Non-fragile switching tracking control for a flexible air-breathing hypersonic vehicle based on polytopic LPV model

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Abstract This article proposes a linear parameter varying (LPV) switching tracking control scheme for a flexible air-breathing hypersonic vehicle (FAHV). First, a polytopic LPV model is constructed to represent the complex nonlinear longitudinal model of the FAHV by using Jacobian linearization and tensor-product (T-P) model transformation approach. Second, for less conservative controller design purpose, the flight envelope is divided into four sub-regions and a non-fragile LPV controller is designed for each parameter sub-region. These non-fragile LPV controllers are then switched in order to guarantee the closed-loop FAHV system to be asymptotically stable and satisfy a specified performance criterion. The desired non-fragile LPV switching controller is found by solving a convex constraint problem which can be efficiently solved using available linear matrix inequality (LMI) techniques, and robust stability analysis of the closed-loop FAHV system is verified based on multiple Lyapunov functions (MLFs). Finally, numerical simulations have demonstrated the effectiveness of the proposed approach.

1. Introduction

The scramjet-powered air-breathing hypersonic vehicles (AHVs) present a more cost-efficient way to make access to space routine, or even make the space travel routine and intercontinental travel as easy as intercity travel. The kind of concept aircraft in astronautics fields, it has been studied far and wide in recent years for its ability of long-distance voyage, global deployment in a short time, high-speed overload and repetitive tasks in remote. Flight control design of AHVs poses a challenge due to strong coupling effects between the aerodynamics, propulsion system and the elastic vibrations. One of the earlier studies in this area was performed in Ref. 5. This preliminary
study employed multivariable linear control for the longitudinal model of hypersonic vehicles. A variety of different control methods have been presented in subsequent research efforts. Based on input-output linearization technique, Xu et al.5 presented the sliding mode control for altitude and velocity tracking control considering model uncertainties and varying flight conditions. A state feedback controller was designed in Ref.7, which guarantees a prescribed performance cost with the simultaneous consideration of poles assignment for the closed-loop system. However, in practice, only the vehicle’s velocity and altitude are measurable, therefore, the output feedback control problem for a genetic hypersonic vehicle was addressed in Refs.8,9. In addition, adaptive control10,11 back-stepping method12,13 feed-forward linearization14 and neural control15 have also been intensively concerned in hypersonic vehicles control.

More recently, linear parameter varying (LPV) systems have been widely investigated and they are ubiquitous in chemical processes, robotics systems, and many manufacturing processes.16,17 LPV control has emerged as an effective control technique to accommodate plants that exhibit parameter-dependent dynamics. Based on D-K iteration algorithm, a robust LPV controller which is scheduled on Mach number and altitude for AHVs is presented in Ref.18. The authors design a novel model predictive controller for the complicated aerodynamics of a hypersonic vehicle in Ref.19. More especially, in Ref.20, a self-scheduled control structure is presented for a nonlinear longitudinal model of hypersonic vehicle.

Note that previous results show that a single LPV controller may not be effective in cases of plants with drastic dynamic changes or when highly demanding specifications must be fulfilled only in certain sectors of the parameter space. Recently, switched systems have drawn increased attention and controller switching provides an effective mechanism to cope with highly complex systems.21,22 Especially, switching LPV control techniques have been widely used in the area ranging from aerospace to process control.23–25 Generally speaking, an implicit assumption in the controller design is that the controller will be implemented exactly. However, in practice, the parameters of the controller are possible to accrue some parameter variations or gain variations due to finite word length, the existence of the parameter drift and round-off errors in numerical computations by computers is frequently encountered.26 This is the so-called fragility problem of controllers that has attracted widespread attention and some meaningful results are presented.27,28 As is well-known, a relatively small perturbation of the controller parameters might degrade the performance of the closed-loop system. Therefore, to design a controller which is insensitive to uncertainties, the research of non-fragile control is important and significant.

Motivated by the aforementioned reasons, this paper is concerned with the problem of non-fragile LPV switching control for the flexible air-breathing hypersonic vehicle (FAHV). Based on Jacobian linearization and tensor-product (T-P) model transformation approach, a polytopic LPV model is constructed to represent the complex nonlinear longitudinal model of the FAHV. Then, a novel non-fragile LPV switching controller is designed for the FAHV. The existence conditions for the admissible controller are formulated in the form of linear matrix inequalities (LMIs). Finally, nonlinear simulation comparisons demonstrate the effectiveness and advantage of the proposed control design methods.

2. Model description

2.1. Nonlinear longitudinal model for FAHV

The FAHV model considered in this paper is developed by Bolender and Doman.29 Assuming a flat Earth and

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$m$</td>
<td>vehicle mass</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of air</td>
</tr>
<tr>
<td>$\dot{q}$</td>
<td>dynamic pressure</td>
</tr>
<tr>
<td>$S$</td>
<td>reference area</td>
</tr>
<tr>
<td>$h$</td>
<td>altitude</td>
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<tr>
<td>$V$</td>
<td>velocity</td>
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<td>$L$</td>
<td>lift</td>
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<td>$D$</td>
<td>drag</td>
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<tr>
<td>$\alpha$</td>
<td>angle of attack</td>
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<tr>
<td>$\eta$</td>
<td>pitch angle</td>
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<tr>
<td>$Q$</td>
<td>pitch rate</td>
</tr>
<tr>
<td>$I_{yy}$</td>
<td>moment of inertia</td>
</tr>
<tr>
<td>$M_{yy}$</td>
<td>pitching moment</td>
</tr>
<tr>
<td>$\delta_e$</td>
<td>elevator angular deflection</td>
</tr>
<tr>
<td>$\dot{1}/h_a$</td>
<td>air density decay rate</td>
</tr>
<tr>
<td>$N_i$</td>
<td>$i$th generalized moment</td>
</tr>
<tr>
<td>$N_{i0}$</td>
<td>$i$th order contribution of $\alpha$ to $N_i$</td>
</tr>
<tr>
<td>$N_{i2}$</td>
<td>contribution of $\delta_e$ to $N_i$</td>
</tr>
<tr>
<td>$\beta_i(h,\dot{\alpha})$</td>
<td>$i$th thrust fit parameter</td>
</tr>
<tr>
<td>$\eta_i$</td>
<td>$i$th generalized elastic coordinate</td>
</tr>
<tr>
<td>$\zeta_i$</td>
<td>damping ratio for elastic mode $\eta_i$</td>
</tr>
<tr>
<td>$\omega_i$</td>
<td>natural frequency for elastic mode $\eta_i$</td>
</tr>
<tr>
<td>$C_L(\alpha,\delta_e)$</td>
<td>lift coefficient</td>
</tr>
<tr>
<td>$C_D(\alpha,\delta_e)$</td>
<td>drag coefficient</td>
</tr>
<tr>
<td>$C_D^i$</td>
<td>$i$th order coefficient of $\alpha$ contribution to $C_D(\alpha,\delta_e)$</td>
</tr>
<tr>
<td>$C_D^0$</td>
<td>constant term in $C_D(\alpha,\delta_e)$</td>
</tr>
<tr>
<td>$C_L^0$</td>
<td>$i$th order coefficient of $\alpha$ contribution to $C_L(\alpha,\delta_e)$</td>
</tr>
<tr>
<td>$C_L^i$</td>
<td>$i$th order coefficient of $\delta_e$ contribution to $C_L(\alpha,\delta_e)$</td>
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<tr>
<td>$C_{M,\alpha}(\alpha,\delta_e)$</td>
<td>contribution to moment due to pitch rate</td>
</tr>
<tr>
<td>$C_{M,\beta}(\alpha,\delta_e)$</td>
<td>control surface contribution to moment</td>
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<tr>
<td>$C_{M,\alpha}(\alpha,\delta_e)$</td>
<td>$i$th order coefficient of $\alpha$ contribution to $C_{M,\alpha}(\alpha,\delta_e)$</td>
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<tr>
<td>$C_{M,\beta}(\alpha,\delta_e)$</td>
<td>$i$th order coefficient of $\alpha$ to $C_{M,\beta}(\alpha,\delta_e)$</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>stoichiometrically normalized fuel-to-air ratio</td>
</tr>
<tr>
<td>$\bar{\epsilon}$</td>
<td>mean aerodynamic chord</td>
</tr>
<tr>
<td>$c_c$</td>
<td>canard coefficient in $C_{M,\beta}(\alpha,\delta_e)$</td>
</tr>
<tr>
<td>$e_c$</td>
<td>elevator coefficient in $C_{M,\beta}(\alpha,\delta_e)$</td>
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normalizing the span of the vehicle to unit depth, the equations of motion of the longitudinal dynamics are written in the stability axes as

\[
\begin{align*}
\dot{h} &= V \sin(\theta - \alpha) \\
\dot{V} &= \frac{T \cos \alpha - D}{m} - g \sin(\theta - \alpha) \\
\dot{\alpha} &= \frac{1}{mV} (-T \sin \alpha - L) + Q + \frac{g}{V} \cos(\theta - \alpha) \\
\dot{\theta} &= Q \\
\dot{Q} &= M - T \eta_1 \\
\eta_1 &= -2\zeta_1 \omega_1 \eta_1 - \omega_1^2 \eta_1 + N_1 \\
\eta_2 &= -2\zeta_2 \omega_2 \eta_2 - \omega_2^2 \eta_2 + N_2
\end{align*}
\]

where \( T, D, L, M \) and \( N_i \) are defined as follows:

\[
\begin{align*}
T &\approx C_f^1 x^3 + C_f^2 x^2 + C_f^0 x + C_f^0 \\
D &\approx \frac{1}{2} \rho V^2 S C_d(x, \delta_e) \\
L &\approx \frac{1}{2} \rho V^2 S C_l(x, \delta_e) \\
M &\approx z_T T + \frac{1}{2} \rho V^2 S (C_{M,\alpha}(x) + C_{M,\delta_e}(\delta_e)) \\
C_{M,\alpha} &= C_{M,\alpha}^1 x^2 + C_{M,\alpha}^0 x + C_{M,\alpha}^0 \\
C_{M,\delta_e} &= C_{M,\delta_e}^0 \\
N_1 &\approx N_1^1 x^2 + N_1^0 x + N_1^0 \\
N_2 &\approx N_2^1 x^2 + N_2^0 x + N_2^0 \delta_e + N_2^0 \\
\end{align*}
\]

Thrust, drag, lift coefficients and the density of air are denoted by

\[
\begin{align*}
C_f^1 &= \beta_1(h, \eta) \Phi + \beta_2(h, \eta) \\
C_f^2 &= \beta_3(h, \eta) \Phi + \beta_4(h, \eta) \\
C_f^0 &= \beta_5(h, \eta) \Phi + \beta_6(h, \eta) \\
\bar{q} &= \frac{1}{2} \rho V^2 \\
C_d &= C_d^1 x^2 + C_d^0 x + C_d^0 \delta_e + C_d^0 \\
C_l &= C_l^1 x + C_l^0 \delta_e + C_l^0 \\
\rho &= \rho_0 \exp \left( \frac{-h - h_0}{h_s} \right)
\end{align*}
\]

**Remark 1.** This model contains five rigid modes \((h, V, \alpha, \theta, Q)\) and two flexible modes \((\eta_1, \eta_2)\). The control input \(\phi\) and \(\delta_e\) do not occur explicitly in the equations of general longitudinal dynamics for the FAHV model in Eq. (1); however, they appear through the forces and moments which are denoted by \(T, L, D, M, N_1\) and \(N_2\). For more details, the reader could refer to Ref. [29].

2.2. Polytopic LPV model of FAHV

By Jacobian linearization method, the LPV model of the FAHV can be written as

\[
\begin{align*}
\dot{x}(t) &= A(V, h)x(t) + B(V, h)u(t) \\
y(t) &= C(V, h)x(t)
\end{align*}
\]

where

\[
x = \begin{bmatrix} h & V & \alpha & \theta & Q \end{bmatrix} \quad \Phi = \begin{bmatrix} \delta_1 \end{bmatrix}
\]

and the expression of elements of matrices \(A(V, h), B(V, h)\) and \(C(V, h)\) are given in Appendix A.

LPV system (2) can be written as a polytopic LPV system by T-P model transformation approach. The goal of the T-P model transformation is to transform the given state-space model (2) into T-P model form and it has three key steps. The first step is the discretization, the second step is extracting the linear time invariant (LTI) vertex systems from the discretized systems and the third step is defining the continuous weighting functions to the LTI vertex systems. Based on T-P model transformation approach, the LPV system (2) can be transformed as

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{N} z_i(V, h)S_i \Phi + \sum_{i=1}^{N} z_i(V, h)B_i u(t) \\
y(t) &= \sum_{i=1}^{N} z_i(V, h)C_i x(t)
\end{align*}
\]

where row vector \(A_i(p_a(t)) \in \mathbb{R}^n(n = 1, 2, \ldots, N)\) contains one bounded variable and continuous weighting functions \(z_i(p_a(t))\), \(p_a(t) \in [V_{min}, V_{max}] \times [h_{min}, h_{max}]\), \(S_i \in \mathbb{R}^{n \times (i = 1, 2, \ldots, N)}\) and \(I\) denote the dimension of the tensor.

According to Eq. (3), the LPV system (2) can be transformed into the following polytopic system:

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{N} z_i(V, h)S_i \Phi + \sum_{i=1}^{N} z_i(V, h)B_i u(t) \\
y(t) &= \sum_{i=1}^{N} z_i(V, h)C_i x(t)
\end{align*}
\]

where \(z_i(V, h) = \prod_{j=1}^{N} z_{a,j}(p_a(t))\).

2.3. Open-loop simulation verification of the developed polytopic LPV model

In this paper, a polytopic LPV model is developed to represent the original nonlinear model of the FAHV. In order to check if the obtained polytopic LPV model captures the local nonlinearities of the origin nonlinear plant. Here, we provided open-loop simulation verification results. The flight condition is chosen as \(V = 3200\) m/s and \(h = 25900\) m. The two command inputs of the vehicle are defined as

\[
\Phi = \begin{cases} 
0.15 & 0 \leq t < 2 \\
0.25 & t \geq 2 
\end{cases} \quad \delta_e = \begin{cases} 
6.50' & 0 \leq t < 2 \\
7.55' & t \geq 2 
\end{cases}
\]

The open-loop simulation verification results are shown in Fig. 1. From these simulation results, it is observed that the developed polytopic LPV model follows the origin nonlinear model quite closely. So the developed polytopic LPV model captures the local nonlinearities of the origin nonlinear plant.
3. Non-fragile LPV switching tracking control

In this paper, the flight envelope is divided into $M$ smaller sub-regions; for the $s$th ($s = 1, 2, \ldots, M$) sub-region which is denoted by $R_s$, the polytopic LPV system can be written as

$$
\dot{x}(t) = \sum_{i=1}^{N_s} a_{si}(V, h)(A_{si}x(t) + B_{si}u)\
y(t) = \sum_{i=1}^{N_s} a_{si}(V, h)C_{si}x(t)
$$

(5)

where $a_{si}(V, h)$ is weighting function of the $s$th sub-region.

For polytopic LPV system (5), consider the following reference model:

$$
\dot{x}_r(t) = A_{r}x_r(t) + r(t)
$$

(6)

where $x_r(t)$ is the state of reference model and $r(t)$ input vector.

Define tracking error vector as follows:

$$
e(t) = Cx(t) - x_r(t)
$$

(7)

where $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. 

![Fig. 1 Open-loop simulation verification results.](image-url)
According to Eqs. (5) and (6), the polytopic LPV augmented system can be written as
\[
\dot{x}(t) = \sum_{i=1}^{N} a_{ij}(V,h)(A_{ij}x(t) + B_{ij}u_{i} + \bar{r}(t))
\]  
where \(\bar{A}_{ij} = \begin{bmatrix} A_{ij} & 0 \\ 0 & A_{ij} \end{bmatrix}\), \(\bar{B}_{ij} = \begin{bmatrix} B_{ij}^{T} & 0 \end{bmatrix}^{T}\), \(\bar{r}(t) = [0 \quad r'(t)]^{T}\)
and \(\bar{x}(t) = [x'(t) \quad x'(t)]^{T}\).

We define switching characteristic function \(\sigma_{s}\) as follows:
\[
\sigma_{s} = \begin{cases} 1 & (V,h) \in R_{s} \\ 0 & (V,h) \notin R_{s} \end{cases} \quad s = \{1,2,\ldots,m\}
\]  
(9)

Under the switching function \(\sigma_{s}\), the polytopic LPV system (8) can be written as
\[
\dot{x} = \sum_{i=1}^{N} \sum_{j=1}^{m} \sigma_{sij}(V,h)(A_{ij}x(t) + B_{ij}u_{i} + \bar{r}(t))
\]  
(10)

Our control objective is to design a non-fragile LPV switching controller
\[
u_{i} = \sum_{j=1}^{m} \sigma_{sij}(V,h) (K_{ij} + \Delta K)(C_{i}x(t) - x_{i}(t))
\]  
which guarantees the polytopic LPV system (10) to be asymptotically stable and satisfies the following performance index:
\[
\int_{0}^{T_{f}} e^{T}(t)P e(t)dt \leq \gamma
\]  
(12)

In Eqs. (11) and (12), \(\tilde{K}_{ij} = [(K_{ij} + \Delta K)C - (K_{ij} + \Delta K)]\), \(\Delta K = HEF\), \(H,E\) and \(F\) are constant matrices, \(F^{T}F \leq I\). \(e(t)\) is tracking error, \(P\) a symmetry positive definite matrix and \(\gamma\) a constant.

Also, the designed controller (11) can be written as
\[
\nu_{i} = \sum_{j=1}^{M} \sum_{l=1}^{N} \sigma_{ali}(V,h) (\tilde{K}_{ij} + \Delta \tilde{K}) \tilde{x}(t)
\]  
(13)

where \(\tilde{K}_{ij} = [K_{ij}C - K_{ij}]\) and \(\Delta \tilde{K} = [\Delta KC - \Delta K]\).

Controller (13) is applied in system (10), and the closed-loop system can be written as
\[
\dot{x}(t) = \sum_{i=1}^{M} \sum_{j=1}^{N} \sigma_{ali}(V,h) a_{j}(V,h) (A_{ij}x(t) + \bar{r}(t))
\]  
(14)

where \(A_{ij} = \tilde{A}_{ij} + \tilde{B}_{ij}\tilde{K}_{ij} + \Delta \tilde{K}\).

Lemma 1 31. Let \(P, Q\) and \(\Sigma(t)\) be real matrices of appropriate dimension with \(\Sigma(t)\) being a matrix function. Then, for any \(\varepsilon > 0\) and \(\Sigma(t)\Sigma^{T}(t) \leq I\), the following inequality holds.

\[
\text{Continued.}
\]
The following polytopic LPV system is asymptotically stable
\[ U + U_{SW} + W_{p}^{T} \frac{W_{p}}{W_{p}} < 0 \]
for all \( SS^{T} \leq R, \) if and only if there exists a scalar \( \xi > 0 \) such that
\[ U + VV^{T} + \xi^{-1} W_{R}^{T} W_{R} < 0 \]

**Theorem 1.** For the \( sth (s = 1, 2, \ldots, m) \) sub-region and any \( \xi_{s} > 0, \delta_{s} > 0, \) if there exists a symmetry positive definite matrix \( P_{s} \) such that the following inequality holds.
\[
(\bar{A}_{s} + \bar{B}_{s} \bar{K}_{s} \bar{P}_{s}) + P_{s}(\bar{A}_{s} + \bar{B}_{s} \bar{K}_{s}) + \frac{1}{\xi_{s}} P_{s} \bar{B}_{s} \bar{H} \bar{H}^{T} \bar{B}_{s}^{T} P_{s} + \delta_{s} M M^{T} < 0
\]

where \( \bar{Z} = \begin{bmatrix} C^{T} \Pi & -C^{T} \Pi \\ -\Pi C & \Pi \end{bmatrix} \) and \( M = [E C \ -E] \). Then, the following polytopic LPV system is asymptotically stable and satisfies the performance index (12).

\[
\dot{x}(t) = \sum_{s=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{si}(V, h)a_{ij}(V, h)A_{si}x(t)
\]

**Proof.**

For polytopic LPV system (10), there exist matrices \( A_{s}, F_{s}, q \) and \( F_{s,q} \) such that
\[
\begin{bmatrix} E_{s} & x(t) \end{bmatrix} \geq 0 \quad x(t) \in \bar{x}_{s}(t)
\]
\[
F_{s,q} \bar{x}(t) = F_{s} \bar{x}(t) \quad \bar{x}(t) \in \bar{x}_{s}(t) \cap \bar{x}_{s}(t)
\]
where \( q \in (1, 2, \ldots, m) \) and \( q \neq t, \bar{x}(t) \) is the common state vector of the \( qth \) sub-region and the \( tth \) sub-region, \( \bar{x}_{s}(t) \) and \( \bar{x}_{t}(t) \) denote the \( qth \) sub-region and the \( tth \) sub-region which contained \( x(t) \), respectively.

Consider the following Lyapunov function candidate:
\[
V(\bar{x}(t)) = \sum_{s=1}^{M} V_{s}(\bar{x}(t)) = \sum_{s=1}^{M} \bar{x}^{T}(t)P_{s}\bar{x}(t)
\]
where \( P_{s} = P_{s}^{T} F_{s} \) and \( P_{s} - E_{s}^{T} W_{s} E_{s} > 0 \). Here, \( T_{s} \) and \( W_{s} \) are symmetric matrices and they have nonnegative entries.

Substituting Eq. (19) into the derivative of the Lyapunov function candidate \( V(\bar{x}(t)) \), it follows that
\[
\dot{V}(\bar{x}(t)) = \sum_{s=1}^{M} \dot{V}_{s}(\bar{x}(t)) \]
\[
= \sum_{s=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{si}(V, h)a_{ij}(V, h)\bar{x}^{T}(t)\left(\bar{A}_{si}^{T}P_{s} + P_{s}\bar{A}_{si}\right)\bar{x}(t)
\]

From Lemma 1 and substituting \( \bar{A}_{si} = \bar{A}_{si} + \bar{B}_{si} \bar{K}_{si} + \Delta \bar{K} \) into Eq. (22), for any \( \xi_{s} > 0 \), we have
\[
\dot{V}(\bar{x}(t)) \leq \sum_{s=1}^{M} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{si}(V, h)a_{ij}(V, h)\bar{x}^{T}(t)\left(\bar{A}_{si}^{T}P_{s} + P_{s}\bar{A}_{si}\right)\bar{x}(t)
\]
\[
\left[\bar{A}_{si} + \bar{B}_{si} \bar{K}_{si}\right]^{T}P_{s} + P_{s}\left[\bar{A}_{si} + \bar{B}_{si} \bar{K}_{si}\right] + \frac{1}{\xi_{s}} P_{s} \bar{B}_{si} \bar{H} \bar{H}^{T} \bar{B}_{si}^{T} P_{s} + \delta_{s} M M^{T} < 0
\]
Therefore, if only Theorem 1 holds, we have
\[
\left(\bar{A}_{si} + \bar{B}_{si} \bar{K}_{si}\right)^{T}P_{s} + P_{s}\left[\bar{A}_{si} + \bar{B}_{si} \bar{K}_{si}\right] + \frac{1}{\xi_{s}} P_{s} \bar{B}_{si} \bar{H} \bar{H}^{T} \bar{B}_{si}^{T} P_{s} + \delta_{s} M M^{T} < 0
\]
\[e_{s} M M^{T} < 0, \]
then \( \dot{V}(\bar{x}(t)) < 0 \).

Moreover, for the \( sth (s = 1, 2, \ldots, m) \) sub-region, the performance index (12) can be written as
\[
\int_{t_{0}}^{t_{f}} e_{s}^{T}(t) I e_{s}(t) dt = \int_{t_{0}}^{t_{f}} \left[ x^{T}(t) - x(t) \right]^{T} M \left[ x(t) - x(t) \right] dt
\]
\[
= \int_{t_{0}}^{t_{f}} \left[ x^{T}(t) - x(t) \right] M \left[ x(t) - x(t) \right] dt
\]
\[
= \int_{t_{0}}^{t_{f}} \left[ x^{T}(t) - x(t) \right] M \left[ x(t) - x(t) \right] dt
\]
\[
\leq \int_{t_{0}}^{t_{f}} \left[ x^{T}(t) - x(t) \right] \left[ x(t) - x(t) \right] dt
\]
\[
\leq \int_{t_{0}}^{t_{f}} \left[ x^{T}(t) - x(t) \right] \left[ x(t) - x(t) \right] dt
\]
\[
\leq \int_{t_{0}}^{t_{f}} \left[ x^{T}(t) - x(t) \right] \left[ x(t) - x(t) \right] dt
\]
Substituting \( \hat{A}_{ij} = \hat{A}_{ij} + \hat{B}_{ji}(\hat{K}_{ij} + \Delta \hat{K}) \) into Eq. (27), for the scalar \( \varepsilon_1 > 0 \), we have

\[
\int_0^T e_i^*(t)\Pi e_i(t)\,dt \leq \int_0^T \sum_{i=1}^n \sum_{j=1}^n \sigma_i a_{ij}(V, h) a_{ji}(V, h) \cdot \left\{ \frac{1}{\varepsilon_2} \hat{r}(t)\hat{r}^T(t) - \Delta \hat{K}_i \right\} \,dt \leq \int_0^T \sum_{i=1}^n \sum_{j=1}^n \sigma_i a_{ij}(V, h) a_{ji}(V, h) \cdot \left\{ \frac{1}{\varepsilon_2} \hat{r}(t)\hat{r}^T(t) - \Delta \hat{K}_i \right\} \,dt + \int_0^T \Delta \hat{K}_i \,dt.
\]

According to Eq. (28), if Eq. (18) holds, it follows that

\[
\int_0^T e_i^*(t)\Pi e_i(t)\,dt \leq \int_0^T \sum_{i=1}^n \sum_{j=1}^n \sigma_i a_{ij}(V, h) a_{ji}(V, h) \cdot \left\{ \frac{1}{\varepsilon_2} \hat{r}(t)\hat{r}^T(t) - \Delta \hat{K}_i \right\} \,dt + \int_0^T \Delta \hat{K}_i \,dt.
\]

Therefore, by choosing \( \gamma = \max_{i=1,2,...,M} \left\{ \int_0^T \sum_{i=1}^n \sum_{j=1}^n \sigma_i a_{ij}(V, h) a_{ji}(V, h) \cdot \left\{ \frac{1}{\varepsilon_2} \hat{r}(t)\hat{r}^T(t) \right\} \,dt \right\} \), the closed-loop system (14) satisfies performance index (12). This completes the proof of Theorem 1. □

**Theorem 2.** For polytopic LPV system (10), there exists a switching controller (11) which makes the closed-loop system (14) asymptotically stable and satisfy the performance index (12). If there exist matrices \( F_i \geq 0, G_{ij}(k) \in \{1, 2, ..., m, j = 1, 2, ..., n \} \) and scalars \( \varepsilon_2, \varepsilon_3, \varepsilon_4 \) such that the following LMI holds,

\[
\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} < 0
\]

where

\[
\Omega_{11} = \begin{bmatrix} \varepsilon_2 I + E_1 A_{ij}^T + A_{ij} E_1^T & E_1 C^T \\ E_1^T & -\Pi^{-1} \end{bmatrix}, \quad \Omega_{12} = \begin{bmatrix} B_{ij} F_3 & 0 & 0 & 0 \\ 0 & B_{ij} H & 0 & 0 \\ 0 & 0 & B_{ij} H & 0 \\ 0 & 0 & 0 & B_{ij} H \end{bmatrix}, \quad \Omega_{21} = \begin{bmatrix} F_3 B_{ij}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Omega_{22} = \begin{bmatrix} C E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Then, the nominal controller gain of non-fragile LPV switching controller \( \hat{K}_{ij} = [\varepsilon_3 F_3 C \varepsilon_4 F_3^T] \).

**Proof.** Define

\[
P_i = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}
\]

where \( P_1 \) and \( P_2 \) are two positive definite matrices.

By Lemma 1, inequality (18) is equivalent to the following inequality:

\[
\hat{A}_{ij}^T P_i + P_i \hat{A}_{ij}^T + e_i^2 P_i + P_i \varepsilon_1^2 < 0
\]

Therefore, substituting Eq. (31) and \( \hat{A}_{ij} = [A_{ij} 0] + B_{ij} [ (K_{ij} + \Delta K) C (K_{ij} + \Delta K) ] \) into Eq. (32), we have

\[
\begin{bmatrix} \hat{A}_{ij} & \hat{B}_{ji} \end{bmatrix} \begin{bmatrix} P_i & 0 \\ 0 & P_i \end{bmatrix} \begin{bmatrix} \hat{A}_{ij}^T & \hat{B}_{ji}^T \end{bmatrix} + \begin{bmatrix} P_i \varepsilon_1^2 & 0 \\ 0 & P_i \varepsilon_1^2 \end{bmatrix} + \begin{bmatrix} e_i^2 & 0 \\ 0 & e_i^2 \end{bmatrix} < 0
\]

where

\[
A_i = e_i \varepsilon_1 P_i + C^T(C^T + \hat{K}_{ij}^T B_{ij} - \hat{K}_{ij}^T B_{ij})
\]

According to Eq. (33), we have

\[
A_i = e_i \varepsilon_1 P_i + C^T(C^T + \hat{K}_{ij}^T B_{ij} - \hat{K}_{ij}^T B_{ij})
\]

Now, pre- and post-multiplying Eq. (34) by \( P_i^{-1} \), the following inequality can be obtained:

\[
\begin{bmatrix} A_i & e_i \varepsilon_1 P_i^T \end{bmatrix} < 0
\]

By applying the Schur complement to Eq. (35), we get

\[
\begin{bmatrix} A_i & e_i \varepsilon_1 P_i^T \end{bmatrix} < 0
\]

where

\[
A_i = e_i \varepsilon_1 P_i + C^T(C^T + \hat{K}_{ij}^T B_{ij} - \hat{K}_{ij}^T B_{ij})
\]

From Eq. (36), by Lemmas 1 and 2, there exist scalars \( \varepsilon_3 \) and \( \varepsilon_4 \) such that

\[
\varepsilon_3 I + P_i^{-1} A_i + P_i^{-1} e_i \varepsilon_1 B_{ij} K_{ij}^T B_{ij} + e_i \varepsilon_1 P_i^{-1} C^T C P_i^{-1} P_i^{-1} C^T < 0
\]

By applying the Schur complement to Eq. (37), we have

\[
\begin{bmatrix} A_i & e_i \varepsilon_1 P_i^T \end{bmatrix} < 0
\]

where

\[
A_i = e_i \varepsilon_1 P_i + C^T(C^T + \hat{K}_{ij}^T B_{ij} - \hat{K}_{ij}^T B_{ij})
\]

Inequality (38) can be written as

\[
\begin{bmatrix} e_i \varepsilon_1 B_{ij} K_{ij}^T B_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_i \varepsilon_1 I \\ e_i \varepsilon_1 I \\ e_i \varepsilon_1 I \end{bmatrix} < 0
\]
By applying the Schur complement to Eq. (39), we have
\[
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix} < 0
\]
where
\[
\Sigma_{11} = \begin{bmatrix}
\varepsilon_2 I + P_1 A_1^T + A_1 P_1 + P_1 C^T \varepsilon_3 B_1 H P_1 E^T C^T \\
CP_1 \Pi^{-1} & -\Pi^{-1}
\end{bmatrix}
\]
\[
\Sigma_{12} = \begin{bmatrix}
0 & 0 \\
0 & -\varepsilon_3 I
\end{bmatrix},
\Sigma_{21} = \begin{bmatrix}
0 & 0 \\
ECP_1 & 0
\end{bmatrix},
\Sigma_{22} = \text{diag}(-\varepsilon_4 K, -\varepsilon_4 K, -\varepsilon_4 I, -\varepsilon_4 I)
\]
Denote \( P_1 = E \) and \( -\varepsilon_4 K = F \), then the Theorem 2 can be obtained.

**Remark 2.** There are other control methods for FAHVs (see Refs. 9, 11). However, in practice, the parameters of the controller are possible to accrue some parameter variations or gain variations due to finite word length, parameter drift and round-off errors in numerical computations, which is the so-called controller fragility. Therefore, how to design a non-fragile controller for complex flight control systems has significant value. This article investigates the non-fragile switching tracking control problem for a FAHV using LPV techniques.

**Remark 3.** Conventionally, linear model is used to design controller for hypersonic vehicles. For example, in Ref. 9, the authors developed a linear model for the FAHV at specified trim condition by using small deviation linearized method. Then, based on the linear model, the reference output velocity and altitude tracking control design problem is addressed. However, in our method, a polytopic LPV model is constructed to represent the complex nonlinear longitudinal model of the FAHV. Furthermore, the open-loop simulation verification results illustrate that the developed polytopic LPV can capture the local nonlinearities of the origin nonlinear plant. Therefore, compared with the linear model which is derived at specified trim condition, the adopted polytopic LPV model is less conservative than the linear model.

**Remark 4.** Over the past decade, LPV switching control has been extensively investigated by many researchers. In Ref. 23, the authors employ a common Lyapunov function to obtain sufficient LMI conditions for the desired switching LPV controller. But the common Lyapunov function may not exist. If it does exist, it is often necessary to sacrifice the performance in some parameter sub-regions. In Refs. 24, 25 the authors present a switching LPV control method based on multiple parameter-dependent Lyapunov functions (MPLFs). However, this technique restricts the changing rates of scheduling parameters and requires additional LMI constraints. In this paper, robust stability analysis of the closed-loop system is proved via multiple Lyapunov functions (MLFs), which is less conservative than the employment of a common Lyapunov function and MPLFs. It is worth mentioning that when \( P_1 = P(s = 1.2, \cdots, m) \), our results can be further contended based on a common Lyapunov function method.

**4. Nonlinear numerical simulation** Simulation results are presented to demonstrate the effectiveness of the proposed techniques in this section. For the purpose of this study, the flight envelop covers altitude of \( h \in [1500, 35000] \) m and velocity \( V \in [3000, 3400] \) m/s, it is divided into four sub-regions, and each sub-region \( R_s(s = 1,2,3,4) \) is denoted as
\[
R_1: h \in [1500, 25000] \text{ m and } V \in [3000, 3200] \text{ m/s}
\]
\[
R_2: h \in [1500, 25000] \text{ m and } V \in [3200, 3400] \text{ m/s}
\]
\[
R_3: h \in [25000, 35000] \text{ m and } V \in [3000, 3200] \text{ m/s}
\]
\[
R_4: h \in [25000, 35000] \text{ m and } V \in [3200, 3400] \text{ m/s}
\]
Based on Eq. (3), by using the undetermined coefficient method, \( x_i(V,h)(ij = 1,2,3,4) \) is given by
\[
x_{a1}(V,h) = \frac{(h_{max} - h)(V_{max} - V)}{(h_{max} - h_{min})(V_{max} - V_{min})}
\]
\[
x_{a2}(V,h) = \frac{(h_{max} - h)(V_{max} - V)}{(h_{max} - h_{min})(V_{max} - V_{min})}
\]
\[
x_{a3}(V,h) = \frac{(h_{max} - h)(V_{max} - V)}{(h_{max} - h_{min})(V_{max} - V_{min})}
\]
System parameters and the trimmed cruise conditions of the nominal flight of the vehicle are set as follows:
\[
\Pi = A = 0.5I_3, h = \text{diag}(0.2, 0.6), E = I_3, F = I_3 \sin t, M_a = 9; h = 3.4 \times 10^4 \text{ m}; V = 3200 \text{ m/s}; x = 0.02^\circ; \psi = 0^\circ; \Phi = 0.35; \delta_e = 9.65^\circ.
\]
The reference inputs are chosen as multiple step signals of 80 m and 15 m/s. For the altitude tracking reference command, the first step of 80 m starts from 3400 m at \( t = 0 \) s and after each 125 s another step will be applied. Similarly, for the velocity tracking reference command, the first step of 15 m/s is starting from 3200 m/s at \( t = 0 \) s and after each 125 s another step will be applied.

By solving LMI (30), \( K_{ij} \) are given as follows:
\[
K_{11} = \begin{bmatrix}
-0.9356 \times 10^{-4} \\
0.1652 \times 10^{-4}
\end{bmatrix},
K_{12} = \begin{bmatrix}
-0.9327 \times 10^{-4} \\
-1.2480 \times 10^{-3}
\end{bmatrix},
K_{13} = \begin{bmatrix}
-0.6835 \times 10^{-4} \\
-1.2480 \times 10^{-3}
\end{bmatrix},
K_{14} = \begin{bmatrix}
-0.4260 \times 10^{-4} \\
-0.3064 \times 10^{-4}
\end{bmatrix},
K_{21} = \begin{bmatrix}
-2.3575 \times 10^{-3} \\
4.3472 \times 10^{-4}
\end{bmatrix},
K_{22} = \begin{bmatrix}
-0.5732 \times 10^{-3} \\
2.6338 \times 10^{-4}
\end{bmatrix},
K_{23} = \begin{bmatrix}
-6.0024 \times 10^{-4} \\
-0.0003 \times 10^{-4}
\end{bmatrix},
K_{24} = \begin{bmatrix}
-0.8732 \times 10^{-4} \\
-0.5024 \times 10^{-4}
\end{bmatrix},
K_{31} = \begin{bmatrix}
0.1835 \times 10^{-3} \\
-0.1832 \times 10^{-3}
\end{bmatrix},
K_{32} = \begin{bmatrix}
0.2347 \times 10^{-3} \\
-0.1248 \times 10^{-3}
\end{bmatrix},
K_{33} = \begin{bmatrix}
0.2347 \times 10^{-3} \\
-0.1248 \times 10^{-3}
\end{bmatrix},
K_{34} = \begin{bmatrix}
0.2347 \times 10^{-3} \\
-0.1248 \times 10^{-3}
\end{bmatrix},
K_{41} = \begin{bmatrix}
0.1835 \times 10^{-3} \\
-0.1832 \times 10^{-3}
\end{bmatrix},
K_{42} = \begin{bmatrix}
0.2347 \times 10^{-3} \\
-0.1248 \times 10^{-3}
\end{bmatrix},
K_{43} = \begin{bmatrix}
0.2347 \times 10^{-3} \\
-0.1248 \times 10^{-3}
\end{bmatrix},
K_{44} = \begin{bmatrix}
0.2347 \times 10^{-3} \\
-0.1248 \times 10^{-3}
\end{bmatrix}
\]
Fig. 2 illustrates the altitude and velocity tracking performances. We can see that the tracking performance of our method is better than robust adaptive control and switching LPV control in keeping stable tracking of the reference commands. Meanwhile, the altitude and velocity tracking errors are shown in Fig. 3. It can be seen that a smaller tracking error neighborhood can be achieved by the proposed non-fragile switching LPV control method. Figs. 4 and 5 demonstrate angle of attack, pitch angle, pitch rate and the control inputs of the FAHV. It can be seen that they also show satisfactory performances. Thrust, drag and lift are shown in Fig. 6. In summary, the simulation results demonstrate that the presented non-fragile switching tracking control scheme can achieve higher control precision than the existing robust adaptive control and switching LPV control method.

Remark 5. In Ref. 33, a linearized model is developed around a trim point for the FAHV. Then, the authors design a robust adaptive tracking controller which guarantees the property of asymptotical stability for the linearized model. In this section, simulation comparisons under the presented non-fragile switching LPV control, switching LPV control without considering controller gain perturbations and robust adaptive control which is presented in Ref. 33 are given to illustrate the effectiveness of our method.

\[
K_{24} = \begin{bmatrix}
-3.3402 \times 10^{-3} & 0.2376 \times 10^7 \\
0.4273 \times 10^{-4} & 5.4371 \times 10^{-4}
\end{bmatrix}
\]

\[
K_{31} = \begin{bmatrix}
1.3758 \times 10^{-4} & 1.3732 \times 10^4 \\
9.2361 \times 10^{-5} & 0.8274 \times 10^{-4}
\end{bmatrix}
\]

\[
K_{32} = \begin{bmatrix}
6.4582 \times 10^{-4} & 5.4583 \times 10^3 \\
0.8634 \times 10^{-3} & 6.7752 \times 10^{-4}
\end{bmatrix}
\]

\[
K_{33} = \begin{bmatrix}
3.6793 \times 10^{-4} & 6.0034 \times 10^3 \\
1.3684 \times 10^{-5} & 4.5786 \times 10^{-4}
\end{bmatrix}
\]

\[
K_{34} = \begin{bmatrix}
7.9257 \times 10^{-4} & -0.3685 \times 10^3 \\
3.3572 \times 10^{-3} & 2.3158 \times 10^{-3}
\end{bmatrix}
\]

\[
K_{41} = \begin{bmatrix}
-3.2274 \times 10^{-5} & 2.4722 \times 10^4 \\
7.2468 \times 10^{-5} & 0.5381 \times 10^{-4}
\end{bmatrix}
\]

\[
K_{42} = \begin{bmatrix}
-4.8472 \times 10^{-5} & 0.3857 \times 10^4 \\
6.5472 \times 10^{-5} & 3.6542 \times 10^{-4}
\end{bmatrix}
\]

\[
K_{43} = \begin{bmatrix}
-8.2264 \times 10^{-5} & 0.0037 \times 10^4 \\
5.3476 \times 10^{-5} & 1.2469 \times 10^{-4}
\end{bmatrix}
\]

\[
K_{44} = \begin{bmatrix}
-2.5721 \times 10^{-4} & 7.6348 \times 10^3 \\
6.3853 \times 10^{-3} & 3.4752 \times 10^{-4}
\end{bmatrix}
\]
5. Conclusions

(1) The non-fragile LPV switching tracking control problem for the FAHV is investigated in this paper. The longitudinal model of FAHV is modeled as a polytopic LPV model and open-loop simulation results show that the developed model captures the local nonlinearities of the origin nonlinear model.

(2) Based on the developed model, a family of non-fragile LPV controllers is designed over different parameter subspace. By switching characteristic function, these non-fragile LPV controllers are switched in order to guarantee the closed-loop system to be asymptotically stable and satisfy a specified performance criterion.

(3) Our approach is based on multiple Lyapunov functions, which is less conservative than the employment of multiple parameter-dependent Lyapunov functions. Compared with traditional LPV control method, nonlinear simulation comparisons have validated the superiority of the proposed control method.

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\[ b_{31} = - \frac{1}{m} \frac{\partial T}{\partial \theta_1(V, h)} \sin \theta_1(V, h), \quad b_{32} = - \frac{1}{m V} \frac{\partial L}{\partial \delta_2(V, h)} \]
\[ b_{33} = \frac{1}{I_{yy}} \frac{\partial M}{\partial \delta_1(V, h)}, \quad b_{35} = \frac{1}{I_{yy}} \frac{\partial M}{\partial \delta_2(V, h)} \]

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