Stochastic integral with respect to set-valued square integrable martingales

Shoumei Li, J. Jungang Li, Xiaohua Li

Department of Applied Mathematics, Beijing University of Technology, 100 Pingleyuan, Chaoyang District, Beijing 100124, PR China
Mainbo Education, Science and Technology Company, B 1507, 3 Danling Street, Haidian District, Beijing 100080, PR China
School of Science, Beijing University of Posts and Telecommunications, 10 Xitucheng, Haidian District, Beijing 100876, PR China

Abstract

In this paper, we shall firstly illustrate why we should consider integral of a stochastic process with respect to a set-valued square integrable martingale. Secondly, we shall prove the representation theorem of set-valued square integrable martingale. Thirdly, we shall give the definition of stochastic integral of a stochastic process with respect to a set-valued square integrable martingale and the representation theorem of this kind of integrals. Finally, we shall prove that the stochastic integral is a set-valued sub-martingale.

1. Introduction

Classical stochastic differential equations have widely been used in optimal control problems (cf. [34]), mathematical finance (cf. [13]) and so on. Stochastic inclusions as a special form of stochastic differential equations appear in a natural way as a reduced or as a theoretical description of stochastic control problems (cf. [17]). For example, in the Hull–White interest rate model (cf. [40]), the short-rate \( r(t) \) obeys a Gaussian diffusion process of the following form

\[
dr(t) = \left[ \theta(t) - a(t)r(t) \right] dt + \sigma(t) dB_t,
\]

for any \( t \in [0, T] \)

where \( (B_t)_{0 \leq t \leq T} \) is a Brownian motion, \( \theta(t) \) is a deterministic function of time and chosen with the purpose to fit the theoretical bond prices to yield the curve observed on the market, \( \sigma(t) \) determines the overall level of volatility, the reversion rate \( a(t) \) determines the relative volatilities of long and short rate. Or more general form

\[
dr(t) = \left[ \theta(t) - a(t)r(t) \right] dt + \sigma(t) dM(t) \tag{1.1}
\]

where \( M_t \) is a martingale. In the real world, however, this model is not always precise. If we know \( m_t \) takes values in or fluctuates within one interval, for example \( M(t) = [m_1(t), m_2(t)] \), where \( m_1, m_2 \) are martingales with \( m_1 \leq m_2 \), then (1.1) can be reformed as a differential inclusion

\[
dr(t) \in \left[ \theta(t) - a(t)r(t) \right] dt + \sigma(t) dM(t) \tag{1.2}
\]
or stochastic integral form

\[ r_t - r_s \in \text{cl} \left( \int_s^t [\dot{\theta}(\tau) - a(\tau) r(\tau)] \, d\tau + \sigma(\tau) \, dM(\tau) \right), \quad s, t \in [0, T] \]  

(1.2')

where the closure is taken in the sense of \( L^1 \). \( M(t) \) is a set-valued martingale. Concerning \( M(t) \) and its selection, we could refer to Example 4.1 in [26]. In (1.2), there are two parts: one part is \( \int_s^t [\dot{\theta}(\tau) - a(\tau) r(\tau)] \, d\tau \) which is related to classical Lebesgue integral of a stochastic process with respect to time \( t \), i.e. \( \int_0^t [\dot{\theta}(s) - a(s) r(s)] \, ds \), and the other part is \( \sigma(\tau) \, dM(\tau) \) which is related to the integral of a stochastic process with respect to a set-valued martingale. How to define this set-valued integral suitably is the first problem in the theory of set-valued stochastic analysis. What properties does it have? These problems are what we shall consider. These would be useful for us to study the solution of set-valued stochastic differential inclusion (1.2) and its properties (cf. [15, 16]).

There are many related former works about set-valued Lebesgue integral. Based on the work of Richter [39] and Kudo [21], Aumann introduced the Lebesgue integral of set-valued functions and discussed its properties in [3]. Kisielewicz introduced Aumann type Lebesgue integral of set-valued stochastic processes in [15]. Kisielewicz with his colleagues did a lot of nice works about stochastic differential inclusions, especially their solution problems in [15–20]. In [22], Li and Li discussed more properties of the Lebesgue integral of set-valued stochastic processes. We also would like to refer to related works such as [5, 23, 24, 32] and so on.

Concerning set-valued martingales, it was first introduced by Van Cutsem in the case of convex compact values in [7]. Hiai and Umegaki gave more general definition of conditional expectation of a set-valued random variable in [10] so that the theory of set-valued martingales could be developed deeply and extensively. There are many works in this area, for instance, [9–12, 25–31, 35–37, 41, 44, 45]. But it is necessary to investigate set-valued square integrable martingales.

Concerning set-valued works on classical integral of a stochastic process with respect to a Brownian motion and differential inclusions, Kisielewicz introduced the definitions in [15]. More related works have [1, 2, 8, 14–20, 29, 33] and so on.

There are many good works on classical stochastic integral of a stochastic process with respect to a classical martingale, especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38]. Qi and Wang gave a definition of integral of a stochastic process especially to a square integrable martingale (e.g. [6]). But there are not so many works on stochastic integral with respect to a set-valued martingale. Until now, we only have [38].

In this paper, we shall give a new definition of the stochastic integral of a classical stochastic process with respect to a set-valued martingale and prove that the integral is a set-valued submartingale. Finally, we shall discuss other properties of the stochastic integral.

We organize our paper as follows: in Section 2, we shall introduce some necessary notations, definitions and results about set-valued stochastic processes. In Section 3, we shall investigate set-valued martingales and set-valued square integrable martingales, especially we give the representation theorem of set-valued square integrable martingales. In Section 4, we shall give a new definition of stochastic integral of a predictable stochastic process with respect to a set-valued square integrable martingale, prove the representation theorem and discuss some properties of set-valued stochastic integral, especially set-valued submartingale property.

2. Preliminary on set-valued stochastic processes

Throughout this paper, assume that \( R \) is the set of all real numbers, \( I = [0, T] \), \( N \) is the set of all natural numbers, \( R^d \) is the \( d \)-dimensional Euclidean space with usual norm \( \| \cdot \| \), \( B(E) \) is the Borel \( \sigma \)-field of the metric space \( E, (\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \in I}, \mu) \) is a complete filtration probability space, the \( \sigma \)-field filtration \( \{ \mathcal{A}_t : t \in I \} \) satisfies the usual conditions (i.e. complete, non-decreasing and right continuous). Let \( L^p[\Omega, \mathcal{A}_t, \mu; R^d] \) be the set of \( R^d \)-valued \( \mathcal{A}_t \)-measurable random variables \( \xi \) with \( E[\| \xi \|^p] < \infty \) (\( 1 \leq p < \infty \)), and write \( \| \xi \|_p = [E[\| \xi \|^p]]^{1/p} \). When \( \mathcal{A}_t \) is replaced by \( \mathcal{A}_r \), \( L^p[\Omega; R^d] \) can be written as \( L^p[\Omega; R^d] \). Let \( L^p(R^d) \) be the family of all \( \mathcal{A}_t \)-adapted \( R^d \)-valued measurable stochastic processes \( f = \{ f(t), \mathcal{A}_t : t \in I \} \) such that for each \( t \geq 0 \), \( f(t) \in L^p[\Omega; \mathcal{A}_t, \mu; R^d] \).

Now we review notations and concepts of set-valued stochastic processes.

Assume that \( K(R^d) \) is the family of all non-empty, closed subsets of \( R^d \), and \( K_c(R^d) \) (resp. \( K_\text{ck}(R^d), K_\text{ rele}(R^d) \)) is the family of all non-empty closed convex (resp. compact, compact convex) subsets of \( R^d \).

For any \( x \in R^d \), \( A \) is a non-empty subset of \( R^d \), define the distance between \( x \) and \( A \),

\[ d(x, A) = \inf_{y \in A} \| x - y \|. \]

The Hausdorff metric on \( K(R^d) \) is defined as

\[ d_H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\} \]

(2.1)

for \( A, B \in K(R^d) \). Note that the Hausdorff metric between two closed sets \( A, B \) may be infinite when they are unbounded. But it is known (cf. Theorem 1.1.2 [28]) that the family of all bounded elements in \( K(R^d) \), i.e. \( K_\text{rel}(R^d) \), is a complete separable
space with respect to the Hausdorff metric $d_H$, and $K_{sc}(R^d)$ is its closed convex subset. For $B \in K(R^d)$, define $\|B\|_K = d_H([0], B) = sup_{x \in B} \|x\|_K$.

If $F : (\Omega, \mathcal{A}) \to K(R^d)$ satisfies that for any open set $O \subset R^d$, $F^{-1}(O) = \{\omega \in \Omega: F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}$, then $F$ is called $(\mathcal{A} \cdot \mathcal{A})$-measurable (or a set-valued random variable, random set, multivalued function (cf. [4,10,28])). If $\mathcal{F}$ is a sub-$\sigma$-field of $\mathcal{A}$, let $\mathcal{M}(\Omega, \mathcal{F}, \mu; K(R^d))$ be the family of all $K(R^d)$-valued $\mathcal{F}$-measurable random variables. When $\sigma$-field $\mathcal{F} = \mathcal{A}$, it is denoted briefly by $\mathcal{M}(\Omega, K(R^d))$. Similarly, we have notations $\mathcal{M}(\Omega, \mathcal{F}, \mu; K_{sc}(R^d))$, $\mathcal{M}(\Omega, \mathcal{F}, \mu; K_{sc}(R^d))$ and so on. Let

$$S^p_F(\mathcal{F}) = \{f \in L^p[\Omega, \mathcal{F}, \mu; R^d]: f(\omega) \in F(\omega), \text{ a.e. } \omega \in \Omega\}.$$ 

When $\mathcal{F} = \mathcal{A}$, it is written $S^p_F$ for short.

A set-valued random variable $F : \Omega \to K(R^d)$ is called integrable if $S^1_F$ is non-empty. $F$ is $L^p$-bounded if and only if the real-valued random variable $\|F\|_K \in L^p[\Omega; R]$. If $F$ is $L^1$-bounded, then $F$ is also called integrably bounded. Let $L^p[\Omega, \mathcal{F}, \mu; K(R^d)]$ be the family of all $K(R^d)$-valued $L^p$-bounded $\mathcal{F}$-measurable random variables. Similarly, we have notations $L^p[\Omega, \mathcal{F}, \mu; K_{sc}(R^d)]$, $L^p[\Omega, \mathcal{F}, \mu; K_{sc}(R^d)]$, $L^p[\Omega, K(R^d)]$ and so on.

**Definition 2.1.** A non-empty set $\Gamma \subset L^p[\Omega, \mathcal{F}, \mu; R^d]$ is called decomposable with respect to the $\sigma$-field $\mathcal{F}$, if for any $f, g \in \Gamma$, any $U \in \mathcal{F}$, we have $I_U f + I_U g \in \Gamma$.

Firstly, we know that for any set-valued random variable $F \in L^p[\Omega; K(R^d)]$, $S^p_F$ is decomposable with respect to $\mathcal{A}$. We also have the following opposite result.

**Theorem 2.2.** (Cf. [10] or [28]) Let $\Gamma$ be a non-empty closed subset of $L^p[\Omega, R^d]$. Then there exists an $F \in \mathcal{M}(\Omega, K(R^d))$ such that $\Gamma = S^p_F$ if and only if $\Gamma$ is decomposable with respect to $\mathcal{A}$. Furthermore, $\Gamma$ is bounded if and only if $F$ is integrably bounded, and $\Gamma$ is convex if and only if $F$ is convex.

In general, if $\Gamma$ is a non-empty subset of $L^p[\Omega, \mathcal{A}_I, \mu; R^d]$, we have the following definition.

**Definition 2.3.** For any non-empty subset $\Gamma \subset L^p[\Omega, \mathcal{A}_I, \mu; R^d]$, define the decomposable closure of $\Gamma$ with respect to $\mathcal{A}_I$ as

$$\overline{d_{\mathcal{A}_I}} \Gamma = \left\{ f \in L^p[\Omega, \mathcal{A}_I, \mu; R^d]: \text{ for any } \varepsilon > 0, \text{ there exists an } \mathcal{A}_I\text{-measurable finite partition } \{A_1, \ldots, A_n\} \text{ of } \Omega \text{ and } f_1, \ldots, f_n \in \Gamma \text{ such that } \left\| f - \sum_{i=1}^n I_{A_i} f_i \right\|_p < \varepsilon \right\}.$$ 

The expectation of $F$ is defined as $E[F] = \{E[f]: f \in S^1_F\}$. It is called Aumann integral introduced by Aumann in 1965 (cf. [3]). Hai and Umegaki gave the following theorem and definition of conditional expectation of a set-valued random variable when $p = 1$ in [10]. Similarly, they hold for $p \geq 1$.

**Theorem 2.4.** Let $F \in \mathcal{M}(\Omega, K(R^d))$ with $S^p_F \neq \emptyset$, $\mathcal{F}$ be a sub-$\sigma$-field of $\mathcal{A}$. Then there exists a unique $\mathcal{F}$-measurable element $E[F|\mathcal{F}]$ of $\mathcal{M}(\Omega, \mathcal{F}, \mu; K(R^d))$ such that

$$S^p_{E[F|\mathcal{F}]}(\mathcal{F}) = cl\{E[f|\mathcal{F}]: f \in S^p_F\},$$

where the closure is taken in $L^p[\Omega, R^d]$.

**Definition 2.5.** For $F \in \mathcal{M}(\Omega, K(R^d))$, $E[F|\mathcal{F}] \in \mathcal{M}(\Omega, \mathcal{F}, \mu; K(R^d))$ satisfying (2.2) is called the conditional expectation of $F$ relative to $\mathcal{F}$ (in $L^p$).

Concerning more definitions and more results of set-valued random variables, readers may refer to the excellent paper [10] or the book [28].
Let $S^p(F(\omega))$ or $S^p(F)$ denote the family of all $L^p$-selections of $F = \{F(t), A_t: t \in I\}$, i.e.

$$S^p(F) = \{ f \in L^p[I \times \Omega; R^d]: f(t, \omega) \in F(t, \omega), \text{ a.e. } (t, \omega) \in I \times \Omega \}.$$ 

$L^p[I \times \Omega, B(I) \times \Omega, \lambda \times \mu; R^d]$ is the space of all $B(I) \times \Omega$-measurable elements $f : I \times \Omega \to R^d$ with

$$\| f \|_p := \left( E \left( \int_0^T \| f(s) \|_p^p \ ds \right) \right)^{1/p} < \infty.$$ 

and denoted as $L^p[I \times \Omega; R^d]$ for short.

Let $f, f' \in L^p[I \times \Omega; R^d]$, $f = f'$ if and only if $\| f - f' \|_p = 0$. Then $(L^p[I \times \Omega; R^d], \| \cdot \|_p)$ is complete. Further, we have the following theorem.

**Theorem 2.6.** If $S^p(F) \neq \emptyset$, then $S^p(F)$ is a closed set of $(L^p[I \times \Omega; R^d], \| \cdot \|_p)$.

**Proof.** Let $[f_n(t), A_t: t \in I] \in S^p(F)$, $n \geq 1$, and $f_n$ converge to $[f(t), A_t: t \in I]$ in $(L^p[I \times \Omega; R^d], \| \cdot \|_p)$, then we have

$$\| f_n - f \|_p = \left( E \left( \int_0^T \| f_n(s) - f(s) \|_p^p \ ds \right) \right)^{1/p} \to 0 \quad (n \to \infty),$$

i.e. $\{f_n: n \geq 1\}$ converges to $f$ in the sense of $L^p$ with respect to $\lambda \times \mu$ in the product space $I \times \Omega$, where $\lambda$ is a Lebesgue measure on $I$. So there exists subsequence $\{f_{n_k}: k \geq 1\}$ of $\{f_n: n \geq 1\}$ such that $\{f_{n_k}(s, \omega)\}$ almost everywhere converges to $f(s, \omega)$ in the space $I \times \Omega$ when $k \to \infty$.

For any $k \geq 1$, $f_{n_k}(t, \omega) \in F(t, \omega)$ a.e., and $F(t, \omega) \in K(R^d)$, we have $f(t, \omega) \in F(t, \omega)$ a.e. So $[f(t), A_t: t \in I] \in S^p(F)$. The proof is completed. $\Box$

Let $L^p[I \times \Omega; K(R^d)]$ be the space of all elements $F(s, \omega) \in \mathcal{M}(I \times \Omega; K(R^d))$ with

$$\| F \|_p := \left( E \left( \int_0^T \| F(s) \|_{K(R^d)}^p \ ds \right) \right)^{1/p} < \infty.$$ 

Then we have the following theorem:

**Theorem 2.7.** Let $F \in L^p[I \times \Omega; K(R^d)]$, then $S^1(F) = S^p(F)$, where $p \geq 1$.

**Proof.** $S^p(F) \subseteq S^1(F)$ is obvious. Now we prove the converse. For any $f \in S^1(F)$, we have $\| f(s, \omega) \|_K \leq \| F(s, \omega) \|_K$ since $f(s, \omega) \in F(s, \omega)$ for a.e. $(t, \omega) \in I \times \Omega$. Note that $F \in L^p[I \times \Omega; K(R^d)]$, so that we have $f \in L^p[I \times \Omega; R^d]$, which implies $S^1(F) \subseteq S^p(F)$. $\Box$

Now we start to discuss set-valued martingale and set-valued square integrable martingale.

### 3. Set-valued martingale and set-valued square integrable martingale

**Definition 3.1.** A set-valued stochastic process $F = \{F(t), A_t: t \in I\}$ is called a set-valued martingale if

(i) $F = \{F(t), A_t: t \in I\}$ is adapted and for any $t \in I$, $F(t)$ is $L^1$-bounded;

(ii) for any $t \geq s, t, s \in I$, $E[F(t)|A_s] = F(s)$, a.e. $(\mu)$.

**Definition 3.2.** A set-valued stochastic process $F = \{F(t), A_t: t \in I\}$ is called lower semicontinuous at $t_0 \in I$ for any $\omega \in \Omega$ if for any fixed $\omega \in \Omega$ and for any open set $G$ satisfying $F(\omega, t_0) \cap G \neq \emptyset$, there exists $\delta > 0$ such that when $|t - t_0| < \delta$, we have $F(\omega, t) \cap G \neq \emptyset$.

**Definition 3.3.** A set-valued stochastic process $F = \{F(t), A_t: t \in I\}$ is called Hausdorff lower semicontinuous at $t_0 \in I$ for any $\omega \in \Omega$ if for any fixed $\omega \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that when $|t - t_0| < \delta$, we have $F(\omega, t_0) \subseteq \varepsilon + F(\omega, t)$.

Note that lower semicontinuous is briefly denoted by l.s.c., Hausdorff lower semicontinuous is briefly denoted by h.l.s.c. Similarly, upper semicontinuous is briefly denoted by u.s.c., Hausdorff upper semicontinuous is briefly denoted by h.u.s.c.
Definition 3.4. Assume \((X, d)\) is a separable metric space, \(Y\) is a metric space, \(f : \Omega \times X \to Y\) is called a Caratheodory function, if

(a) for any given \(x \in X\), \(\omega \to f(\omega, x)\) is measurable;
(b) for any given \(\omega \in \Omega\), \(x \to f(\omega, x)\) is continuous.

Theorem 3.5. (Cf. [45].) Assume that \(I : \Omega \times X \to Y\) is a Caratheodory function, then \(f(\cdot, \cdot)\) is product measurable.

Theorem 3.6. (Cf. [11] or [45].) Assume \(X\) is a complete separable metric space, \(Y\) is a separable Banach space, the set-valued mapping \(F : \Omega \times X \to K(Y)\) satisfies

(i) \((\omega, x) \to F(\omega, x)\) is measurable;
(ii) for every \(\omega \in \Omega\), \(x \to F(\omega, x)\) is lower semicontinuous.

Then there exists a sequence of Caratheodory selections \(\{f_n : \Omega \times X \to Y, n \geq 1\}\) of \(F\), such that for any \((\omega, x) \in \Omega \times X\),

\[
F(\omega, x) = \text{cl}\{f_n(\omega, x) : n \geq 1\}.
\]

Now we shall discuss separable set-valued stochastic process. Assume that \(X\) is a separable Banach space, \(\{x_n : n \geq 1\}\) is a countable dense subset of \(X\), \(\{r_k : k \geq 1\}\) is the set of all the rational numbers, \(B(x_n, r_k)\) is a ball. Let \(F\) be the set of finite intersection of \(\bigl\{B(x_n, r_k), B(x_n, r_k)^c : n, k \geq 1\bigr\}\), then \(F\) is countable.

Definition 3.7. Assume that \(Y\) is a complete separable metric space, \(F : \Omega \times X \to K(Y)\) is called separable, if there exist a countable set \(D \subset X\) and \(N \in A\) with \(\mu(N) = 0\) such that for any \(\omega \in \Omega \setminus N\) and \(A \in F\), \(\text{cl} F(\omega, A \cap D) = \text{cl} F(\omega, A)\), where \(F(\omega, B) = \bigcup\{F(\omega, x) : x \in B\}(B \subset X)\). \(F\) is called product measurable, if for any open set \(G \subset Y\), \(F^{-1}(G) = \{(\omega, x) \in \Omega \times X : F(\omega, x) \cap G \neq \emptyset\} \in A \times B(X)\).

Theorem 3.8. If \(F : \Omega \times X \to K(X)\) satisfies:

(1) for any \(x \in X\), \(\omega \to F(\omega, x)\) is measurable, i.e. for any open set \(G \subset X\), \(\{\omega \in \Omega : F(\omega, x) \cap G \neq \emptyset\} \in A\); 
(2) for any given \(\omega \in \Omega\), \(x \to F(\omega, x)\) is l.s.c. or h.l.s.c. (u.s.c. or h.u.s.c.);
(3) \(F(\cdot, \cdot)\) is separable,

then \((\omega, x) \to F(\omega, x)\) is product measurable.

Corollary 3.9. Assume that \(F = \{F(t), A_t : t \in I\}\) is a separable set-valued martingale, for any fixed \(\omega \in \Omega\), \(F(\omega, t)\) is lower semicontinuous, then \(F\) is product measurable.

Definition 3.10. An \(R^d\)-valued stochastic process \(f = \{f(t), A_t : t \in I\}\) is called an \(L^p\)-martingale selection of \(F = \{F(t), A_t : t \in I\}\) if

(1) \(f \in S^p(F)\),
(2) \(\{f(t), A_t : t \in I\}\) is a martingale in \(L^p[I \times \Omega; R^d]\).

Let \(\text{MS}^p(F)\) be the set of all \(L^p\)-martingale selections of \(F = \{F(t), A_t : t \in I\}\). If \(p = 1\), \(\text{MS}^1(F)\) can be written as \(\text{MS}(F)\).

Theorem 3.11. Let \(F = \{F(t), A_t : t \in I\} \subset L^1[I; K_c(R^d)]\) be an adapted product-measurable set-valued stochastic process, then the following propositions are equivalent:

(1) \(\{F(t), A_t : t \in I\}\) is a set-valued martingale;
(2) for any \(s, t \in I\), \(s \leq t\), we have
\[
S^1_{F(t)}(A_t) = \text{cl}\{E[g | A_t] : g \in S^1_{F(t)}(A_t)\};
\]
(3) for any \(s \in I\), \(S^1_{F(t)}(A_t) = \text{cl}\{g(s) : [g(t) : t \in I] \in \text{MS}(F)\}\).
Moreover, if $F = \{ F(t), \mathcal{A}_t : t \in I \}$ be an interval-valued stochastic process (i.e. $d = 1$), (1) is also equivalent to

(4) there exist two real-valued $L^1$-martingale selections $\xi = \{ \xi(t), \mathcal{A}_t : t \in I \}$ and $\eta = \{ \eta(t), \mathcal{A}_t : t \in I \}$, such that for each $t$, $F(t, \omega) = [\xi(t, \omega), \eta(t, \omega)]$ a.e.

Proof. Since $F$ is product measurable, then the selection of $F$ is product measurable. By Theorem 3.1 in [29] or [43], we have the result. □

Corollary 3.12. (See [29].) Assume that $F = \{ F(t), \mathcal{A}_t : t \in I \} \subset L^1[\Omega; K_p(R^d)]$ is a product-measurable set-valued martingale, then there exists a sequence of $R^d$-valued martingales $\{ g^i = [g^i(t) : t \in I] : i \geq 1 \} \subset \text{MS}(F)$, such that for any $t \in I$,

$$F(t, \omega) = \text{cl}\{ g^i(t, \omega) : i \geq 1 \}, \quad \text{a.e.}$$

Theorem 3.13. Assume that $\mathcal{A}$ is $\mu$-separable and let $F = \{ F(t), \mathcal{A}_t : t \in I \}$ be a separable square integrable set-valued martingale and for any fixed $\omega \in \Omega$, $F(\omega, t)$ is lower semicontinuous, then

$$S^1(F) = \text{cl}\{ g_i : \{ g_i(t) : t \in I \} \in \text{MS}(F), i \geq 1 \},$$

where the closure is taken in the space $(L^1[I \times \Omega; R^d], \| \cdot \|_1)$. 

Proof. For any $f \in S^1(F)$, by virtue of Theorem 2.6, there exists $f_n \in S^1(F)$ such that

$$E \int_0^T \| f_n(s, \omega) - f(s, \omega) \| ds \to 0, \quad \text{as } n \to \infty.$$ 

So there exists a subsequence $\{ f_{n_k} : k \geq 1 \}$ of $\{ f_n : n \geq 1 \}$ such that for almost everywhere $s \in I$, $E\| f_{n_k}(s, \omega) - f(s, \omega) \|$ converges to 0, when $k \to \infty$. By Theorem 3.11, for any $s \in I$, $S^1_F(s, \mathcal{A}_s) = \text{cl}\{ g(s) : \{ g(t) : t \in I \} \in \text{MS}(F) \}$, and $\text{MS}(F)$ is a subset of separable space $L^1[I \times \Omega; R^d]$. Thus, there exists a sequence $\{ [g_i(t) : t \in I] : i \geq 1 \} \subset \text{MS}(F)$ such that for a.e. $s \in I$,

$$E\| g_i(s, \omega) - f(s, \omega) \| \to 0, \quad \text{as } i \to \infty.$$ 

By bounded convergence theorem, we have

$$E \int_0^T \| g_i(s, \omega) - f(s, \omega) \| ds \to 0, \quad \text{as } i \to 0.$$ 

The proof is completed. □

Remark 3.14. In above theorem and corollary, $L^1$, $S^1$, $\text{MS}(F)$ can be replaced by $L^p$, $S^p$, $\text{MS}^p(F)$ with $p > 1$, respectively.

In the following, we assume that stochastic process $F = \{ F(t) : t \in I \}$ takes values in $K_p(R^d)$ and $\mathcal{A}$ is $\mu$-separable without special statement.

Definition 3.15. A set-valued martingale $F = \{ F(t), \mathcal{A}_t : t \in I \}$ is called square integrable, if $\sup_{t \in I} E[\| F(t) \|_K^2] < \infty$.

Note that a set-valued square integrable martingale is $L^2$-bounded.

Theorem 3.16. Assume that $F = \{ F(t), \mathcal{A}_t : t \in I \}$ is a separable square integrable set-valued martingale and for any fixed $\omega \in \Omega$, $F(\omega, t)$ is lower semicontinuous, then there exists a sequence of continuous martingale selections $\{ f_i : i \geq 1 \}$ such that for any $(t, \omega) \in I \times \Omega$,

$$F(t, \omega) = \text{cl}\{ f_i(t, \omega) : i \geq 1 \}. \quad (3.1)$$

Proof. Since for any given $\omega \in \Omega$, $F(t, \omega)$ is lower semicontinuous, by Theorem 3.6, there exists a sequence of Caratheodory selections $\{ f_i : I \times \Omega \to R^d, i \geq 1 \}$ such that for any $(t, \omega) \in I \times \Omega$, we have

$$F(t, \omega) = \text{cl}\{ f_i(t, \omega) : i \geq 1 \}.$$
Further, for any given $f_i$, by Theorem 3.13, there exists a sequence \( \{g_j; \ j \geq 1\} \subset \text{MS}_2^f(F) \) such that
\[
E \int_0^T \|g_j(s, \omega) - f_i(s, \omega)\|^2 \, ds \to 0, \quad \text{as } j \to \infty.
\]
Therefore, there exists a subsequence \( \{g_{j_k}; \ k \geq 1\} \) of \( \{g_j; \ j \geq 1\} \) such that for a.e. \( s \in I, \)
\[
E \|g_{j_k}(s, \omega) - f_i(s, \omega)\|^2 \to 0, \quad \text{as } k \to \infty.
\]
Since \( A \) is \( \mu \)-separable, by Proposition 1.3 in [6], \( \{f_i(t) \cap A_i; \ a.e. \ t \in I\} \) is a square integrable martingale for any \( i \geq 1 \). Without loss of generality, we assume \( \{f_i(t) \cap A_i; \ a.e. \ t \in I\} \) is a square integrable martingale for any \( i \geq 1 \) in the following. Thus, we get (3.1). The proof is completed. \( \Box \)

By this theorem and Theorem 1.3.3 in [28], we have the following.

**Corollary 3.17.** Assume that \( F = \{F(t), A_i; \ t \in I\} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega, F(\omega, t) \) is lower semicontinuous, then there exists a sequence of \( \mathbb{R}^d \)-valued continuous martingales selections \( \{f^i = \{f^i(t), A_i; \ t \in I\}; \ i \geq 1\} \) such that for any \( t \in I, \)
\[
S^2_{f(t)}(A_i) = \overline{d e_{A_i}} \{f_i(t); \ i \geq 1\},
\]
where the decomposable is respect to \( A_i \)-measurable finite partition of \( \Omega \).

**Remark 3.18.** The set of \( \mathbb{R}^d \)-valued continuous martingale selections of \( F \) is denoted as \( \text{CMS}(F) \) and unfortunately we cannot directly infer whether \( \text{CMS}(F) \) is a closed set in \( L^2[I \times \Omega; \mathbb{R}^d] \) or not. In general, we cannot directly get \( S^2_{f(t)}(A_i) = \{f(t); \ f \text{ is a continuous martingale selection of } F\} \), but only get (3.2) from Theorem 3.16.

Now we shall discuss stochastic integral with respect to a set-valued square integrable martingale.

### 4. Stochastic integral with respect to a set-valued square integrable martingale

**Definition 4.1.** Assume that \( F = \{F(t), A_i; \ t \in I\} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega, F(\omega, t) \) is lower semicontinuous with \( F(0) = 0 \) a.e., \( g \) is a predictable bounded stochastic process. For any \( \omega \in \Omega, \ t \in I, \) define
\[
(A) \int_0^t g(s, \omega) \, dF(s, \omega) = \left\{ \int_0^t g(s, \omega) \, df(s, \omega): \ f = \{f(t); \ t \in I\} \in \text{CMS}(F) \right\},
\]
(4.1)
\( (A) \int_0^t g(s, \omega) \, dF(s, \omega) \) is said to be the Aumann type stochastic integral of \( g \) with respect to the set-valued square integrable martingale \( F \).

**Theorem 4.2.** Assume that \( F = \{F(t), A_i; \ t \in I\} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega, F(\omega, t) \) is lower semicontinuous, \( g \) is a predictable bounded stochastic process, and for any \( t \in I \) and \( \omega \in \Omega, \Gamma(t, \omega) =: (A) \int_0^t g(s, \omega) \, dF(s, \omega) \) defined above. Then for any \( t \in I, \Gamma(t) =: \int_0^t g(s, \omega) \, dF(s, \omega) \) is a non-empty convex subset of \( L^2[\Omega, A_i, \mu; \mathbb{R}^d] \).

**Proof.** Since \( \text{CMS}(F) \) is non-empty by Theorem 3.15, it is obvious that \( \Gamma(t) \) is a non-empty subset of \( L^2[I \times \Omega, A_i, \mu; \mathbb{R}^d] \) by properties of classical stochastic integral. Furthermore, \( F \) takes in \( K_c(\mathbb{R}^d) \) from Theorem 3.11, then \( \text{CMS}(F) \) is convex. Thus \( \Gamma(t) \) is convex. \( \Box \)

**Remark 4.3.** In [38], authors introduced a definition by taking convex closure in (4.1). However, they did not discuss whether the integral is a set-valued random variable or not. From above theorem, we only know that \( \Gamma(t) \) is a non-empty subset of \( L^2[I \times \Omega, A_i, \mu; \mathbb{R}^d] \) for any \( t \in I \). It is natural to hope that the result of integral is a set-valued stochastic process taking values in \( K(\mathbb{R}^d) \) rather than in \( L^2[\Omega, A_i, \mu; \mathbb{R}^d] \). According to Theorem 2.2, \( \Gamma(t) \) should be decomposable with respect to \( A_i \), if we want to decide an \( A_i \)-measurable set-valued random variable. Unfortunately, we cannot prove it directly. Hence we will take the decomposable closure of \( \Gamma(t) \).

**Theorem 4.4.** Assume that \( F = \{F(t), A_i; \ t \in I\} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega, F(\omega, t) \) is lower semicontinuous and \( g \) is a predictable bounded stochastic process, \( \Gamma(t, \omega) = \int_0^t g(s, \omega) \, dF(s, \omega) \), then for any \( t \in I, \) there exists \( M_t(g) \in M[\Omega, A_i, \mu; K_c(\mathbb{R}^d)] \) such that
\[ S_{M_t(g)}^2(A_t) = \overline{d e}_{A_t} \Gamma(t), \]
where the closure is taken in \( L^2[\Omega, A_t, \mu; R^d] \).

**Proof.** For any \( t \in I \), by Theorem 4.2, \( \Gamma(t) \) is a non-empty subset of \( L^2[\Omega, A_t, \mu; R^d] \). For any \( x(t) \in \Gamma(t) \), there exists \( f \in \text{CMS}(F) \) such that \( x(t, \omega) = \int_0^t g(s, \omega) d f(s, \omega) \), \( \forall \omega \in \Omega \). Let

\[
N(t) = \overline{d e}_{A_t} \Gamma(t) = \overline{d e}_{A_t} \left\{ h(t): h(t, \omega) = \int_0^t g(s, \omega) d f(s, \omega), \ f \in \text{CMS}(F) \right\}.
\]

Then \( N(t) \) is a closed subset of \( L^2[\Omega, A_t, \mu; R^d] \) and it is decomposable with respect to \( A_t \). By Theorem 2.2, there exists a set-valued random variable \( M_t(g) \in \mathcal{M}[\Omega, A_t, \mu; K(R^d)] \) such that \( S_{M_t(g)}^2(A_t) = N_t \).

Since \( F \) is convex, \( \Gamma(t) \) is convex by Theorem 4.2. To finish the proof of the theorem, it needs only to prove that \( N_t = \overline{d e}_{A_t} \Gamma(t) \) is convex. Indeed, for any \( \phi, \psi \in N_t \), any \( \varepsilon > 0 \), there exists two \( A_t \)-measurable partitions \( \{A_i: i = 1, 2, \ldots, n\} \), \( \{B_j: j = 1, 2, \ldots, m\} \) of \( \Omega \) and \( \{\phi_i: i = 1, 2, \ldots, n\} \), \( \{\psi_j: j = 1, 2, \ldots, m\} \subset U(t) := \{h(t): h(t) = \int_0^t g(s) df(s), \ f \in \text{CMS}(F)\} \) such that

\[
\left\| \phi - \sum_{i=1}^n I_{A_i} \phi_i \right\|_2 < \varepsilon,
\]
\[
\left\| \psi - \sum_{j=1}^m I_{B_j} \psi_j \right\|_2 < \varepsilon.
\]

For any \( \alpha \in [0, 1] \), we have

\[
\left\| \alpha \phi + (1 - \alpha) \psi - \alpha \sum_{i=1}^n I_{A_i} \phi_i - (1 - \alpha) \sum_{j=1}^m I_{B_j} \psi_j \right\|_2 \leq \alpha \left\| \phi - \sum_{i=1}^n I_{A_i} \phi_i \right\|_2 + (1 - \alpha) \left\| \psi - \sum_{j=1}^m I_{B_j} \psi_j \right\|_2 \leq \alpha \varepsilon + (1 - \alpha) \varepsilon = \varepsilon,
\]

and

\[
\alpha \sum_{i=1}^n I_{A_i} \phi_i + (1 - \alpha) \sum_{j=1}^m I_{B_j} \psi_j = \sum_{i=1}^n \sum_{j=1}^m I_{A_i \cap B_j} (\alpha \phi_i + (1 - \alpha) \psi_j).
\]

Since \( \{A_i \cap B_j: i = 1, \ldots, n; j = 1, \ldots, m\} \) is also an \( A_t \)-measurable partition of \( \Omega \) and \( \text{CMS}(F) \) is convex, \( \{\alpha \phi_i + (1 - \alpha) \psi_j: i = 1, \ldots, n; j = 1, \ldots, m\} \subset U(t) \), so that \( \alpha \phi + (1 - \alpha) \psi \in \overline{d e}_{A_t} U(t) = N(t) \), the proof is completed. \( \square \)

**Definition 4.5.** The set-valued stochastic process \( M(g) = \{M_t(g): t \in I\} \) defined in Theorem 4.4 is called stochastic integral of \( g \) with respect to a set-valued square integral martingale \( F \) and denoted as \( M_t(g) = (M) \int_0^t g d F \) or \( M_t(g; F) = (M) \int_0^t g d F \).

**Lemma 4.6.** Assume that \( F = \{F(t), A_t: t \in I\} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega \), \( F(\omega, t) \) is lower semicontinuous, then there exists a sequence \( \{f^n: n \in N\} \subset \text{CMS}(F) \), such that for every \( t \in I \),

\[
S_{M_t(g)}^2(A_t) = \overline{d e}_{A_t} \left\{ \int_0^t g(s) df^n(s): n \in N \right\},
\]

where the closure is taken in \( L^2 \).

**Proof.** Since \( A \) is \( \mu \)-separable and \( B(I) \) is separable with respect to \( \lambda \), \( B(I) \times A \) is separable with respect to product measure \( \lambda \times \mu \), where \( \lambda \) is the Lebesgue measure in \( I \). Then the space \( L^2[I \times \Omega, B(I) \times A, \lambda \times \mu; R^d] \) is separable (cf. Theorem 3.4.9 of [42]). From the fact that \( \text{CMS}(F) \) is a subset of \( L^2[I \times \Omega, B(I) \times A, \lambda \times \mu; R^d] \), we have that \( S^2(F) \) is a separable. Thus, we can choose a sequence \( \{f^n: n \in N\} \) such that

\[
\{f^n: n \in N\} \subseteq \text{CMS}(F) \subseteq \text{cl}\{f^n: n \in N\},
\]

where the closure is taken in \( L^2[I \times \Omega, B(I) \times A, \lambda \times \mu; R^d] \). Because for every \( t \in I \),
\[ S_{M_t}^2(A_t) = d\epsilon_{A_t} \left\{ \int_0^t g(s) \, df(s): \, f \in CMS(F) \right\}. \]

it suffices to prove

\[ d\epsilon_{A_t} \left\{ \int_0^t g(s) \, df(s): \, f \in CMS(F) \right\} \subset d\epsilon_{A_t} \left\{ \int_0^t g(s) \, df^n(s): \, n \in N \right\}. \]

It only needs to show

\[ d\epsilon_{A_t} \left\{ \int_0^t g(s) \, df(s): \, f \in CMS(F) \right\} \subset d\epsilon_{A_t} \left\{ \int_0^t g(s) \, df^n(s): \, n \in N \right\}. \]

since the right hand is closed. By the definition of decomposability, take \( f_1, f_2 \in CMS(F), A \in A_t \), the proof will be finished if we show

\[ I_A \int_0^t g(s) \, df_1(s) + I_{A'} \int_0^t g(s) \, df_2(s) \in d\epsilon_{A_t} \left\{ \int_0^t g(s) \, df^n(s): \, n \in N \right\}. \]  \hspace{1cm} (4.4)

Indeed, note

\[ I_A \int_0^t g(s) \, df_1(s) + I_{A'} \int_0^t g(s) \, df_2(s) \in d\epsilon_{A_t} \left\{ \int_0^t g(s) \, df(s): \, f \in CMS(F) \right\}. \]

On the other hand, by (4.3), there exist two subsequences \( \{f_i^n: \, i \in N\}, \{f_j^n: \, j \in N\} \) of \( \{f^n: \, n \in N\} \), such that

\[ E \int_0^T \| f_i^n(t) - f_1(t) \|^2 \, dt \to 0 \quad (i \to \infty), \]  \hspace{1cm} (4.5)

and

\[ E \int_0^T \| f_j^n(t) - f_2(t) \|^2 \, dt \to 0 \quad (j \to \infty). \]  \hspace{1cm} (4.6)

Thus, we can choose two subsequences of \( \{f_i^n: \, i \in N\}, \{f_j^n: \, j \in N\} \), still denote by them without loss of generality, such that

\[ f_i^n(t, \omega) - f_1(t, \omega) \to 0, \quad \text{a.e.} \quad (t, \omega) \in I \times \Omega, \quad \text{as} \quad i \to \infty, \]  \hspace{1cm} (4.7)

and

\[ f_j^n(t, \omega) - f_2(t, \omega) \to 0, \quad \text{a.e.} \quad (t, \omega) \in I \times \Omega, \quad \text{as} \quad j \to \infty. \]  \hspace{1cm} (4.8)

Since all \( f_1, f_1^n, f_2, f_2^n \) are square integrable martingales, so are \( f_1^n - f_1, f_2^n - f_2 \). By Doob–Meyer decomposition theorem of submartingales, it holds

\[ (f_1^n - f_1)^2 = M_1^n + (f_1^n - f_1), \]

\[ (f_2^n - f_2)^2 = M_2^n + (f_2^n - f_2), \]

where \( M_1^n, M_2^n \) are uniformly integrable martingales starting from 0 and \( (f_1^n - f_1), (f_2^n - f_2) \) are predictable integrable increasing processes starting from 0. Hence, for any \( t \in I \), \( E(f_1^n(t) - f_1(t))^2 = E(f_1^n(t) - f_1(t)) \) and \( E(f_2^n(t) - f_2(t))^2 = E(f_2^n(t) - f_2(t)) \). With (4.7) and (4.8), we can choose two subsequences of \( \{f_i^n: \, i \in N\}, \{f_j^n: \, j \in N\} \), still denote by them without loss of generality, such that

\[ \langle f_i^n(t, \omega) - f_1(t, \omega) \rangle \to 0, \quad \text{a.e.} \quad (t, \omega) \in I \times \Omega, \quad \text{as} \quad i \to \infty, \]

\[ \langle f_j^n(t, \omega) - f_2(t, \omega) \rangle \to 0, \quad \text{a.e.} \quad (t, \omega) \in I \times \Omega, \quad \text{as} \quad j \to \infty. \]
Then, by properties of classical stochastic integral with respect to square integrable martingales and Lebesgue–Stiltjes integral, we have

\[
E \left\| I_A \int_0^t \left( g(s) df_1(s) + I_{A^c} \int_0^t g(s) df_2(s) - \int_0^t g(s) df_i^h(s) \right) \right\|^2 \leq 2E \left\| I_A g(s) d\left( f_1(s) - f_1^h(s) \right) \right\|^2 + 2E \left\| I_{A^c} g(s) d\left( f_2(s) - f_2^h(s) \right) \right\|^2
\]

\[
= 2E \int_0^t \left\| g(s) \right\|^2 d\left( f_1(s) - f_1^h(s) \right) + 2E \int_0^t \left\| g(s) \right\|^2 d\left( f_2(s) - f_2^h(s) \right)
\]

\[
\leq 2E \int_0^t \left\| g(s) \right\|^2 d\left( f_1(s) - f_1^h(s) \right) + 2E \int_0^t \left\| g(s) \right\|^2 d\left( f_2(s) - f_2^h(s) \right)
\]

\[
\rightarrow 0 \quad (i, j \rightarrow \infty),
\]

which means that (4.4) is right. The desired result is obtained. \( \square \)

**Theorem 4.7** (Castaing representation theorem). Assume that \( F = \{ F(t), \ A_t: \ t \in I \} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega \), \( F(\omega, t) \) is lower semicontinuous and \( g \) is a predictable bounded stochastic process, then there exists a sequence of \( R^d \)-valued martingales \( \{ f^i = \{ f^i(t); \ t \in I; \ i \geq 1 \} \subset CMS(F) \) such that for any \( t \in I \),

\[
F(t, \omega) = cl\left\{ f^i(t, \omega); \ i \geq 1 \right\}, \quad a.e. \ \omega \in \Omega,
\]

and

\[
M_t(g)(\omega) = cl\left\{ \int_0^t g(s, \omega) df^i(s, \omega); \ i \geq 1 \right\}, \quad a.e. \ \omega \in \Omega.
\]

**Proof.** For each \( t \in I \), \( M_t(g) \in \mathcal{M}(\Omega; K(R^d)) \) and \( S^2_{M_t(g)}(A_t) \) is non-empty, then by Theorem 1.3.1 in [28], there exists a sequence \( \{ g^i(t); \ i \in N \} \subset S^2_{M_t(g)}(A_t) \) such that

\[
M_t(g)(\omega) = cl\left\{ g^i(t, \omega); \ i \in N \right\}, \quad \text{for all } \omega \in \Omega. \quad (4.9)
\]

On the other hand, by Theorem 4.6, there exists a sequence \( \{ h^i; \ n \in N \} \subset CMS(F) \), such that for every \( t \in I \),

\[
S^2_{M_t(g)}(A_t) = \overline{d}e_{A_t} \left\{ \int_0^t g(s) dh^i(s); \ n \in N \right\}.
\]

Then, for every \( i \geq 1 \), \( g^i(t) \in S^2_{M_t(g)}(A_t) \),

\[
g^i(t, \omega) \in cl\left\{ \int_0^t g(s, \omega) dh^i(s, \omega); \ n \in N \right\}, \quad a.e. \ \omega \in \Omega.
\]

Thus, there exists an \( N_1 \in A_t \) with \( \mu(N_1) = 0 \), such that for \( \omega \in \Omega \setminus N_1 \),

\[
\{ g^i(t, \omega); \ i \in N \} \subset cl\left\{ \int_0^t g(s, \omega) dh^i(s, \omega); \ n \in N \right\}.
\]

This with (4.9) implies

\[
M_t(g)(\omega) = cl\left\{ g^i(t, \omega); \ i \in N \right\} \subset cl\left\{ \int_0^t g(s, \omega) dh^i(s, \omega); \ n \in N \right\} \subset M_t(g)(\omega).
\]
that is,

\[ M_t(g)(\omega) = \text{cl} \left\{ \int_0^t g(s)(\omega) \, dh^n(s)(\omega); \ n \in N \right\}, \quad \omega \in \Omega \setminus N_1. \tag{4.10} \]

On the other hand, according to Theorem 3.16, we can choose a sequence \( \{\phi^j: j \in N\} \subset \text{CMS}(F) \) such that for every \( t \), there exists some \( N_2 \in \mathcal{A}_t \) with \( \mu(N_2) = 0 \), such that

\[ F(t, \omega) = \text{cl}\{\phi^j(t, \omega): j \in N\}, \quad \text{for all } \omega \in \Omega \setminus N_2. \tag{4.11} \]

Let

\[ \{ f^i: i \in N \} = \{ h^n, \phi^j: n, j \in N \}. \tag{4.12} \]

From \( \{h^n: n \in N\} \subset \text{CMS}(F) \), for each \( t \in I \), there exists \( N_3 \in \mathcal{A}_t \) with \( \mu(N_3) = 0 \), such that

\[ \{h^n(t, \omega): n \in N\} \subset F(t, \omega), \quad \text{for all } \omega \in \Omega \setminus N_2. \tag{4.13} \]

From \( \{\phi^j: j \in N\} \subset \text{CMS}(F) \), by the definition, there exists an \( N_3 \in \mathcal{A}_t \) with \( \mu(N_3) = 0 \), such that

\[ \left\{ \int_0^t g(s, \omega) \, df^i(s, \omega); \ j \in N \right\} \subset M_t(g)(\omega), \quad \text{for all } \omega \in \Omega \setminus N_3. \tag{4.14} \]

Put \( N = N_1 \cup N_2 \cup N_3 \). Due to (4.10)–(4.14), we have for \( \omega \in \Omega \setminus N \),

\[ F(t, \omega) = \text{cl}\{f^i(t, \omega): i \in N\} \]

and

\[ M_t(g)(\omega) = \text{cl}\left\{ \int_0^t g(s, \omega) \, df^i(\omega); i \in N \right\}, \]

i.e. \( \{f^i: i \in N\} \) is the desired sequence. \( \square \)

**Theorem 4.8.** Assume that \( F = \{F(t), \mathcal{A}_t: t \in I\} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega \), \( F(\omega, t) \) is lower semicontinuous, then the stochastic integral \( [M_t(g), \mathcal{A}_t: t \in I] \) is a set-valued submartingale.

**Proof.** For any \( s, t \in I, s < t \), by Theorem 4.4 and the definition of conditional expectation, we have

\[ S_{M_t(g)}^2(\mathcal{A}_t) = \mathbb{E}_{\mathcal{A}_t} \left\{ \int_0^t g(s) \, df(s); \ f \in \text{CMS}(F) \right\}. \tag{4.15} \]

and

\[ S_{E[M_t(g), \mathcal{A}_t]}^2(\mathcal{A}_t) = \text{cl}\{E[m_t(g), \mathcal{A}_t]: m_t(g) \in S_{M_t(g)}^2(\mathcal{A}_t)\}. \tag{4.16} \]

where both closures are taken in \( L^2 \). Now, take an \( f \in \text{CMS}(F) \), according to the square integrable martingale property of stochastic integral, we have

\[ \int_0^s g(u) \, df(u) = E\left[ \int_0^t g(u) \, df(u) \bigg| \mathcal{A}_s \right] \]

and

\[ \int_0^t g(u) \, df(u) \in S_{M_t(g)}^2(\mathcal{A}_t). \]
Then from (4.16), we know
\[ \int_0^s g(u) \, df(u) \in S^{2}_{L[M_t(g)|A_t]}(A_s). \]

Hence, we have
\[ \left\{ \int_0^s g(u) \, df(u) : f \in \text{CMS}(F) \right\} \subset S^{2}_{L[M_t(g)|A_t]}(A_s). \]

Since \( S^{2}_{L[M_t(g)|A_t]}(A_s) \) is decomposable with respect to \( A_s \) and closed in \( L^2 \), we obtain
\[ S^{2}_{M_t(g)}(A_s) = \text{d}e_{A_t} \left\{ \int_0^s g(u) \, df(u) : f \in \text{CMS}(F) \right\} \subset S^{2}_{L[M_t(g)|A_t]}(A_s), \]

which implies
\[ M_s(g)(\omega) \subset E[M_t(g)|A_s](\omega), \quad \text{a.e. } \omega \in \Omega. \]
i.e. \( \{M_t(g), A_t : t \in I\} \) is a set-valued submartingale. \( \square \)

**Theorem 4.9.** Assume that \( g \) and \( h \) are two predictable bounded stochastic processes, \( \alpha, \beta \in \mathbb{R}_+ \), \( F = \{F(t), A_t : t \in I\} \) is a separable square integrable set-valued martingale and for any fixed \( \omega \in \Omega \), \( F(\omega, t) \) is lower semicontinuous, then
\[ \int_0^t (\alpha g + \beta h) \, dF \subset \text{cl} \left( \alpha \int_0^t g \, dF + \beta \int_0^t h \, dF \right) \quad \text{a.e.} \]

**Proof.** Since
\[ \int_0^t (\alpha g + \beta h) \, dF = \text{cl} \left\{ \int_0^t (\alpha g + \beta h) \, dm^i : i \geq 1 \right\} \]
\[ = \text{cl} \left\{ \alpha \int_0^t g \, dm^i + \beta \int_0^t h \, dm^i : i \geq 1 \right\} \]
\[ \subset \text{cl} \left\{ \text{cl} \left\{ \alpha \int_0^t g \, dm^i : i \geq 1 \right\} + \beta \int_0^t h \, dm^i : i \geq 1 \right\} \]
\[ = \text{cl} \left( \alpha \int_0^t g \, dF + \beta \int_0^t h \, dF \right). \]
we have
\[ \int_0^t (\alpha g + \beta h) \, dF \subset \text{cl} \left( \alpha \int_0^t g \, dF + \beta \int_0^t h \, dF \right) \quad \text{a.e.} \quad \square \]

**Acknowledgments**

We deeply thank the referees for their valuable remarks and suggestions.
References