose a simpler and elementary method to avoid the Karush–Kuhn–Tucker theorem and the associated calculus and to prove some properties, like continuity. We give examples and algorithms of calculus in terms of width, left-hand and right-hand ambiguity. In the paper [11] it was proved that, in the case of trapezoidal approximation without conditions and in the case of trapezoidal approximation preserving the expected interval, there is no difference whether

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the trapezoidal approximation is performed before or after aggregation with respect to average. The property is valid in the case of approximations treated in the present paper too.

2. Preliminaries

We consider the following well-known description of a fuzzy number A:

$$A(x)=\begin{cases}0, & \text{if } x\le a_1,\\ l_A(x), & \text{if } a_1\le x\le a_2,\\ 1 & \text{if } a_2\le x\le a_3,\\ r_A(x), & \text{if } a_3\le x\le a_4,\\ 0, & \text{if } a_4\le x,\end{cases} \tag{1}$$

where $a_1,a_2,a_3,a_4\in\mathbb{R}$, $l_A:[a_1,a_2]\longrightarrow[0,1]$ is a nondecreasing upper semicontinuous function, $l_A(a_1)=0, l_A(a_2)=1$, called the left side of the fuzzy number A and $r_A:[a_3,a_4]\longrightarrow[0,1]$ is a nonincreasing upper semicontinuous function, $r_A(a_3)=1, r_A(a_4)=0$, called the right side of the fuzzy number A. The $\alpha-$cut, $\alpha\in(0,1]$, of a fuzzy number A is a crisp set defined as

$$A_\alpha=\{x\in\mathbb{R}:A(x)\ge\alpha\}.$$

The support or $0-$cut A_0 of a fuzzy number is defined as

$$A_0=\overline{\{x\in\mathbb{R}:A(x)>0\}}.$$

Every $\alpha-$cut, $\alpha\in[0,1]$, of a fuzzy number is a closed interval

$$A_\alpha=[A_L(\alpha),A_U(\alpha)]$$

where

$$A_L(\alpha)=\inf\{x\in\mathbb{R}:A(x)\ge\alpha\},$$
$$A_U(\alpha)=\sup\{x\in\mathbb{R}:A(x)\ge\alpha\}$$

for any $\alpha\in(0,1]$. If the sides of the fuzzy number A are strictly monotone then one can see easily that A_L and A_U are inverse functions of l_A and r_A, respectively. We denote by $F(\mathbb{R})$ the set of all fuzzy numbers.

The ambiguity $Amb(A)$ of a fuzzy number A, $A_\alpha=[A_L(\alpha),A_U(\alpha)]$, $\alpha\in[0,1]$ is given by (see [24])

$$Amb(A)=\int_0^1\alpha(A_U(\alpha)-A_L(\alpha))\,\mathrm{d}\alpha. \tag{2}$$

The typical value of A, called the expected value of A, is given by (see [25,33])

$$EV(A)=\frac{1}{2}\left(\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha+\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha\right). \tag{3}$$

The nonspecificity of a fuzzy number A, called the width of A, is introduced by (see [18])

$$w(A)=\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha-\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha. \tag{4}$$

To describe the spread of the left-hand and right-hand part of a fuzzy number with respect to the expected value, the left-hand ambiguity and right-hand ambiguity of a fuzzy number A, were introduced in [31] as follows

$$Amb_L(A)=\int_0^1\alpha\,(EV(A)-A_L(\alpha))\,\mathrm{d}\alpha, \tag{5}$$

$$Amb_U(A)=\int_0^1\alpha\,(A_U(\alpha)-EV(A))\,\mathrm{d}\alpha. \tag{6}$$

A metric on the set of fuzzy numbers, which is an extension of the Euclidean distance, is defined by [26]

$$d^2(A,B)=\int_0^1 (A_L(\alpha)-B_L(\alpha))^2\,\mathrm{d}\alpha+\int_0^1 (A_U(\alpha)-B_U(\alpha))^2\,\mathrm{d}\alpha. \tag{7}$$

Fuzzy numbers with simple membership functions are preferred in practice (see e.g. [15,17,19,20,35,38,41,42]). The most used such fuzzy numbers are so-called trapezoidal fuzzy numbers. A trapezoidal fuzzy number T, $T_\alpha=[T_L(\alpha), T_U(\alpha)]$, $\alpha\in[0,1]$ is given by

$$T_L(\alpha)=t_1+(t_2-t_1)\alpha,$$
$$T_U(\alpha)=t_4-(t_4-t_3)\alpha,$$

where $t_1,t_2,t_3,t_4\in\mathbb{R}$, $t_1\le t_2\le t_3\le t_4$. We denote by $T=(t_1,t_2,t_3,t_4)$ a trapezoidal fuzzy number and by $F^T(\mathbb{R})$ the set of all trapezoidal fuzzy numbers. When $t_2=t_3$ we obtain a triangular fuzzy number. We denote by $F^t(\mathbb{R})$ the set of all triangular fuzzy numbers. When $t_2-t_1=t_4-t_3$ we obtain a symmetric trapezoidal fuzzy number. We denote by $F^S(\mathbb{R})$ the set of all symmetric trapezoidal fuzzy numbers. When $t_2=t_3$ and $t_2-t_1=t_4-t_3$ we obtain a symmetric triangular fuzzy number. We denote by $F^s(\mathbb{R})$ the set of all symmetric triangular fuzzy numbers.

Sometimes (see [45]) it is useful to denote a trapezoidal fuzzy number by

$$T=[l,u,x,y]$$

with $l,u,x,y\in\mathbb{R}$ such that

$$x\ge 0, \tag{8}$$
$$y\ge 0, \tag{9}$$
$$x+y\le 2(u-l). \tag{10}$$

Then

$$T_L(\alpha)=l+x\left(\alpha-\frac{1}{2}\right),$$
$$T_U(\alpha)=u-y\left(\alpha-\frac{1}{2}\right),$$

for every $\alpha\in[0,1]$.

It is immediate that

$$l=\frac{t_1+t_2}{2}, \tag{11}$$
$$u=\frac{t_3+t_4}{2}, \tag{12}$$
$$x=t_2-t_1, \tag{13}$$
$$y=t_4-t_3 \tag{14}$$

and

$$Amb(T)=\frac{6u-6l-x-y}{12}. \tag{15}$$

It is easy to obtain that a trapezoidal fuzzy number $T=[l,u,x,y]$ is symmetric if and only if $x=y$ and triangular if and only if $2u-2l=x+y$. The distance between $T=[l,u,x,y]$ and $T'=[l',u',x',y']$ becomes [44]

$$d^2(T,T')=(l-l')^2+(u-u')^2+\frac{1}{12}(x-x')^2+\frac{1}{12}(y-y')^2. \tag{16}$$

Another important kind of fuzzy number was introduced in [16] as follows. Let $a,b,c,d\in\mathbb{R}$ such that $a\le b\le c\le d$. A fuzzy number A such that

$$A_\alpha=[A_L(\alpha), A_U(\alpha)]=\left[a+(b-a)\alpha^{1/r}, d-(d-c)\alpha^{1/s}\right], \alpha\in[0,1], \tag{17}$$

where $r, s > 0$, is denoted by $A = (a, b, c, d)_{r,s}$. When $r = s = 1$ we obtain a trapezoidal fuzzy number. We denote by $F^{r,s}(\mathbb{R})$ the set of such fuzzy numbers.

Let $A, B \in F(\mathbb{R})$, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$ and $\lambda \in \mathbb{R}$. We consider the sum $A + B$ and the scalar multiplication $\lambda \cdot A$ by (see e.g. [23], p. 40)

$$(A+B)_\alpha = A_\alpha + B_\alpha = [A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)]$$

and

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)], & \text{if } \lambda \geq 0 \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)], & \text{if } \lambda < 0, \end{cases}$$

respectively, for every $\alpha \in [0,1]$. In the case of the trapezoidal fuzzy numbers $T = (t_1, t_2, t_3, t_4)$ and $S = (s_1, s_2, s_3, s_4)$ we obtain

$$T + S = (t_1 + s_1, t_2 + s_2, t_3 + s_3, t_4 + s_4),$$
$$\lambda \cdot T = (\lambda t_1, \lambda t_2, \lambda t_3, \lambda t_4) \text{ if } \lambda \geq 0,$$
$$\lambda \cdot T = (\lambda t_4, \lambda t_3, \lambda t_2, \lambda t_1) \text{ if } \lambda < 0.$$

We also mention that

$$(a, b, c, d)_{r,s} + \left(a', b', c', d'\right)_{r,s} = \left(a + a', b + b', c + c', d + d'\right)_{r,s} \tag{18}$$

and

$$\lambda \cdot (a, b, c, d)_{r,s} = (\lambda a, \lambda b, \lambda c, \lambda d)_{r,s}, \tag{19}$$

for every $\lambda \geq 0$.

An extended trapezoidal fuzzy number [44,45] (see also [12,46]) is an ordered pair of polynomial functions of degree less than or equal to 1. An extended trapezoidal fuzzy number may not be a fuzzy number, that is $t_1 \leq t_2 \leq t_3 \leq t_4$ (or equivalently (8)–(10)) are not satisfied, but the distance between two extended trapezoidal fuzzy numbers is similarly defined as in (7) or (16). In addition, we define the ambiguity of an extended trapezoidal fuzzy number in the same way as in the classical case of a fuzzy number. We denote by $F_e^T(\mathbb{R})$ the set of all extended trapezoidal fuzzy numbers.

The extended trapezoidal approximation

$$T_e(A) = [l^e(A), u^e(A), x^e(A), y^e(A)] = [l^e, u^e, x^e, y^e]$$

of a fuzzy number A is the extended trapezoidal fuzzy number which minimizes the distance $d(A, X)$ where X is an extended trapezoidal fuzzy number. In the paper [5] the authors proved that $T_e(A)$ is not always a fuzzy number. The extended trapezoidal approximation $T_e(A) = [l^e, u^e, x^e, y^e]$ of a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0,1]$ is determined (see [44]) by the following equalities

$$l^e = \int_0^1 A_L(\alpha)\, d\alpha, \tag{20}$$

$$u^e = \int_0^1 A_U(\alpha)\, d\alpha, \tag{21}$$

$$x^e = 12 \int_0^1 \left(\alpha - \frac{1}{2}\right) A_L(\alpha)\, d\alpha, \tag{22}$$

$$y^e = -12 \int_0^1 \left(\alpha - \frac{1}{2}\right) A_U(\alpha)\, d\alpha. \tag{23}$$

The real numbers x^e and y^e are non-negative (see [44]) and from the definition of a fuzzy number we have $l^e \leq u^e$.

In the paper [43] the author proved two very important distance properties for the extended trapezoidal approximation operator $T_e : F(\mathbb{R}) \to F_e^T(\mathbb{R})$, as follows.

Proposition 1 ([43], Proposition 4.2)**.** *Let A be a fuzzy number. Then*

$$d^2(A, B) = d^2(A, T_e(A)) + d^2(T_e(A), B),$$

for any trapezoidal fuzzy number B.

Proposition 2 ([43], Proposition 4.4). *$d(T_e(A), T_e(B)) \leq d(A, B)$ for all fuzzy numbers A and B.*

Remark 3. Let A and B be two fuzzy numbers and $T_e(A) = [l^e, u^e, x^e, y^e]$, $T_e(B) = [l'^e, u'^e, x'^e, y'^e]$ the extended trapezoidal approximations of A and B. From (16) and from Proposition 2, it is immediate that

$$(l^e - l'^e)^2 + (u^e - u'^e)^2 \leq d^2(A, B)$$

and

$$(x^e - x'^e)^2 + (y^e - y'^e)^2 \leq 12d^2(A, B).$$

Proposition 4. *If A is a fuzzy number and $T_e(A) = [l^e, u^e, x^e, y^e]$ is the extended trapezoidal approximation of A then*

$$Amb(A) = Amb(T_e(A)).$$

Proof. By direct calculation, we get

$$\begin{aligned}
&Amb(T_e(A)) \\
&= \frac{1}{12}(6u^e - 6l^e - x^e - y^e) = \frac{1}{2}\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - \frac{1}{2}\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha \\
&\quad - \int_0^1 \left(\alpha - \frac{1}{2}\right) A_L(\alpha)\,\mathrm{d}\alpha + \int_0^1 \left(\alpha - \frac{1}{2}\right) A_U(\alpha)\,\mathrm{d}\alpha \\
&= \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))\,\mathrm{d}\alpha = Amb(A). \qquad \square
\end{aligned}$$

The below version of the well-known Karush–Kuhn–Tucker theorem is an important tool in approximation of fuzzy numbers by trapezoidal or triangular fuzzy numbers (see [6,7,9,29]).

Theorem 5 ([34,36], see also [39], pp. 281–283). *Let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable functions. Then $\bar{x}$ solves the convex programming problem*

$$\begin{aligned}
&\min f(x) \\
&s.t.\ g_i(x) \leq h_i,\ i \in \{1, \ldots, m\}
\end{aligned}$$

if and only if there exists ξ_i, $i \in \{1, \ldots, m\}$, such that

(i) $\nabla f(\bar{x}) + \sum_{i=1}^{m} \xi_i \nabla g_i(\bar{x}) = 0$;
(ii) $g_i(\bar{x}) - h_i \leq 0$;
(iii) $\xi_i \geq 0$;
(iv) $\xi_i(h_i - g_i(\bar{x})) = 0$.

3. Approximation of fuzzy numbers by real intervals preserving ambiguity

Because a real interval $[v, w]$ can be represented as a fuzzy number with the α-cuts $[v, w]$, for every $\alpha \in [0, 1]$, from (2) we get

$$Amb([v, w]) = \frac{w - v}{2}.$$

We find I_A, the nearest interval of a given fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0, 1]$ such that its ambiguity is preserved, by solving the following problem

$$\begin{aligned}
&\min d([v, w], A), \\
&\int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))\,\mathrm{d}\alpha = \frac{w - v}{2}.
\end{aligned}$$

The problem is equivalent to find v, w such that

$$\min\left(\int_0^1 (A_L(\alpha) - v)^2 \, d\alpha + \int_0^1 (A_U(\alpha) - w)^2 \, d\alpha\right), \tag{24}$$

$$\frac{w-v}{2} = \int_0^1 \alpha (A_U(\alpha) - A_L(\alpha)) \, d\alpha \tag{25}$$

$$v \le w. \tag{26}$$

After simple calculations we get the problem (24)–(26) is reduced to

$$\min\left(2v^2 + \left(4\int_0^1 \alpha A_U(\alpha)\, d\alpha - 4\int_0^1 \alpha A_L(\alpha)\, d\alpha \right.\right.$$
$$\left.\left. -2\int_0^1 A_U(\alpha)\, d\alpha - 2\int_0^1 A_L(\alpha)\, d\alpha\right) v\right),$$
$$w = 2\int_0^1 \alpha (A_U(\alpha) - A_L(\alpha))\, d\alpha + v$$

The following result is immediate.

Theorem 6. *The nearest interval to $A \in F(\mathbb{R})$, which preserves the ambiguity of A, $I_A = [v, w]$, is given by*

$$v = \int_0^1 \left(\alpha + \frac{1}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(-\alpha + \frac{1}{2}\right) A_U(\alpha)\, d\alpha \tag{27}$$

$$w = \int_0^1 \left(-\alpha + \frac{1}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(\alpha + \frac{1}{2}\right) A_U(\alpha)\, d\alpha. \tag{28}$$

According with (2) and (3) we obtain

$$I_A = [EV(A) - Amb(A), EV(A) + Amb(A)].$$

Corollary 7. *If $A = (a, b, c, d)_{r,s}$ then $I_A = [v, w]$ is given by*

$$v = \frac{2+3r}{2(1+r)(1+2r)} a + \frac{3r+4r^2}{2(1+r)(1+2r)} b + \frac{s}{2(1+s)(1+2s)}(d-c),$$

$$w = \frac{r}{2(1+r)(1+2r)}(a-b) + \frac{3s+4s^2}{2(1+s)(1+2s)} c + \frac{2+3s}{2(1+s)(1+2s)} d.$$

Example 8. We have

$$I_{(1,2,3,4)} = \left[\frac{5}{3}, \frac{21}{6}\right]$$

$$I_{(1,2,3,4)_{2,2}} = \left[\frac{9}{5}, \frac{10}{3}\right].$$

4. Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving ambiguity

In this section we prove that for any fuzzy number A there exists an unique trapezoidal fuzzy number T_A such that $Amb(A) = Amb(T_A)$ and which is the nearest to A with respect to the metric d. By Proposition 1 and Proposition 4 it follows that the problem to find the trapezoidal approximation preserving the ambiguity of a fuzzy number A is equivalent with the problem to find a trapezoidal fuzzy number T_A such that $Amb(T_A) = Amb(T_e(A))$ and $d(T_A, T_e(A)) \le d(B, T_e(A))$ for all $B \in F^T(\mathbb{R})$ satisfying $Amb(B) = Amb(T_e(A))$. Therefore

$$T_A = [l_T(A), u_T(A), x_T(A), y_T(A)] = [l_T, u_T, x_T, y_T]$$

is a solution of the discussed problem if and only if the quadruple $(l_T, u_T, x_T, y_T) \in \mathbb{R}^4$ is a solution of the minimization problem

$$\min\left((l-l^e)^2 + (u-u^e)^2 + \frac{1}{12}(x-x^e)^2 + \frac{1}{12}(y-y^e)^2\right) \tag{29}$$

under the conditions

$$x \geq 0, \tag{30}$$
$$y \geq 0, \tag{31}$$
$$x + y \leq 2(u-l), \tag{32}$$
$$6u - 6l - x - y = 6u^e - 6l^e - x^e - y^e. \tag{33}$$

Condition (33) implies

$$u - l = u^e - l^e + \frac{1}{6}(x+y) - \frac{1}{6}(x^e + y^e)$$

and

$$l - l^e = u - u^e - \frac{1}{6}(x - x^e) - \frac{1}{6}(y - y^e), \tag{34}$$

therefore problem (29)–(33) becomes

$$\min F(l, u, x, y),$$

where

$$F(l,u,x,y) = 2(u-u^e)^2 + \frac{1}{9}(x-x^e)^2 + \frac{1}{9}(y-y^e)^2 - \frac{1}{3}(u-u^e)(x-x^e) - \frac{1}{3}(u-u^e)(y-y^e) + \frac{1}{18}(x-x^e)(y-y^e)$$

under the conditions

$$x \geq 0, \tag{35}$$
$$y \geq 0, \tag{36}$$
$$2x + 2y \leq 6u^e - 6l^e - x^e - y^e. \tag{37}$$

After elementary calculus we get

$$F(l,u,x,y) = 2\left(u - u^e - \frac{1}{12}(x - x^e + y - y^e)\right)^2 + \frac{7}{72}(x-x^e)^2 + \frac{7}{72}(y-y^e)^2 + \frac{1}{36}(x-x^e)(y-y^e).$$

Because conditions (35)–(37) are independent of u and taking into account (34), $T_A = [l_T, u_T, x_T, y_T]$ is the nearest trapezoidal fuzzy number to fuzzy number A such that $Amb(A) = Amb(T_A)$ if and only if

$$u_T = u^e + \frac{1}{12}(x_T - x^e + y_T - y^e), \tag{38}$$
$$l_T = l^e - \frac{1}{12}(x_T - x^e + y_T - y^e) \tag{39}$$

and (x_T, y_T) is the solution of the minimization problem

$$\min\left((7(x-x^e)^2+7(y-y^e)^2+2(x-x^e)(y-y^e)\right), \tag{40}$$
$$x \ge 0, \tag{41}$$
$$y \ge 0, \tag{42}$$
$$2x+2y \le 6u^e-6l^e-x^e-y^e. \tag{43}$$

Let us denote (see Fig. 1)

$$M_A = \{(x,y)\in\mathbb{R}^2 : x\ge 0, y\ge 0, 2x+2y \le 6u^e-6l^e-x^e-y^e\} \tag{44}$$

and d_E the Euclidean metric on $\mathbb{R}^2$.

To solve the above problem we give the following result.

Theorem 9. *The problem (40)–(43) has an unique solution (x_T, y_T), where*

$$(x_T, y_T) = P_{M_A}(x^e, y^e)$$

and $P_M(a,b)$ denotes the orthogonal projection of $(a,b)\in\mathbb{R}^2$ on nonempty set $M\subset\mathbb{R}^2$, with respect to the Euclidean metric d_E.

Proof. First, we prove that M_A is nonempty. Indeed, taking into account (20)–(23) we have

$$6u^e-6l^e-x^e-y^e = 12\int_0^1 \alpha(A_U(\alpha)-A_L(\alpha))\,\mathrm{d}\alpha \ge 0,$$

therefore $M_A \ne \emptyset$.

Let us define an inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^2$ by

$$\langle (x_1,y_1),(x_2,y_2)\rangle = 7x_1x_2+7y_1y_2+x_1y_2+x_2y_1.$$

If $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$ then

$$\begin{aligned} D^2((x_1,y_1),(x_2,y_2)) &= \langle (x_1-x_2, y_1-y_2),(x_1-x_2,y_1-y_2)\rangle \\ &= 7(x_1-x_2)^2+7(y_1-y_2)^2+2(x_1-x_2)(y_1-y_2) \end{aligned}$$

introduces a distance on $\mathbb{R}^2$, induced by the above inner product. Since $\mathbb{R}^2$ is a finite dimensional space it follows that $\left(\mathbb{R}^2,\langle\cdot,\cdot\rangle\right)$ is a Hilbert space. Because M_A is a nonempty closed convex subset of the Hilbert space $\left(\mathbb{R}^2,\langle\cdot,\cdot\rangle\right)$ it follows (see e.g. [40], Theorem 4. 10, p. 79) that for any $(x,y)\in\mathbb{R}^2$ there exists an unique element denoted $(P_{M_A})_D(x,y)$ such that

$$D((x,y),(P_{M_A})_D(x,y)) = \inf_{C\in M_A} D((x,y),C),$$

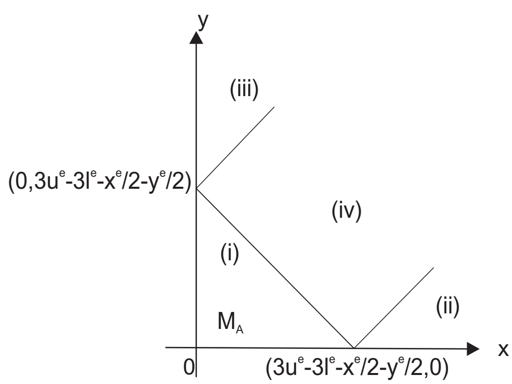

Fig. 1. M_A and cases for (x^e, y^e) in the finding of T_A.

that is

$$(x_T, y_T) = (P_{M_A})_D (x^e, y^e) \tag{45}$$

is the unique solution of problem (40)–(43). Now, we distinguish the following two cases:
(i) $(x^e, y^e) \in M_A$. Then it is obvious that

$$(x_T, y_T) = (x^e, y^e) = (P_{M_A})_D (x^e, y^e) = P_{M_A} (x^e, y^e).$$

(ii) $(x^e, y^e) \notin M_A$. We prove that $(x_T, y_T) \in M_1$, where

$$M_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 2x + 2y = 6u^e - 6l^e - x^e - y^e\}.$$

Let us assume $(x_T, y_T) \notin M_1$. Because $(x_T, y_T) \in M_A \backslash M_1$, there exists $B \in M_1$ such that B is between (x_T, y_T) and (x^e, y^e), therefore

$$D((x^e, y^e), (x_T, y_T)) > D((x^e, y^e), B),$$

a contradiction with (45).

Since $(x_T, y_T) \in M_1$ we get that (x_T, y_T) is the solution of the problem

$$\min G(x, y),$$

where

$$G(x, y) = 7(x - x^e)^2 + 7(y - y^e)^2 + 2(x - x^e)(y - y^e),$$

under conditions

$$x \geq 0, \tag{46}$$
$$y \geq 0, \tag{47}$$
$$2x + 2y = 6u^e - 6l^e - x^e - y^e. \tag{48}$$

Taking into account (48) we get

$$G(x, y) = 6(x - x^e)^2 + 6(y - y^e)^2 + \left(3u^e - 3l^e - \frac{3}{2}x^e - \frac{3}{2}y^e\right)^2$$

and since the expression $(3u^e - 3l^e - \frac{3}{2}x^e - \frac{3}{2}y^e)^2$ is constant we obtain that (x_T, y_T) is the solution of the problem

$$\min\left((x - x^e)^2 + (y - y^e)^2\right)$$

under the conditions

$$x \geq 0,$$
$$y \geq 0,$$
$$2x + 2y = 6u^e - 6l^e - x^e - y^e,$$

therefore $(x_T, y_T) = P_{M_1}(x^e, y^e)$. Because $(x^e, y^e) \notin M_A$ we easily obtain $(x_T, y_T) = P_{M_A}(x^e, y^e)$. $\square$

Theorem 9, together (38) and (39), suggest us the following method to compute $T_A = [l_T, u_T, x_T, y_T]$, the nearest trapezoidal fuzzy number of a fuzzy number A preserving the ambiguity (see Fig. 1).

(i) $(x^e, y^e) \in M_A$, that is

$$-2l^e + 2u^e - x^e - y^e \geq 0.$$

Then

$$\begin{aligned} x_T &= x^e, \\ y_T &= y^e, \\ l_T &= l^e, \\ u_T &= u^e. \end{aligned}$$

If $(x^e, y^e) \notin M_A$ then the following cases are possible:
(ii) If

$$6l^e - 6u^e + 3x^e - y^e > 0$$

then

$$\begin{aligned} x_T &= -3l^e + 3u^e - \frac{1}{2}x^e - \frac{1}{2}y^e, \\ y_T &= 0, \\ u_T &= -\frac{1}{4}l^e + \frac{5}{4}u^e - \frac{1}{8}x^e - \frac{1}{8}y^e, \\ l_T &= \frac{5}{4}l^e - \frac{1}{4}u^e + \frac{1}{8}x^e + \frac{1}{8}y^e. \end{aligned}$$

(iii) If

$$-6l^e + 6u^e + x^e - 3y^e < 0$$

then

$$\begin{aligned} x_T &= 0, \\ y_T &= -3l^e + 3u^e - \frac{1}{2}x^e - \frac{1}{2}y^e, \\ u_T &= -\frac{1}{4}l^e + \frac{5}{4}u^e - \frac{1}{8}x^e - \frac{1}{8}y^e, \\ l_T &= \frac{5}{4}l^e - \frac{1}{4}u^e + \frac{1}{8}x^e + \frac{1}{8}y^e. \end{aligned}$$

(iv) If

$$\begin{aligned} -2l^e + 2u^e - x^e - y^e &< 0, \\ 6l^e - 6u^e + 3x^e - y^e &\le 0, \\ -6l^e + 6u^e + x^e - 3y^e &\ge 0, \end{aligned}$$

then

$$\begin{aligned} x_T &= -\frac{3}{2}l^e + \frac{3}{2}u^e + \frac{1}{4}x^e - \frac{3}{4}y^e, \\ y_T &= -\frac{3}{2}l^e + \frac{3}{2}u^e - \frac{3}{4}x^e + \frac{1}{4}y^e, \\ u_T &= -\frac{1}{4}l^e + \frac{5}{4}u^e - \frac{1}{8}x^e - \frac{1}{8}y^e, \\ l_T &= \frac{5}{4}l^e - \frac{1}{4}u^e + \frac{1}{8}x^e + \frac{1}{8}y^e. \end{aligned}$$

Taking into account (11)–(14) and (20)–(23) we obtain the following result to compute

$$T_A = (T_1(A), T_2(A), T_3(A), T_4(A)) = (T_1, T_2, T_3, T_4),$$

the nearest trapezoidal fuzzy number of a fuzzy number A preserving the ambiguity.

Theorem 10. (i) *If*

$$\int_0^1 (1-3\alpha)A_L(\alpha)\ d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)\ d\alpha \geq 0 \tag{49}$$

then

$$T_1 = \int_0^1 (-6\alpha+4)A_L(\alpha)\,d\alpha, \tag{50}$$

$$T_2 = \int_0^1 (6\alpha-2)A_L(\alpha)\,d\alpha, \tag{51}$$

$$T_3 = \int_0^1 (6\alpha-2)A_U(\alpha)\,d\alpha, \tag{52}$$

$$T_4 = \int_0^1 (-6\alpha+4)A_U(\alpha)\,d\alpha. \tag{53}$$

(ii) *If*

$$\int_0^1 (3\alpha-1)A_L(\alpha)\ d\alpha + \int_0^1 (\alpha-1)A_U(\alpha)\ d\alpha > 0 \tag{54}$$

then

$$T_1 = \frac{1}{2}\int_0^1 (1+9\alpha)A_L(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 (1-9\alpha)A_U(\alpha)\,d\alpha, \tag{55}$$

$$T_2 = T_3 = T_4 = \frac{1}{2}\int_0^1 (1-3\alpha)A_L(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 (1+3\alpha)A_U(\alpha)\,d\alpha. \tag{56}$$

(iii) *If*

$$\int_0^1 (\alpha-1)A_L(\alpha)\ d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)\ d\alpha < 0 \tag{57}$$

then

$$T_1 = T_2 = T_3 = \frac{1}{2}\int_0^1 (1+3\alpha)A_L(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 (1-3\alpha)A_U(\alpha)\,d\alpha, \tag{58}$$

$$T_4 = \frac{1}{2}\int_0^1 (1-9\alpha)A_L(\alpha)\,d\alpha + \frac{1}{2}\int_0^1 (1+9\alpha)A_U(\alpha)\,d\alpha. \tag{59}$$

(iv) *If*

$$\begin{aligned}
&\int_0^1 (1-3\alpha)A_L(\alpha)\ d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)\ d\alpha < 0,\\
&\int_0^1 (3\alpha-1)A_L(\alpha)\ d\alpha + \int_0^1 (\alpha-1)A_U(\alpha)\ d\alpha \leq 0,\\
&\int_0^1 (\alpha-1)A_L(\alpha)\ d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)\ d\alpha \geq 0,
\end{aligned}$$

then

$$T_1 = 2\int_0^1 A_L(\alpha)\,d\alpha - \int_0^1 (6\alpha-2)A_U(\alpha)\,d\alpha, \tag{60}$$

$$T_2 = T_3 = \int_0^1 (3\alpha-1)A_L(\alpha)\ d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)\ d\alpha, \tag{61}$$

$$T_4 = \int_0^1 (2-6\alpha)A_L(\alpha)\,d\alpha + 2\int_0^1 A_U(\alpha)\,d\alpha. \tag{62}$$

Taking into account the representation (17) of the fuzzy numbers $(a, b, c, d)_{r,s}$ we get

$$\int_0^1 ((a,b,c,d)_{r,s})_L(\alpha)\,\mathrm{d}\alpha = \frac{a+rb}{r+1},$$
$$\int_0^1 ((a,b,c,d)_{r,s})_U(\alpha)\,\mathrm{d}\alpha = \frac{sc+d}{s+1},$$
$$\int_0^1 \alpha((a,b,c,d)_{r,s})_L(\alpha)\,\mathrm{d}\alpha = \frac{a+2rb}{2\,(2r+1)},$$
$$\int_0^1 \alpha((a,b,c,d)_{r,s})_U(\alpha)\,\mathrm{d}\alpha = \frac{2sc+d}{2(2s+1)}$$

and replacing in (49)–(62) we can obtain an important consequence of Theorem 10 (see [10], Corollary 15, in the case of the nearest trapezoidal fuzzy number preserving value and ambiguity). Due to the length of the result we prefer to give some examples instead.

Example 11. Case (i) in Theorem 10 is applicable to fuzzy number $(1, 2, 3, 4)_{2,2}$ and

$$T_{(1,2,3,4)_{2,2}} = \left(\frac{19}{15}, \frac{31}{15}, \frac{44}{15}, \frac{56}{15}\right).$$

The fuzzy numbers $(1, 200, 201, 220)_{2,2}$ and $(1, 20, 30, 320)_{2,2}$ satisfy conditions (54) and (57), respectively, and

$$T_{(1,200,201,220)_{2,2}} = \left(\frac{1\,403}{20}, \frac{4\,079}{20}, \frac{4\,079}{20}, \frac{4\,079}{20}\right),$$
$$T_{(1,20,30,320)_{2,2}} = \left(\frac{979}{60}, \frac{979}{60}, \frac{979}{60}, \frac{13\,903}{60}\right).$$

Case (iv) in Theorem 10 is applicable to fuzzy number $(1, 2, 4, 35)_{2,2}$ and

$$T_{(1,2,4,35)_{2,2}} = \left(\frac{7}{5}, 2, 2, \frac{133}{5}\right).$$

5. Approximation of fuzzy numbers by symmetric trapezoidal fuzzy numbers preserving ambiguity

The uncertain or imprecise information is often represented by symmetric fuzzy numbers, particularly by symmetric trapezoidal/triangular fuzzy numbers. Taking into account the considerations in Section 4 we obtain that

$$S_A = [l_S(A), u_S(A), x_S(A), y_S(A)] = [l_S, u_S, x_S, y_S]$$

is the symmetric trapezoidal fuzzy number nearest to fuzzy number A with respect to d (see (16)) such that $Amb(A) = Amb(S_A)$ if and only if $(x_S, y_S) \in \mathbb{R}^2$ is the solution of the problem

$$\begin{aligned}
&\min\left((7(x-x^e)^2 + 7(y-y^e)^2 + 2(x-x^e)(y-y^e)\right),\\
&x \geq 0,\\
&y \geq 0,\\
&2x+2y \leq 6u^e - 6l^e - x^e - y^e,\\
&x = y,
\end{aligned} \tag{63}$$

and

$$u_S = u^e + \frac{1}{12}\left(x_S - x^e + y_S - y^e\right), \tag{64}$$
$$l_S = l^e - \frac{1}{12}\left(x_S - x^e + y_S - y^e\right). \tag{65}$$

We recall, $T_e(A) = [l^e, u^e, x^e, y^e]$ is the extended trapezoidal approximation of A (see (20)–(23)). The problem (63) is equivalent with

$$\min\left(16x^2 - 16\left(x^e + y^e\right)x + 7\left(x^e\right)^2 + 7\left(y^e\right)^2 + 2x^e y^e\right),$$
$$x \geq 0,$$
$$x \leq \frac{3}{2}u^e - \frac{3}{2}l^e - \frac{1}{4}x^e - \frac{1}{4}y^e.$$

Because the function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h\left(x\right) = 16x^2 - 16\left(x^e + y^e\right)x + 7\left(x^e\right)^2 + 7\left(y^e\right)^2 + 2x^e y^e$$

attains its minimum if and only if $x = \frac{x^e+y^e}{2}$ and taking into account $x^e \geq 0, y^e \geq 0$ (see [44]), the following cases are possible:

(i) If

$$\frac{x^e + y^e}{2} \leq \frac{3}{2}u^e - \frac{3}{2}l^e - \frac{1}{4}x^e - \frac{1}{4}y^e,$$

that is

$$x^e + y^e \leq 2\left(u^e - l^e\right)$$

then

$$x_S = y_S = \frac{x^e + y^e}{2},$$
$$u_S = u^e,$$
$$l_S = l^e.$$

(ii) If

$$\frac{x^e + y^e}{2} > \frac{3}{2}u^e - \frac{3}{2}l^e - \frac{1}{4}x^e - \frac{1}{4}y^e,$$

that is

$$x^e + y^e > 2\left(u^e - l^e\right)$$

then

$$x_S = y_S = \frac{3}{2}u^e - \frac{3}{2}l^e - \frac{1}{4}x^e - \frac{1}{4}y^e,$$
$$u_S = \frac{5}{4}u^e - \frac{1}{4}l^e - \frac{1}{8}x^e - \frac{1}{8}y^e,$$
$$l_S = -\frac{1}{4}u^e + \frac{5}{4}l^e + \frac{1}{8}x^e + \frac{1}{8}y^e.$$

From (20)–(23) we obtain the following result to compute

$$S_A = \left(S_1\left(A\right), S_2\left(A\right), S_3\left(A\right), S_4\left(A\right)\right) = \left(S_1, S_2, S_3, S_4\right),$$

the nearest symmetric trapezoidal fuzzy number of a fuzzy number A preserving the ambiguity.

Theorem 12. (i) *If*

$$\int_0^1 (1 - 3\alpha)A_L\left(\alpha\right)\, d\alpha + \int_0^1 (3\alpha - 1)A_U\left(\alpha\right)\, d\alpha \geq 0 \tag{66}$$

then

$$S_1 = \int_0^1 \left(-3\alpha + \frac{5}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(3\alpha - \frac{3}{2}\right) A_U(\alpha)\, d\alpha, \tag{67}$$

$$S_2 = \int_0^1 \left(3\alpha - \frac{1}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(-3\alpha + \frac{3}{2}\right) A_U(\alpha)\, d\alpha, \tag{68}$$

$$S_3 = \int_0^1 \left(-3\alpha + \frac{3}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(3\alpha - \frac{1}{2}\right) A_U(\alpha)\, d\alpha, \tag{69}$$

$$S_4 = \int_0^1 \left(3\alpha - \frac{3}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(-3\alpha + \frac{5}{2}\right) A_U(\alpha)\, d\alpha. \tag{70}$$

(ii) *If*

$$\int_0^1 (1 - 3\alpha) A_L(\alpha)\; d\alpha + \int_0^1 (3\alpha - 1) A_U(\alpha)\; d\alpha < 0 \tag{71}$$

then

$$S_1 = \int_0^1 \left(3\alpha + \frac{1}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(-3\alpha + \frac{1}{2}\right) A_U(\alpha)\, d\alpha, \tag{72}$$

$$S_2 = S_3 = \frac{1}{2} \int_0^1 A_L(\alpha)\, d\alpha + \frac{1}{2} \int_0^1 A_U(\alpha)\, d\alpha, \tag{73}$$

$$S_4 = \int_0^1 \left(-3\alpha + \frac{1}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(3\alpha + \frac{1}{2}\right) A_U(\alpha)\, d\alpha. \tag{74}$$

Example 13. Case (i) in Theorem 12 is applicable to fuzzy number $(1, 2, 3, 4)_{2,2}$ and

$$S_{(1,2,3,4)_{2,2}} = \left(\frac{19}{15}, \frac{31}{15}, \frac{44}{15}, \frac{56}{15}\right).$$

The fuzzy numbers $(1, 200, 201, 220)_{2,2}$, $(1, 20, 30, 320)_{2,2}$ and $(1, 2, 4, 35)_{2,2}$ satisfy (71) in Theorem 12. We get

$$S_{(1,200,201,220)_{2,2}} = \left(\frac{518}{5}, \frac{341}{2}, \frac{341}{2}, \frac{1187}{5}\right),$$
$$S_{(1,20,30,320)_{2,2}} = \left(-\frac{563}{15}, \frac{421}{6}, \frac{421}{6}, \frac{2668}{15}\right),$$
$$S_{(1,2,4,35)_{2,2}} = \left(-\frac{23}{5}, 8, 8, \frac{103}{5}\right).$$

6. Approximation of fuzzy numbers by triangular fuzzy numbers preserving ambiguity

Taking into account (38)–(43),

$$t_A = [l_t(A), u_t(A), x_t(A), y_t(A)] = [l_t, u_t, x_t, y_t]$$

is the nearest triangular fuzzy number to fuzzy number A such that $Amb(A) = Amb(t_A)$ if and only if

$$u_t = u^e + \frac{1}{12}\left(x_t - x^e + y_t - y^e\right), \tag{75}$$

$$l_t = l^e - \frac{1}{12}\left(x_t - x^e + y_t - y^e\right) \tag{76}$$

and (x_t, y_t) is the solution of the minimization problem

$$\min\left((7(x - x^e)^2 + 7(y - y^e)^2 + 2(x - x^e)(y - y^e)\right), \tag{77}$$

$$x \geq 0, \tag{78}$$

$$y \geq 0, \tag{79}$$

$$2x + 2y = 6u^e - 6l^e - x^e - y^e, \tag{80}$$

where $T_e(A) = [l^e, u^e, x^e, y^e]$ is the extended trapezoidal approximation of A. Let us denote (see Fig. 2)

$$N_A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 2x + 2y = 6u^e - 6l^e - x^e - y^e\} \tag{81}$$

and d_E the Euclidean metric on $\mathbb{R}^2$.

Theorem 14. *The problem (77)–(80) has an unique solution (x_t, y_t), where*

$$(x_t, y_t) = P_{N_A}(x^e, y^e)$$

and $P_M(a, b)$ denotes the orthogonal projection of $(a, b) \in \mathbb{R}^2$ on nonempty set $M \subset \mathbb{R}^2$ with respect to d_E.

Theorem 14 suggests ((75) and (76) are important here) the following method to compute $t_A = [l_t, u_t, x_t, y_t]$, the nearest triangular fuzzy number of a fuzzy number A preserving the ambiguity (see Fig. 2).

(i) If

$$6l^e - 6u^e + 3x^e - y^e \leq 0$$

and

$$-6l^e + 6u^e + x^e - 3y^e \geq 0$$

then

$$\begin{aligned} x_t &= -\frac{3}{2}l^e + \frac{3}{2}u^e + \frac{1}{4}x^e - \frac{3}{4}y^e, \\ y_t &= -\frac{3}{2}l^e + \frac{3}{2}u^e - \frac{3}{4}x^e + \frac{1}{4}y^e, \\ u_t &= -\frac{1}{4}l^e + \frac{5}{4}u^e - \frac{1}{8}x^e - \frac{1}{8}y^e, \\ l_t &= \frac{5}{4}l^e - \frac{1}{4}u^e + \frac{1}{8}x^e + \frac{1}{8}y^e. \end{aligned}$$

(ii) If

$$6l^e - 6u^e + 3x^e - y^e > 0$$

then

$$\begin{aligned} x_t &= -3l^e + 3u^e - \frac{1}{2}x^e - \frac{1}{2}y^e, \\ y_t &= 0, \\ u_t &= -\frac{1}{4}l^e + \frac{5}{4}u^e - \frac{1}{8}x^e - \frac{1}{8}y^e, \\ l_t &= \frac{5}{4}l^e - \frac{1}{4}u^e + \frac{1}{8}x^e + \frac{1}{8}y^e. \end{aligned}$$

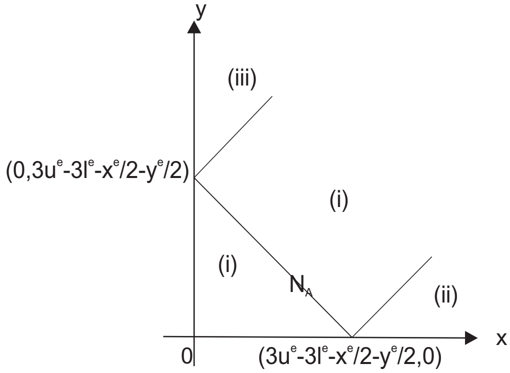

Fig. 2. N_A and cases for (x^e, y^e) in the finding of t_A.

(iii) If

$$-6l^e + 6u^e + x^e - 3y^e < 0$$

then

$$\begin{aligned} x_t &= 0, \\ y_t &= -3l^e + 3u^e - \frac{1}{2}x^e - \frac{1}{2}y^e, \\ u_t &= -\frac{1}{4}l^e + \frac{5}{4}u^e - \frac{1}{8}x^e - \frac{1}{8}y^e, \\ l_t &= \frac{5}{4}l^e - \frac{1}{4}u^e + \frac{1}{8}x^e + \frac{1}{8}y^e. \end{aligned}$$

From (20)–(23) we obtain the following result to compute t_A in the representation

$$t_A = (t_1(A), t_2(A), t_3(A)) = (t_1, t_2, t_3).$$

Theorem 15. (i) *If*

$$\int_0^1 (3\alpha - 1)A_L(\alpha)\, d\alpha + \int_0^1 (\alpha - 1)A_U(\alpha)\, d\alpha \le 0 \tag{82}$$

and

$$\int_0^1 (\alpha - 1)A_L(\alpha)\, d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)\, d\alpha \ge 0 \tag{83}$$

then

$$t_1 = 2\int_0^1 A_L(\alpha)\, d\alpha - \int_0^1 (6\alpha - 2)A_U(\alpha)\, d\alpha, \tag{84}$$

$$t_2 = \int_0^1 (3\alpha - 1)A_L(\alpha)\, d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)\, d\alpha, \tag{85}$$

$$t_3 = -\int_0^1 (6\alpha - 2)A_L(\alpha)\, d\alpha + 2\int_0^1 A_U(\alpha)\, d\alpha. \tag{86}$$

(ii) *If*

$$\int_0^1 (3\alpha - 1)A_L(\alpha) + \int_0^1 (\alpha - 1)A_U(\alpha)\, d\alpha > 0 \tag{87}$$

then

$$t_1 = \frac{1}{2}\int_0^1 (1 + 9\alpha)A_L(\alpha)\, d\alpha + \frac{1}{2}\int_0^1 (1 - 9\alpha)A_U(\alpha)\, d\alpha, \tag{88}$$

$$t_2 = t_3 = \frac{1}{2}\int_0^1 (1 - 3\alpha)A_L(\alpha)\, d\alpha + \frac{1}{2}\int_0^1 (1 + 3\alpha)A_U(\alpha)\, d\alpha. \tag{89}$$

(iii) *If*

$$\int_0^1 (\alpha - 1)A_L(\alpha)\, d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)\, d\alpha < 0 \tag{90}$$

then

$$t_1 = t_2 = \frac{1}{2}\int_0^1 (1 + 3\alpha)A_L(\alpha)\, d\alpha + \frac{1}{2}\int_0^1 (1 - 3\alpha)A_U(\alpha)\, d\alpha, \tag{91}$$

$$t_3 = \frac{1}{2}\int_0^1 (1 - 9\alpha)A_L(\alpha)\, d\alpha + \frac{1}{2}\int_0^1 (1 + 9\alpha)A_U(\alpha)\, d\alpha. \tag{92}$$

As in the case of trapezoidal approximation we prefer to give few examples instead of a theoretical result in the particular case of fuzzy numbers $(a, b, c, d)_{r,s}$ (see (17)).

Example 16. Case (i) in Theorem 15 is applicable to fuzzy numbers $(1, 2, 3, 4)_{2,2}$ and $(1, 2, 4, 35)_{2,2}$. We get

$$t_{(1,2,3,4)_{2,2}} = \left(\frac{2}{5}, \frac{5}{2}, \frac{23}{5}\right),$$
$$t_{(1,2,4,35)_{2,2}} = \left(\frac{7}{5}, 2, \frac{133}{5}\right).$$

The fuzzy numbers $(1, 200, 201, 220)_{2,2}$ and $(1, 20, 30, 320)_{2,2}$ satisfy (87) and (90) in Theorem 15, respectively, and

$$t_{(1,200,201,220)_{2,2}} = \left(\frac{1403}{20}, \frac{4079}{20}, \frac{4079}{20}\right),$$
$$t_{(1,20,30,320)_{2,2}} = \left(\frac{979}{60}, \frac{979}{60}, \frac{13903}{60}\right).$$

From (63), (64) and (65),

$$s_A = [l_s(A), u_s(A), x_s(A), y_s(A)] = [l_s, u_s, x_s, y_s]$$

is the nearest symmetric triangular fuzzy number to fuzzy number A such that $Amb(A) = Amb(s_A)$ if and only if $(x_s, y_s) \in \mathbb{R}^2$ is the solution of the problem

$$\min\left((7(x - x^e)^2 + 7(y - y^e)^2 + 2(x - x^e)(y - y^e)\right), \tag{93}$$
$$x \geq 0,$$
$$y \geq 0,$$
$$2x + 2y = 6u^e - 6l^e - x^e - y^e, \tag{94}$$
$$x = y \tag{95}$$

and

$$u_s = u^e + \frac{1}{12}(x_s - x^e + y_s - y^e),$$
$$l_s = l^e - \frac{1}{12}(x_s - x^e + y_s - y^e),$$

where $T_e(A) = [l^e, u^e, x^e, y^e]$ is the extended trapezoidal approximation of A. In fact, (94) and (95) imply

$$x = y = \frac{3}{2}u^e - \frac{3}{2}l^e - \frac{1}{4}x^e - \frac{1}{4}y^e,$$

therefore the minimization in (93) is not effective because the system of conditions has an unique solution. We obtain the following result taking into account (20)–(23).

Theorem 17. *The nearest symmetric triangular fuzzy number preserving ambiguity of the fuzzy number* A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0, 1]$, *in the representation*

$$s_A = (s_1(A), s_2(A), s_3(A)) = (s_1, s_2, s_3)$$

is given by

$$s_1 = \int_0^1 \left(3\alpha + \frac{1}{2}\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(\frac{1}{2} - 3\alpha\right) A_U(\alpha)\, d\alpha, \tag{96}$$
$$s_2 = \frac{1}{2}\int_0^1 A_L(\alpha)\, d\alpha + \frac{1}{2}\int_0^1 A_U(\alpha)\, d\alpha, \tag{97}$$
$$s_3 = \int_0^1 \left(\frac{1}{2} - 3\alpha\right) A_L(\alpha)\, d\alpha + \int_0^1 \left(3\alpha + \frac{1}{2}\right) A_U(\alpha)\, d\alpha. \tag{98}$$

Example 18. We get

$$s_{(1,2,3,4)_{2,2}} = \left(\frac{2}{5}, \frac{5}{2}, \frac{23}{5}\right),$$
$$s_{(1,2,4,35)_{2,2}} = \left(-\frac{23}{5}, 8, \frac{103}{5}\right),$$
$$s_{(1,200,201,220)_{2,2}} = \left(\frac{518}{5}, \frac{341}{2}, \frac{1187}{5}\right)$$

and

$$s_{(1,20,30,320)_{2,2}} = \left(-\frac{563}{15}, \frac{421}{6}, \frac{2668}{15}\right).$$

7. Algorithms

Because the conditions in Theorem 10 are too technical, we express them in terms of ambiguity, width, right and left-hand ambiguity (see (2), (4)–(6)). The idea was proposed in [30,31] and continued in [7,10].

Since (49) is equivalent with

$$w(A) \leq 3Amb(A),$$

(54) with

$$4Amb_L(A) < Amb(A)$$

and (57) with

$$4Amb_U(A) < Amb(A),$$

we get the following algorithm for computing the nearest trapezoidal approximation preserving the ambiguity:

Algorithm 19. Step 1: If $w(A) \leq 3Amb(A)$ then apply (50)–(53) to compute T_A, else
Step 2: If $4Amb_L(A) < Amb(A)$ then apply (55)–(56) to compute T_A, else
Step 3: If $4Amb_U(A) < Amb(A)$ then apply (58)–(59) to compute T_A, else
Step 4: apply (60)–(62) to compute T_A.

To avoid checking unnecessary requirements and taking into account the properties (see [10], Proposition 14)

$$4Amb_L(A) < Amb(A) \Rightarrow w(A) > 3Amb(A)$$
$$4Amb_U(A) < Amb(A) \Rightarrow w(A) > 3Amb(A)$$

we use Algorithm 19 if the fuzzy number is almost symmetrical or moderately asymmetrical and the below algorithm if the fuzzy number A is strongly asymmetric to the right or to the left.

Algorithm 20. Step 1: If $4Amb_L(A) < Amb(A)$ then apply (55) and (56) to compute T_A, else
Step 2: If $4Amb_U(A) < Amb(A)$ then apply (58) and (59) to compute T_A, else
Step 3: If $w(A) > 3Amb(A)$ then apply (60)–(62) to compute T_A, else
Step 4: apply (50)–(53) to compute T_A.

As a conclusion, for more vague fuzzy numbers the approximation is a trapezoidal fuzzy number computed by (50)–(53). For less vague fuzzy numbers the approximation is a triangular fuzzy number.

The following short algorithm describes the computing of the nearest symmetric trapezoidal fuzzy number preserving the ambiguity of a fuzzy number A (see Theorem 12).

Algorithm 21. Step 1: If $w(A) \leq 3Amb(A)$ then apply (67)–(70) to compute S_A, else
Step 2: apply (72)–(74) to compute S_A.

The computing of the nearest triangular fuzzy number preserving the ambiguity of a fuzzy number A (see Theorem 15) can be given in the following form:

Algorithm 22. Step 1: If $4Amb_L(A) < Amb(A)$ then apply (88)–(89) to compute t_A, else
Step 2: If $4Amb_U(A) < Amb(A)$ then apply (91) and (92) to compute t_A, else
Step 3: Apply (84)–(86) to compute t_A.

8. Properties

Scale invariance, translation invariance and additivity are between the properties which have been studied for trapezoidal or triangular approximations (see [4,6,7,10,46,47]).

Because

$$Amb(A+z) = Amb(A),$$

for every $A \in F(\mathbb{R})$, $z \in \mathbb{R}$, and the distance d in (7) is translation invariant, that is

$$d(A+z, B+z) = d(A, B),$$

for every $A, B \in F(\mathbb{R})$, $z \in \mathbb{R}$, from Theorem 1 in [13] we obtain the translation invariance of the operators given in Theorems 6, 10, 12, 15 and 17. Because

$$Amb(\lambda \cdot A) = |\lambda| \, Amb(A),$$

for every $A \in F(\mathbb{R})$, $\lambda \in \mathbb{R}$, and the distance d is scale invariant, that is

$$d(\lambda \cdot A, \lambda \cdot B) = |\lambda| \, d(A, B),$$

for every $A, B \in F(\mathbb{R})$, $\lambda \in \mathbb{R}$, from Theorem 4 in [13] we obtain the scale invariance of the operators given in Theorems 6, 10, 12, 15 and 17.

It is immediate that the operators given in Theorems 6 and 17 are additive, that is

$$I_A + I_B = I_{A+B}$$

and

$$s_A + s_B = s_{A+B},$$

for every $A, B \in F(\mathbb{R})$. Unfortunately, the trapezoidal approximation operators in Theorems 10, 12 and the triangular approximation operator in Theorem 15 are not additive.

Example 23. Case (ii) in Theorem 10 is applicable to fuzzy number $A = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0, 1]$ given by $A_L(\alpha) = \sqrt{\alpha}$, $A_U(\alpha) = 1$ and

$$T_A = \left(\frac{23}{60}, \frac{59}{60}, \frac{59}{60}, \frac{59}{60}\right).$$

If $B = (0, 0, 0, 1)$ then

$$T_B = (0, 0, 0, 1)$$

and

$$T_A + T_B = \left(\frac{23}{60}, \frac{59}{60}, \frac{59}{60}, \frac{119}{60}\right).$$

On the other hand,

$$(A+B)_L(\alpha) = \sqrt{\alpha},$$
$$(A+B)_U(\alpha) = 2 - \alpha,$$

the case (iv) in Theorem 10 is applicable and

$$T_{A+B} = \left(\frac{1}{3}, \frac{31}{30}, \frac{31}{30}, \frac{29}{15}\right),$$

therefore

$$T_A + T_B \neq T_{A+B}.$$

We easy get

$$t_A = \left(\frac{23}{60}, \frac{59}{60}, \frac{59}{60}\right),$$
$$t_B = (0, 0, 1)$$

and

$$t_{A+B} = \left(\frac{1}{3}, \frac{31}{30}, \frac{29}{15}\right),$$

therefore

$$t_A + t_B \neq t_{A+B}.$$

Example 24. If $A, B \in F(\mathbb{R})$, A as in Example 23 and $B = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0, 1]$ is given by $B_L(\alpha) = 0$, $B_U(\alpha) = 2 - \sqrt{\alpha}$ then we use (*ii*) and respectively (i) in Theorem 12 to compute S_A and S_B. We obtain

$$S_A = \left(\frac{8}{15}, \frac{5}{6}, \frac{5}{6}, \frac{17}{15}\right)$$

and

$$S_B = \left(-\frac{1}{5}, \frac{1}{5}, \frac{17}{15}, \frac{23}{15}\right).$$

On the other hand, Theorem 12, (i) is applicable to compute S_{A+B} and

$$S_{A+B} = \left(\frac{4}{15}, \frac{16}{15}, \frac{29}{15}, \frac{41}{15}\right),$$

therefore

$$S_A + S_B \neq S_{A+B}.$$

Another very important property that a trapezoidal or triangular approximation operator should posses is the continuity (see [28]). In what follows, we prove that the approximation operators given in Sections $4-6$ own this property. In fact we prove that the discussed operators satisfy a stronger condition, namely, they are Lipschitz. In this sense we have the following.

Theorem 25. (i) *The nearest trapezoidal approximation operator preserving the ambiguity given in Theorem 10 satisfies the inequality*

$$d(T_A, T_B) \leq \sqrt{10 + 4\sqrt{2}}d(A, B),$$

for all $A, B \in F(\mathbb{R})$.

(ii) *The nearest symmetric trapezoidal approximation operator preserving the ambiguity given in Theorem 12 satisfies*

$$d(S_A, S_B) \leqslant 2\sqrt{3}d(A, B),$$

for all $A, B \in F(\mathbb{R})$.

(iii) *The nearest triangular approximation operator preserving the ambiguity given in Theorem 15 satisfies*

$$d\left(t_A, t_B\right) \leqslant \sqrt{10+4\sqrt{2}}d\left(A, B\right),$$

for all $A, B \in F(\mathbb{R})$.

(iv) *The nearest symmetric triangular approximation operator preserving the ambiguity given in Theorem 17 satisfies*

$$d\left(s_A, s_B\right) \leqslant \sqrt{6}d\left(A, B\right),$$

for all $A, B \in F(\mathbb{R})$.

Proof. (i) Let us consider two fuzzy numbers A and B,

$$A = \left[A_L(\alpha), A_U(\alpha)\right],$$
$$B = \left[B_L(\alpha), B_U(\alpha)\right], \alpha \in [0, 1],$$

$T_e(A) = [l^e, u^e, x^e, y^e]$, $T_e(B) = [l'^e, u'^e, x'^e, y'^e]$ the extended trapezoidal approximations of A and B and $T_A = [l_T, u_T, x_T, y_T]$, $T_B = [l'_T, u'_T, x'_T, y'_T]$ the trapezoidal approximations preserving the ambiguity of A and B, respectively. We also consider the points $A_e(x^e, y^e)$, $B_e(x'^e, y'^e)$, $A_0(x_T, y_T)$ and $B_0(x'_T, y'_T)$. Relation (16) implies

$$d^2\left(T_A, T_B\right) = \left(l_T - l'_T\right)^2 + \left(u_T - u'_T\right)^2 + \frac{1}{12}\left(x_T - x'_T\right)^2 + \frac{1}{12}\left(y_T - y'_T\right)^2. \tag{99}$$

The Cauchy–Buniakowski–Schwarz inequality, (38) and (39) imply

$$\begin{aligned}
&\left(l_T - l'_T\right)^2 + \left(u_T - u'_T\right)^2 \\
&= \frac{1}{144}[(x'_T - x_T) + (y'_T - y_T) + 12(l^e - l'^e) + (x^e - x'^e) + (y^e - y'^e)]^2 \\
&\quad + \frac{1}{144}[(x_T - x'_T) + (y_T - y'_T) + 12(u^e - u'^e) + (x'^e - x^e) + (y'^e - y^e)]^2 \\
&= \frac{1}{288}[12(l^e - l'^e) + 12(u^e - u'^e)]^2 \\
&\quad + \frac{1}{288}[2(x'_T - x_T) + 2(y'_T - y_T) + 12(l^e - l'^e) - 12(u^e - u'^e) + 2(x^e - x'^e) + 2(y^e - y'^e)]^2 \\
&= \frac{1}{2}[(l^e - l'^e) + (u^e - u'^e)]^2 \\
&\quad + \frac{1}{72}[(x'_T - x_T) + (y'_T - y_T) + 6(l^e - l'^e) - 6(u^e - u'^e) + (x^e - x'^e) + (y^e - y'^e)]^2 \\
&\leq \left[(l^e - l'^e)^2 + (u^e - u'^e)^2\right] \\
&\quad + \frac{1}{12}[(x'_T - x_T)^2 + (y'_T - y_T)^2] + 3[(l^e - l'^e)^2 + (u^e - u'^e)^2] \\
&\quad + \frac{1}{12}[(x^e - x'^e)^2 + (y^e - y'^e)^2] \\
&= \left[(l^e - l'^e)^2 + (u^e - u'^e)^2 + \frac{1}{12}(x^e - x'^e)^2 + \frac{1}{12}(y^e - y'^e)^2\right] \\
&\quad + \frac{1}{12}[(x'_T - x_T)^2 + (y'_T - y_T)^2] + 3[(l^e - l'^e)^2 + (u^e - u'^e)^2] \\
&= d^2(T_e(A), T_e(B)) + \frac{1}{12}[(x'_T - x_T)^2 + (y'_T - y_T)^2] + 3[(l^e - l'^e)^2 + (u^e - u'^e)^2].
\end{aligned}$$

Taking into account Proposition 2 and Remark 3, we easily get

$$\left(l_T - l'_T\right)^2 + \left(u_T - u'_T\right)^2 \leq 4d^2(A, B) + \frac{1}{12}[(x'_T - x_T)^2 + (y'_T - y_T)^2].$$

Substituting in (99) we obtain

$$d^2\,(T_A,T_B)\leq 4d^2(A,B)+\frac{1}{6}[(x'_T-x_T)^2+(y'_T-y_T)^2]$$

or

$$d^2\,(T_A,T_B)\leq 4d^2(A,B)+\frac{1}{6}d_E^2(A_0,B_0), \tag{100}$$

where d_E denotes the Euclidean metric on $\mathbb{R}^2$. Let us assume (contrariwise the proof is similar)

$$6u'^e-6l'^e-x'^e-y'^e\geq 6u^e-6l^e-x^e-y^e.$$

We consider

$$M_A=\left\{(x,y)\in\mathbb{R}^2:x\geq 0,y\geq 0,x+y\leq 3u^e-3l^e-\frac{1}{2}x^e-\frac{1}{2}y^e\right\},$$
$$M_B=\left\{(x,y)\in\mathbb{R}^2:x\geq 0,y\geq 0,x+y\leq 3u'^e-3l'^e-\frac{1}{2}x'^e-\frac{1}{2}y'^e\right\}$$

and

$$C\left(3u^e-3l^e-\frac{1}{2}x^e-\frac{1}{2}y^e,0\right), \tag{101}$$
$$C'\left(0,3u^e-3l^e-\frac{1}{2}x^e-\frac{1}{2}y^e\right), \tag{102}$$
$$G\left(3u'^e-3l'^e-\frac{1}{2}x'^e-\frac{1}{2}y'^e,0\right), \tag{103}$$
$$G'\left(0,3u'^e-3l'^e-\frac{1}{2}x'^e-\frac{1}{2}y'^e\right), \tag{104}$$

the points which define the closed convex sets M_A and M_B in the Euclidean space $\mathbb{R}^2$ (see Fig. 3).

According to Theorem 9 we get

$$A_0=P_{M_A}\,(A_e)$$

and

$$B_0=P_{M_B}\,(B_e)\,.$$

We denote by B_1 the projection of B_0 on the convex set M_A, that is the unique element in M_A which minimizes $D_E\,(B_0,Q)$, where $Q\in M_A$. It is easy to check that B_1 is the projection of B_e on the set M_A, that is $B_1\in M_A$ and $\min_{R\in M_A} D_E\,(B_e,R)=D_E\,(B_e,B_1)$.

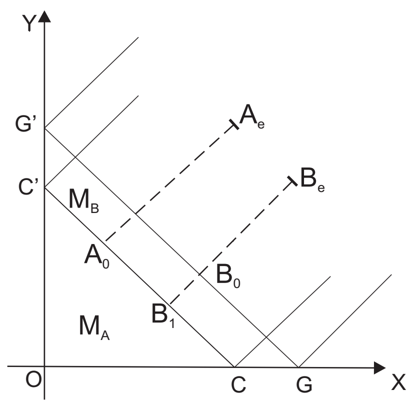

Fig. 3. One case in the evaluating of the Lipschitz constant of T_A.

Also, it is immediate that

$$d_E\left(B_1,B_0\right) \le d_E\left(C,G\right) = d_E\left(C',G'\right).$$

We have ((20)–(23) and the Cauchy–Buniakowski–Schwarz integral inequality are important here)

$$\begin{aligned}
&d_E^2\left(C,G\right)\\
&= [3(u'^e-u^e)-3(l'^e-l^e)-\frac{1}{2}(x'^e-x^e)-\frac{1}{2}(y'^e-y^e)]^2\\
&= \left(6\int_0^1 \alpha\left(B_U(\alpha)-A_U(\alpha)\right)\,\mathrm{d}\alpha - 6\int_0^1 \alpha\left(B_L(\alpha)-A_L(\alpha)\right)\,\mathrm{d}\alpha\right)^2\\
&\le 72\left(\left(\int_0^1 \alpha\left(B_U(\alpha)-A_U(\alpha)\right)\,\mathrm{d}\alpha\right)^2+\left(\int_0^1 \alpha\left(B_L(\alpha)-A_L(\alpha)\right)\,\mathrm{d}\alpha\right)^2\right)\\
&\le 72\left(\int_0^1 \alpha^2\,\mathrm{d}\alpha\int_0^1\left(B_U(\alpha)-A_U(\alpha)\right)^2\,\mathrm{d}\alpha+\int_0^1 \alpha^2\,\mathrm{d}\alpha\int_0^1\left(B_L(\alpha)-A_L(\alpha)\right)^2\,\mathrm{d}\alpha\right)\\
&= 24\left(\int_0^1\left(B_L(\alpha)-A_L(\alpha)\right)^2\,\mathrm{d}\alpha+\int_0^1\left(B_U(\alpha)-A_U(\alpha)\right)^2\,\mathrm{d}\alpha\right)\\
&= 24d^2(A,B),
\end{aligned}$$

that is

$$d_E^2\left(C,G\right) \le 24d^2(A,B) \tag{105}$$

and therefore we get

$$d_E^2\left(B_1,B_0\right) \le 24d^2(A,B).$$

Because M_A is a closed convex subset of $\mathbb{R}^2$ we obtain (see [44], Appendix C)

$$d_E(P_{M_A}(A_e),P_{M_A}(B_e)) \le d_E(A_e,B_e),$$

that is

$$d_E\left(A_0,B_1\right) \le d_E\left(A_e,B_e\right).$$

Since by Remark 3 we get $d_E(A_e,B_e) \le 2\sqrt{3}d(A,B)$, it follows that

$$\begin{aligned}
d_E\left(A_0,B_0\right) &\le d_E\left(A_0,B_1\right)+d_E\left(B_1,B_0\right)\\
&\le d_E\left(A_e,B_e\right)+\sqrt{24}d(A,B)\\
&\le 2\sqrt{3}d(A,B)+2\sqrt{6}d(A,B)\\
&= 2\sqrt{3}\left(1+\sqrt{2}\right)d(A,B).
\end{aligned}$$

Substituting in (100) we obtain

$$d^2\left(T_A,T_B\right) \le 4d^2(A,B)+2(1+\sqrt{2})^2d^2(A,B)$$

which after simple calculus implies

$$d\left(T_A,T_B\right) \le \sqrt{10+4\sqrt{2}}d(A,B)$$

and the proof is complete.

(ii) Let us consider $A,B \in F(\mathbb{R})$,

$$\begin{aligned}
S_A &= [l_S,u_S,x_S,x_S],\\
S_B &= [l'_S,u'_S,x'_S,x'_S]
\end{aligned}$$

the symmetric trapezoidal approximations preserving ambiguity of A and B,

$$T_e(A) = [l^e, u^e, x^e, y^e],$$
$$T_e(B) = [l'^e, u'^e, x'^e, y'^e],$$

the extended trapezoidal approximations of A and B and

$$A_e(x^e, y^e),$$
$$B_e(x'^e, y'^e),$$
$$A_0(x_S, x_S),$$
$$B_0(x'_S, x'_S).$$

Following the same root as in the proof of (i), we get

$$d^2(S_A, S_B) \leq 4d^2(A, B) + \frac{d_E^2(A_0, B_0)}{6}. \quad (106)$$

Without any loss of generality, let us assume that

$$6u'^e - 6l'^e - x'^e - y'^e \geq 6u^e - 6l^e - x^e - y^e.$$

We consider the closed convex sets

$$Q_A = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, x = y, x + y \leq 3u^e - 3l^e - \frac{1}{2}x^e - \frac{1}{2}y^e\right\},$$
$$Q_B = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, x = y, x + y \leq 3u'^e - 3l'^e - \frac{1}{2}x'^e - \frac{1}{2}y'^e\right\}$$

and the points C, C', G, G' given in (101)–(104). It is easy to check that Q_A and Q_B represent the medians of the triangles OCC' and OGG', respectively (see Fig. 3). According to Theorem 9, it follows that

$$A_0 = P_{Q_A}(A_e)$$

and

$$B_0 = P_{Q_B}(B_e).$$

As in the proof of (i), let us consider $B_1 = P_{Q_A}(B_0)$. Again, it is immediate that B_1 is the projection of B_e on the set Q_A. In addition, we have

$$d_E(B_1, B_0) \leq \frac{1}{\sqrt{2}} d_E(C, G) = \frac{1}{\sqrt{2}} d_E\left(C', G'\right).$$

Taking into account that Q_A is closed and convex we obtain ([44])

$$d_E(A_0, B_1) \leq d_E(A_e, B_e).$$

Since in (105) we have $d_E(C, G) \leq 2\sqrt{6}d(A, B)$, it follows that

$$\begin{aligned} d_E(A_0, B_0) &\leq d_E(A_0, B_1) + d_E(B_1, B_0) \\ &\leq d_E(A_e, B_e) + 2\sqrt{3}d(A, B) \\ &\leq 4\sqrt{3}d(A, B). \end{aligned}$$

Substituting in (106) we obtain

$$d(S_A, S_B) \leqslant 2\sqrt{3}d(A, B).$$

(iii) Let us consider $A, B \in F(\mathbb{R})$,

$$t_A = [l_t, u_t, x_t, y_t],$$
$$t_B = [l'_t, u'_t, x'_t, y'_t],$$

the triangular approximations preserving ambiguity of A and B,

$$T_e(A) = [l^e, u^e, x^e, y^e],$$
$$T_e(B) = [l'^e, u'^e, x'^e, y'^e],$$

the extended trapezoidal approximations of A and B and

$$A_e(x^e, y^e),$$
$$B_e(x'^e, y'^e),$$
$$A_0(x_t, y_t),$$
$$B_0(x'_t, y'_t).$$

Again, following the same root as in the proof of (i), we get

$$d^2\,(t_A, t_B) \le 4d^2(A, B) + \frac{d_E^2(A_0, B_0)}{6}.$$

Without any loss of generality, let us assume that

$$6u'^e - 6l'^e - x'^e - y'^e \ge 6u^e - 6l^e - x^e - y^e.$$

We consider the closed convex sets (see Fig. 4)

$$N_A = \left\{(x, y) \in \mathbb{R}^2 : x \ge 0, y \ge 0, x + y = 3u^e - 3l^e - \frac{1}{2}x^e - \frac{1}{2}y^e\right\},$$
$$N_B = \left\{(x, y) \in \mathbb{R}^2 : x \ge 0, y \ge 0, x + y = 3u'^e - 3l'^e - \frac{1}{2}x'^e - \frac{1}{2}y'^e\right\}$$

and the points C, C', G, G' given in (101)–(104) (see Fig. 4). In fact, N_A and N_B represent the closed segments $[CC']$ and $[GG']$ respectively.

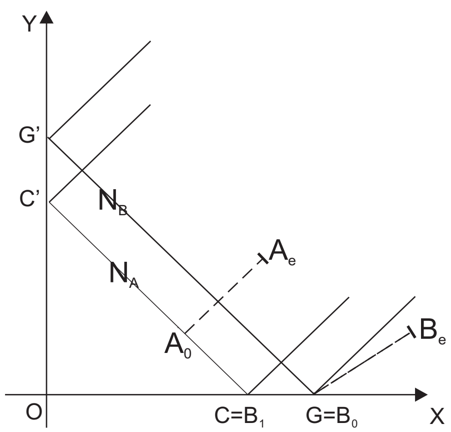

Fig. 4. One case in the evaluating of the Lipschitz constant of t_A.

According to Theorem 14 we get

$$A_0 = P_{N_A}(A_e),$$
$$B_0 = P_{N_B}(B_e).$$

We denote by B_1 the projection of B_0 on the convex set N_A, that is $B_1 = P_{N_A}(B_0)$. It is easy to check that B_1 is the projection of B_e on the set N_A, that is $B_1 \in N_A$ and $\min_{R \in N_A} d_E(B_e, R) = d_E(B_e, B_1)$. Also, it is immediate that

$$d_E(B_1, B_0) \leq d_E(C, G) = d_E\left(C', G'\right).$$

Therefore, from now one, the proof goes exactly on the same pattern with the proof of case (i) and consequently the same type of estimation is obtained, that is

$$d(t_A, t_B) \leq \sqrt{10 + 4\sqrt{2}}d(A, B).$$

(iv) Let us consider $A, B \in F(\mathbb{R})$,

$$s_A = [l_s, u_s, x_s, y_s],$$
$$s_B = [l'_s, u'_s, x'_s, y'_s]$$

the symmetric triangular approximations preserving ambiguity of A and B,

$$T_e(A) = [l^e, u^e, x^e, y^e],$$
$$T_e(B) = [l'^e, u'^e, x'^e, y'^e],$$

the extended trapezoidal approximations of A and B and

$$A_e(x^e, y^e),$$
$$B_e(x'^e, y'^e),$$
$$A_0(x_s, y_s),$$
$$B_0(x'_s, y'_s).$$

Also, we consider the points C, G given in (101), (103). As in the previous cases we get

$$d^2(s_A, s_B) \leq 4d^2(A, B) + \frac{d_E^2(A_0, B_0)}{6}. \tag{107}$$

Because $x_s = y_s$ and $x'_s = y'_s$ we obtain

$$\begin{aligned} & d_E^2(A_0, B_0) \\ &= 2(x_s - x'_s)^2 = \frac{1}{2}\left[\left(3u^e - 3l^e - \frac{1}{2}x^e - \frac{1}{2}y^e\right) - \left(3u'^e - 3l'^e - \frac{1}{2}x'^e - \frac{1}{2}y'^e\right)\right] \\ &= \frac{1}{2}d_E^2(C, G) \end{aligned}$$

which together (105) imply

$$d_E^2(A_0, B_0) \leq 12d^2(A, B).$$

Combining this last inequality with (107) we get

$$d(s_A, s_B) \leq \sqrt{6}d(A, B)$$

and the proof is complete. □

At the end of this section we mention that the best Lipschitz constant of the trapezoidal approximation operator preserving the expected interval was determined in [21]. The calculus of the best Lipschitz constant in the case of approximation operators studied in the present paper is sophisticated too. A future research will be dedicated to this subject.

9. Trapezoidal and triangular approximation preserving ambiguity and aggregation

The important problem whether it is better to simplify initial data before using an aggregation operator or conversely, to aggregate original fuzzy values and then to simplify the output is addressed in [11] with respect to trapezoidal approximation and trapezoidal approximation preserving the expected interval of fuzzy numbers. The aim of this section is to prove that the conclusion in [11] remains valid under the approximations introduced in the previous sections.

The ambiguity of a set $\{A_1, A_2, \ldots, A_n\} \subset F(\mathbb{R})$ can be defined in a natural way (see [32] for the case of an intuitionistic fuzzy number, which is a conjunction of two fuzzy numbers) by

$$Amb\,(A_1, A_2, \ldots, A_n) = \frac{1}{n}\left(Amb\,(A_1) + \cdots + Amb\,(A_n)\right).$$

Let us suppose that given fuzzy numbers $A_1, A_2, \ldots, A_n$ should be efficiently aggregated to a fuzzy number such that the ambiguity of aggregation to be equal with the ambiguity of the initial data set $\{A_1, A_2, \ldots, A_n\}$. Therefore, we try to find a trapezoidal fuzzy number $T_{A_1,A_2,\ldots,A_n} = [l^*, u^*, x^*, y^*]$ which is the nearest one to all members of the set $A_1, A_2, \ldots, A_n$ with respect to the distance d given in (7) and, in addition,

$$Amb\,(T_{A_1,A_2,\ldots,A_n}) = Amb\,(A_1, A_2, \ldots, A_n).$$

In fact, we are looking for a trapezoidal fuzzy number $T_{A_1,A_2,\ldots,A_n}$ such that

$$D^2\left((A_1, \ldots, A_n), T_{A_1,A_2,\ldots,A_n}\right) = \sum_{i=1}^{n} d^2\left(A_i, T_{A_1,A_2,\ldots,A_n}\right) \tag{108}$$

is minimized and the conditions

$$\frac{6u^* - 6l^* - x^* - y^*}{12} = \frac{1}{n}\sum_{i=1}^{n} Amb\,(A_i), \tag{109}$$

$$x^* \geq 0, \tag{110}$$

$$y^* \geq 0, \tag{111}$$

$$x^* + y^* \leq 2\left(u^* - l^*\right), \tag{112}$$

that assure the preservation of ambiguity and our output is really a trapezoidal fuzzy number (see (8)–(10)) are satisfied. According with Proposition 4, after some elementary calculus, (109) can be rewritten as

$$6u^* - 6l^* - x^* - y^* = \frac{6}{n}\sum_{i=1}^{n} u_i^e - \frac{6}{n}\sum_{i=1}^{n} l_i^e - \frac{1}{n}\sum_{i=1}^{n} x_i^e - \frac{1}{n}\sum_{i=1}^{n} y_i^e. \tag{113}$$

Taking into account Proposition 1 this problem becomes to find such $T_{A_1,A_2,\ldots,A_n}$ that

$$\sum_{i=1}^{n} d^2\left(T_e(A_i), T_{A_1,A_2,\ldots,A_n}\right) \to \min \tag{114}$$

with respect to (110)–(113), where $T_e(A_i) = [l_i^e, u_i^e, x_i^e, y_i^e]$ is the extended trapezoidal approximation of A_i, $i \in \{1, \ldots, n\}$.

Because u^* is expressed from (113) it reduces ((16) is important here) to find l^*, x^* and y^* such that

$$\left\{\sum_{i=1}^{n}(l^* - l_i^e)^2 + \frac{1}{12}\sum_{i=1}^{n}(x^* - x_i^e)^2 + \frac{1}{12}\sum_{i=1}^{n}(y^* - y_i^e)^2 \right. + \sum_{i=1}^{n}\left(u_i^e - l^* - \frac{1}{6}x^* - \frac{1}{6}y^* - \frac{1}{n}\sum_{i=1}^{n}u_i^e \right. \left.\left. + \frac{1}{n}\sum_{i=1}^{n}l_i^e + \frac{1}{6n}\sum_{i=1}^{n}x_i^e + \frac{1}{6n}\sum_{i=1}^{n}y_i^e\right)^2\right\} \to \min, \tag{115}$$

$$x^* \geq 0, \tag{116}$$
$$y^* \geq 0, \tag{117}$$
$$x^* + y^* \leq \frac{3}{n}\sum_{i=1}^{n} u_i^e - \frac{3}{n}\sum_{i=1}^{n} l_i^e - \frac{1}{2n}\sum_{i=1}^{n} x_i^e - \frac{1}{2n}\sum_{i=1}^{n} y_i^e. \tag{118}$$

According to the Karush–Kuhn–Tucker theorem (Theorem 5), (l^*, x^*, y^*) is a solution of (115)–(118) if and only if there exist μ_1, μ_2, μ_3 such that the following system holds

$$2\sum_{i=1}^{n}(l^* - l_i^e) - 2\sum_{i=1}^{n}\left(u_i^e - l^* - \frac{1}{6}x^* - \frac{1}{6}y^* - \frac{1}{n}\sum_{i=1}^{n} u_i^e\right. \tag{119}$$
$$\left. + \frac{1}{n}\sum_{i=1}^{n} l_i^e + \frac{1}{6n}\sum_{i=1}^{n} x_i^e + \frac{1}{6n}\sum_{i=1}^{n} y_i^e\right) = 0,$$
$$-\frac{1}{3}\sum_{i=1}^{n}\left(u_i^e - l^* - \frac{1}{6}x^* - \frac{1}{6}y^* - \frac{1}{n}\sum_{i=1}^{n} u_i^e + \frac{1}{n}\sum_{i=1}^{n} l_i^e\right. \tag{120}$$
$$\left. + \frac{1}{6n}\sum_{i=1}^{n} x_i^e + \frac{1}{6n}\sum_{i=1}^{n} y_i^e\right) + \frac{1}{6}\sum_{i=1}^{n}(x^* - x_i^e) - \mu_1 + \mu_3 = 0,$$
$$-\frac{1}{3}\sum_{i=1}^{n}\left(u_i^e - l^* - \frac{1}{6}x^* - \frac{1}{6}y^* - \frac{1}{n}\sum_{i=1}^{n} u_i^e + \frac{1}{n}\sum_{i=1}^{n} l_i^e\right. \tag{121}$$
$$\left. + \frac{1}{6n}\sum_{i=1}^{n} x_i^e + \frac{1}{6n}\sum_{i=1}^{n} y_i^e\right) + \frac{1}{6}\sum_{i=1}^{n}(y^* - y_i^e) - \mu_2 + \mu_3 = 0,$$
$$x^* \geq 0, \tag{122}$$
$$y^* \geq 0, \tag{123}$$
$$x^* + y^* \leq \frac{3}{n}\sum_{i=1}^{n} u_i^e - \frac{3}{n}\sum_{i=1}^{n} l_i^e - \frac{1}{2n}\sum_{i=1}^{n} x_i^e - \frac{1}{2n}\sum_{i=1}^{n} y_i^e, \tag{124}$$
$$\mu_1 \geq 0, \tag{125}$$
$$\mu_2 \geq 0, \tag{126}$$
$$\mu_3 \geq 0, \tag{127}$$
$$\mu_1 x^* = 0, \tag{128}$$
$$\mu_2 y^* = 0, \tag{129}$$
$$\mu_3\left(x^* + y^* - \frac{3}{n}\sum_{i=1}^{n} u_i^e + \frac{3}{n}\sum_{i=1}^{n} l_i^e + \frac{1}{2n}\sum_{i=1}^{n} x_i^e + \frac{1}{2n}\sum_{i=1}^{n} y_i^e\right) = 0. \tag{130}$$

Taking into account Propositions 1 and 4, the problem to find the nearest trapezoidal fuzzy number $T_{\overline{A}} = \left[\overline{l}, \overline{u}, \overline{x}, \overline{y}\right]$ of fuzzy number $\overline{A} = \frac{1}{n} \cdot (A_1 + A_2 + \ldots + A_n)$, with respect to the distance d given in (7), such that the ambiguity is preserved, that is

$$Amb\,(T_{\overline{A}}) = Amb\left(\overline{A}\right)$$

is equivalent to solve

$$d^2(T_e\left(\overline{A}\right), T_{\overline{A}}) \rightarrow \min \tag{131}$$

with respect to

$$\overline{x} \geq 0, \tag{132}$$
$$\overline{y} \geq 0, \tag{133}$$
$$\overline{x} + \overline{y} \leq 2\left(\overline{u} - \overline{l}\right), \tag{134}$$
$$Amb\,(T_{\overline{A}}) = Amb\left(T_e\left(\overline{A}\right)\right). \tag{135}$$

Because

$$T_e\left(\overline{A}\right)=\left(\frac{1}{n}\sum_{i=1}^{n}l_i^e,\ \frac{1}{n}\sum_{i=1}^{n}u_i^e,\ \frac{1}{n}\sum_{i=1}^{n}x_i^e,\ \frac{1}{n}\sum_{i=1}^{n}y_i^e\right),$$

according with (15) and (16), (131)–(135) becomes

$$\left\{\left(\bar{l}-\frac{1}{n}\sum_{i=1}^{n}l_i^e\right)^2+\left(\bar{u}-\frac{1}{n}\sum_{i=1}^{n}u_i^e\right)^2\right.$$
$$\left.+\frac{1}{12}\left(\bar{x}-\frac{1}{n}\sum_{i=1}^{n}x_i^e\right)^2+\frac{1}{12}\left(\bar{y}-\frac{1}{n}\sum_{i=1}^{n}y_i^e\right)^2\right\}\rightarrow \min$$

under (132)–(134) and

$$6\bar{u}-6\bar{l}-\bar{x}-\bar{y}=\frac{6}{n}\sum_{i=1}^{n}u_i^e-\frac{6}{n}\sum_{i=1}^{n}l_i^e-\frac{1}{n}\sum_{i=1}^{n}x_i^e-\frac{1}{n}\sum_{i=1}^{n}y_i^e. \tag{136}$$

It is immediate that $T_{\overline{A}}=\left[\bar{l},\bar{u},\bar{x},\bar{y}\right]$ is a solution of the problem (131)–(135) if and only if $\bar{u}$ satisfies (136) and $\left(\bar{l},\bar{x},\bar{y}\right)$ is a solution of the problem

$$\begin{aligned}
&\left(\bar{l}-\frac{1}{n}\sum_{i=1}^{n}l_i^e\right)^2+\left(\bar{l}+\frac{1}{6}\bar{x}+\frac{1}{6}\bar{y}-\frac{1}{n}\sum_{i=1}^{n}l_i^e-\frac{1}{6n}\sum_{i=1}^{n}x_i^e-\frac{1}{6n}\sum_{i=1}^{n}y_i^e\right)^2\\
&\quad+\frac{1}{12}\left(\bar{x}-\frac{1}{n}\sum_{i=1}^{n}x_i^e\right)^2+\frac{1}{12}\left(\bar{y}-\frac{1}{n}\sum_{i=1}^{n}y_i^e\right)^2\rightarrow \min,\\
&\bar{x}\geq 0,\\
&\bar{y}\geq 0,\\
&\bar{x}+\bar{y}\leq\frac{3}{n}\sum_{i=1}^{n}u_i^e-\frac{3}{n}\sum_{i=1}^{n}l_i^e-\frac{1}{2n}\sum_{i=1}^{n}x_i^e-\frac{1}{2n}\sum_{i=1}^{n}y_i^e.
\end{aligned}\tag{137}$$

According to the Karush–Kuhn–Tucker theorem (Theorem 5), $\left(\bar{l},\bar{x},\bar{y}\right)$ is a solution of (137) if and only if there exist ν_1,ν_2,ν_3 such that the following system holds

$$2\left(\bar{l}-\frac{1}{n}\sum_{i=1}^{n}l_i^e\right)+2\left(\bar{l}+\frac{1}{6}\bar{x}+\frac{1}{6}\bar{y}-\frac{1}{n}\sum_{i=1}^{n}l_i^e\right.\tag{138}$$
$$\left.-\frac{1}{6n}\sum_{i=1}^{n}x_i^e-\frac{1}{6n}\sum_{i=1}^{n}y_i^e\right)=0,$$

$$\frac{1}{3}\left(\bar{l}+\frac{1}{6}\bar{x}+\frac{1}{6}\bar{y}-\frac{1}{n}\sum_{i=1}^{n}l_i^e-\frac{1}{6n}\sum_{i=1}^{n}x_i^e-\frac{1}{6n}\sum_{i=1}^{n}y_i^e\right)\tag{139}$$
$$+\frac{1}{6}\left(\bar{x}-\frac{1}{n}\sum_{i=1}^{n}x_i^e\right)-\nu_1+\nu_3=0,$$

$$\frac{1}{3}\left(\bar{l}+\frac{1}{6}\bar{x}+\frac{1}{6}\bar{y}-\frac{1}{n}\sum_{i=1}^{n}l_i^e-\frac{1}{6n}\sum_{i=1}^{n}x_i^e-\frac{1}{6n}\sum_{i=1}^{n}y_i^e\right)\tag{140}$$

$$+\frac{1}{6}\left(\overline{y}-\sum_{i=1}^{n}y_i^e\right)-\nu_2+\nu_3=0,$$

$$\overline{x}\geq 0, \tag{141}$$

$$\overline{y}\geq 0, \tag{142}$$

$$\overline{x}+\overline{y}\leq \frac{3}{n}\sum_{i=1}^{n}u_i^e-\frac{3}{n}\sum_{i=1}^{n}l_i^e-\frac{1}{2n}\sum_{i=1}^{n}x_i^e-\frac{1}{2n}\sum_{i=1}^{n}y_i^e, \tag{143}$$

$$\nu_1\geq 0, \tag{144}$$

$$\nu_2\geq 0, \tag{145}$$

$$\nu_3\geq 0, \tag{146}$$

$$\nu_1\overline{x}=0, \tag{147}$$

$$\nu_2\overline{y}=0, \tag{148}$$

$$\nu_3\left(\overline{x}+\overline{y}-\frac{3}{n}\sum_{i=1}^{n}u_i^e+\frac{3}{n}\sum_{i=1}^{n}l_i^e+\frac{1}{2n}\sum_{i=1}^{n}x_i^e+\frac{1}{2n}\sum_{i=1}^{n}y_i^e\right)=0. \tag{149}$$

If we compare (113) with (136) and (119)–(130) with (138)–(149) the below result is immediate. Because a trapezoidal fuzzy number $[l, u, x, y]$ is symmetric if and only if $x = y$, the above reasoning can be repeated in the case of symmetric trapezoidal approximation.

Theorem 26. *The (symmetric) trapezoidal fuzzy number nearest to fuzzy numbers $A_1, \ldots, A_n$ which preserves the ambiguity of $\{A_1, \ldots, A_n\}$ is the (symmetric) trapezoidal fuzzy number nearest to fuzzy number $\overline{A} = \frac{1}{n}\cdot(A_1 + A_2 + \ldots + A_n)$ which preserves the ambiguity of $\overline{A}$.*

Example 27. Let us consider the fuzzy numbers A and B, given by their α-cuts, $\alpha \in [0, 1]$,

$$A_\alpha=\left[-1+\alpha^2, 4-2\alpha^2\right]$$

and

$$B_\alpha=\left[1+\alpha^2, 3-\alpha^2\right].$$

Because

$$\left(\frac{1}{2}\cdot(A+B)\right)_\alpha=\left[\alpha^2, \frac{7}{2}-\frac{3}{2}\alpha^2\right], \alpha\in[0,1],$$

according with Theorems 10, (i) and 12, (i), we get

$$T_{\frac{1}{2}\cdot(A+B)}=\left(-\frac{1}{6}, \frac{5}{6}, \frac{9}{4}, \frac{15}{4}\right)$$

and

$$S_{\frac{1}{2}\cdot(A+B)}=\left(-\frac{7}{24}, \frac{23}{24}, \frac{19}{8}, \frac{29}{8}\right).$$

Theorem 26 implies that the trapezoidal fuzzy number nearest to fuzzy numbers A and B, which preserves the ambiguity of $\{A, B\}$ is $\left(-\frac{1}{6}, \frac{5}{6}, \frac{9}{4}, \frac{15}{4}\right)$ too. From the same theorem, we conclude that the symmetric trapezoidal fuzzy number nearest to fuzzy numbers A and B, which preserves the ambiguity of $\{A, B\}$ is $\left(-\frac{7}{24}, \frac{23}{24}, \frac{19}{8}, \frac{29}{8}\right)$ too.

We recall, a trapezoidal fuzzy number $[l, u, x, y]$ is triangular if and only if $x + y = 2(u - l)$. A similar result with Theorem 26 remains valid in the case of the nearest (symmetric) triangular fuzzy number preserving ambiguity of a fuzzy number. We omit the proof.

Theorem 28. *The (symmetric) triangular fuzzy number nearest to fuzzy numbers $A_1, \ldots, A_n$ which preserves the ambiguity of $\{A_1, \ldots, A_n\}$ is the (symmetric) triangular fuzzy number nearest to fuzzy number $\overline{A} = \frac{1}{n}\cdot(A_1 + A_2 + \ldots + A_n)$ which preserves the ambiguity of $\overline{A}$.*

Theorems 26 and 28 show that there is no difference whether the trapezoidal and triangular approximations preserving the ambiguity are performed before or after aggregation with respect to average.

Example 29. If $A, B \in F(\mathbb{R})$ are as in Example 27 then, according with Theorems 15, (i) and 17 we get

$$t_{\frac{1}{2}\cdot(A+B)} = \left(-\frac{19}{12}, \frac{37}{24}, \frac{31}{6}\right)$$

and

$$s_{\frac{1}{2}\cdot(A+B)} = \left(-\frac{41}{24}, \frac{5}{3}, \frac{121}{24}\right).$$

Theorem 28 proves that the triangular fuzzy number nearest to fuzzy numbers A and B, which preserves the ambiguity of $\{A, B\}$ is $\left(-\frac{19}{12}, \frac{37}{24}, \frac{31}{6}\right)$ and the symmetric triangular fuzzy number nearest to fuzzy numbers A and B, which preserves the ambiguity of $\{A, B\}$ is $\left(-\frac{41}{24}, \frac{5}{3}, \frac{121}{24}\right)$.

10. Conclusion

We continue the list of approximations of fuzzy numbers under conditions. The ambiguity of a fuzzy number [24] is the characteristic with a central role in the present paper. Strictly speaking, the results are not better or worse than other results of approximation obtained in this topic. In fact, different approximations can be compared only taking into account their properties and importance from the theoretical and practical point of view. As example, the trapezoidal approximation operator given in [4] preserves the core, it is computationally efficient and additive, but discontinuous (see [14]) and does not preserve the ambiguity or the expected value. The trapezoidal approximation in [44] give us the nearest trapezoidal fuzzy number of a fuzzy number with respect to the average Euclidean distance (7). The operator is continuous [44], but it is non-additive, not preserves any important characteristic (expected value, expected interval, ambiguity, value, etc.) and it is not very good from the computational point of view. The approximations proposed in the present paper are the best in practical or theoretical developments where it is advisable as the data (expressed by fuzzy numbers) to be simplified but the ambiguity must be preserved. Operators of approximation of fuzzy numbers by real intervals, trapezoidal fuzzy numbers, symmetric trapezoidal fuzzy numbers, triangular fuzzy numbers and symmetric triangular fuzzy numbers, preserving the ambiguity, are given in Theorems 6, 10, 12, 15 and 17. Algorithms of calculus, the study of continuity and behavior related to aggregation were debated too. In this way the main properties of the operators were studied even if some improvements are possible. As example, we prove that the operators are Lipschitz and the continuity is an immediate consequence. Nevertheless, the geometrical reasonings used in the paper do not furnish the best Lipschitz constants as in the case of the trapezoidal approximation preserving the expected interval was performed [21]. The method to find the best Lipschitz constant for the operators given in the present paper was already elaborated in [22], but the concrete reasonings would be very sophisticated and the presentation too large to be treated here.

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