JOURNAL OF APPROXIMATION THEORY 1, 391–400 (1968)

Continuity of the Best Approximation Operator for Restricted Range Approximations

H. L. LOEB[1] AND D. G. MOURSUND

Department of Mathematics and Computing Center, University of Oregon, Eugene, Oregon 97403

1. INTRODUCTION

In this paper we examine the behavior of the best uniform rational approximation operator in certain generalized weight function approximation problems. An introduction to this subject is given in [2].

Let X be a compact topological space, and for $f \in C(X)$ let

$$\|f\| = \max_{x \in X} |f(x)|.$$

Let P and Q be two finite dimensional linear subspaces of $C(X)$. In generalized rational approximation one is interested in approximating an $f \in C(X)$ by a function of the form $r = p/q$ where $p \in P$, $q \in Q$ and $q > 0$ on X.

A generalized weight function $W(x, y)$ is defined for $x \in X$, y real, and has values in the extended reals. Specific examples and a number of results concerning generalized weight functions are given in ([1], [2], [3], [4]). In this paper we are concerned with the problem of finding a generalized rational function r which minimizes

$$\sup_{x \in X} |W[x, f(x) - r(x)]|. \tag{1}$$

The sections which follow give a number of results concerning (1), assuming various hypotheses on $W(x, y)$ and on the space of functions $P + rQ$ where r is a solution to the approximation problem. Here $P + rQ = \{p + rq : p \in P, q \in Q\}$.

Certain notations are used throughout the paper. Suppose for a fixed rational function r that $P + rQ$ has a basis $g_1, \ldots, g_n$. Then for $x \in X$ we define a vector $\hat{x}$ by

$$\hat{x} \equiv (g_1(x), g_2(x), \ldots, g_n(x)). \tag{2}$$

The symbol 0 denotes the origin of Euclidean n-space. Suppose Y is a subset of X, and g is a real valued function defined on Y. Then

$$H\{g(y)\,\hat{y} : y \in Y\}$$

denotes the convex hull of the set of vectors $g(y)\,\hat{y}$ with $y \in Y$.

---

[1] Supported by N.S.F. Grant GP-8686.

If G is a linear subspace of $C(X)$, of dimension k, then G is called a Haar subspace iff every nonzero element of G has at most $k-1$ zeros.

2. Restricted Range Approximations

Let l and u be two elements of $C(X)$ satisfying

$$l(x) < u(x) \ \forall\ x \in X.$$

Let $f^* \in C(X)$ be the function to be approximated, and define

$$R = \{r \equiv p/q : p \in P, q \in Q, q > 0, l \leqslant f^* - r \leqslant u\}. \tag{3}$$

In the discussion which follows we always assume that R is nonempty.

We shall consider a generalized weight function $W(x, y)$ with the following properties:

If $D = \{(x, y): x \in X, y \text{ real}, l(x) \leqslant y \leqslant u(x)\}$ then:

$$\left.\begin{array}{ll}
\text{(a)} & W(x, y) \text{ is continuous over } D; \\
\text{(b)} & \partial W(x, y)/\partial y \text{ is continuous over } D \text{ and positive at each point} \\
 & (x, y) \text{ of } D \text{ with } y \neq 0; \\
\text{(c)} & (x, y) \in D \Rightarrow \operatorname{sgn} W(x, y) = \operatorname{sgn} y; \\
\text{(d)} & x \in X \text{ and } y > u(x) \Rightarrow W(x, y) = \infty; \\
\text{(e)} & x \in X \text{ and } y < l(x) \Rightarrow W(x, y) = -\infty.
\end{array}\right\} \tag{4}$$

These hypotheses are satisfied, for example, in the problem considered in [4].

For notational convenience we write

$$E(f^* - r)(x) \equiv W[x, f^*(x) - r(x)].$$

We call $E(f^* - r)$ the weighted error function. Thus the problem (1) is to minimize

$$\sup_x |E(f^* - r)(x)| \equiv \|E(f^* - r)\|.$$

In restricted range approximations there are two types of critical points. For a particular $r \in R$ under consideration define:

$$X_{+1} = \{x \in X: E(f^* - r)(x) = \|E(f^* - r)\|\}$$
$$X_{-1} = \{x \in X: E(f^* - r)(x) = -\|E(f^* - r)\|\}$$
$$X_{+2} = \{x \in X: E(f^* - r)(x) = u(x)\}$$
$$X_{-2} = \{x \in X: E(f^* - r)(x) = l(x)\}$$
$$X_r = X_{+1} \cup X_{-1} \cup X_{+2} \cup X_{-2}.$$

In [1] it was shown that the cases $X_{+1} \cap X_{-2} \neq \varnothing$ and $X_{-1} \cap X_{+2} \neq \varnothing$ are exceptional, and not of general interest. Here we shall assume

$$X_{+1} \cap X_{-2} = X_{-1} \cap X_{+2} = \varnothing .$$

Then if $f^* \not\equiv r$ we can define an integer valued function σ_r on X_r as follows

$$\sigma_r(x) = \begin{cases} \operatorname{sgn} E(f^* - r)(x) & x \in X_{+1} \cup X_{-1} \\ +1 & x \in X_{+2} \\ -1 & x \in X_{-2}. \end{cases}$$

For the remainder of this section we assume $f^* \notin R$. The following characterization theorem and lemma, which we shall need later, are established in [1].

THEOREM 1. *If $P + rQ$ is a Haar subspace then r is a best approximation to f^** iff

$$0 \in H\{\sigma_r(x)\,\hat{x}\colon x \in X_r\}.$$

LEMMA 1. *If $P + rQ$ is a Haar subspace then*

$$0 \in H\{\sigma_r(x)\,\hat{x}\colon x \in X_r\}$$

iff *there is no nonzero $h \in P + rQ$ such that $(\sigma_r h)(x) \geqslant 0$ for all $x \in X_r$.*

If r^* is a best approximation to f^* from R and $P + r^* Q$ is a Haar subspace, then r^* is unique [1]. In this situation we shall denote r^* by τf^*. We shall establish the continuity of the operator τ at a normal point $f^* \in C(X)$.

DEFINITION. $f^* \in C(X)$ is a normal point iff it has a best approximation r^* from R such that $P + r^* Q$ is a Haar subspace whose dimension = dimension P + dimension $Q - 1$.

Results concerning normal points can be found in ([5], [6], [7]). The first result we shall prove here is a *strong uniqueness theorem.*

THEOREM 2. *Let r^* be a best approximation to f^* from R. If f^* is normal then there exists an $\alpha > 0$ such that for all $r \in R$*

$$\|E(f^* - r)\| \geqslant \|E(f^* - r^*)\| + \alpha\|E(f^* - r^*) - E(f^* - r)\|. \qquad (5)$$

Proof. (Note that this result is trivially true if $f^* \in R$.) We assume $f^* \not\equiv r^*$ and that there is no α as stated. Then there exist sequences $\{r_n\} \subset R$ and $\{\alpha_n\}$, where $\alpha_n \to 0$ and

$$\alpha_n\|E(f^* - r^*) - E(f^* - r_n)\| = \|E(f^* - r_n)\| - \|E(f^* - r^*)\|.$$

Here $r_n = p_n/q_n$, $q_n > 0$, $\|p_n\| + \|q_n\| = 1$, and $r_n \not\equiv r^*$. Since $l \leqslant f^* - r_n \leqslant u$, $\{r_n\}$ is bounded. Here there is no loss of generality in assuming that there exist $p \in P, q \in Q$ such that $\|p\| + \|q\| = 1$ and $p_n \to p$, $q_n \to q$. We also can assume $r^* = p^*/q^*$ where $\|p^*\| + \|q^*\| = 1$. For simplicity of notation we shall write $\sigma(x) \equiv \sigma_{r^*}(x)$.

If $x \in X_{+1} \cup X_{-1}$ then

$$\begin{aligned} \alpha_n \| E(f^* - r^*) - E(f^* - r_n) \| & \\ &= \|E(f^* - r_n)\| - \|E(f^* - r^*)\| \\ &\geqslant \sigma(x)\{W[x, f^*(x) - r_n(x)] - W[x, f^*(x) - r^*(x)]\} \\ &= \sigma(x) \frac{\partial W[x, y_n(x)]}{\partial y} [r^*(x) - r_n(x)]. \end{aligned} \tag{6}$$

Here $y_n(x)$ is between $f^*(x) - r_n(x)$ and $f^*(x) - r^*(x)$. For the fixed x under consideration it might happen that zero is a point of accumulation of $\{f^*(x) - r_n(x)\}$. If that happens then by choosing subsequences one can assume $f^*(x) - r_n(x) \to 0$. Then for sufficiently large n,

$$\sigma(x)[r^*(x) - r_n(x)] = \sigma(x)[r^*(x) - f^*(x) + f^*(x) - r_n(x)] \leqslant 0. \tag{7}$$

This uses the fact that

$$\sigma(x)[f^*(x) - r^*(x)] = \|(f^* - r^*)\| > 0.$$

Now by multiplying each side of (7) by $q_n(x)$ and taking limits, one concludes

$$0 \geqslant \sigma(x)[r^*(x) q(x) - p(x)]. \tag{8}$$

If $\{f^*(x) - r_n(x)\}$ does not have zero as a point of accumulation then there exists an N such that

$$d(x) \equiv \inf_{n \geqslant N} \frac{\partial W[x, y_n(x)]}{\partial y} > 0.$$

Hence for sufficiently large n it follows from (6) that

$$\frac{\alpha_n}{d(x)} \|E(f^* - r^*) - E(f^* - r_n)\| \geqslant \sigma(x)[r^*(x) - r_n(x)]. \tag{9}$$

Then by multiplying by $q_n(x)$ and taking limits one again obtains the inequality (8). That is, (8) holds for all $x \in X_{+1} \cup X_{-1}$.

For $x \in X_{+2} \cup X_{-2}$,

$$\sigma(x)[f^*(x) - r^*(x)] \geqslant \sigma(x)[f^*(x) - r_n(x)].$$

Hence

$$\sigma(x)[-r^*(x) q_n(x) + p_n(x)] \geqslant 0. \tag{10}$$

Taking limits we again conclude that (8) holds.

Since (8) holds for all $x \in X_r$ we obtain, using Lemma 1, $-r^* q + p \equiv 0$.

It then follows from ([5], p. 165) that $p^* \equiv p$, $q^* \equiv q$, and hence $r_n \to r^*$. We conclude that zero is not an accumulation point of $\{f(x) - r_n(x)\}$ when $x \in X_{+1} \cup X_{-1}$. Thus, since in any event $r_n \to r^*$ uniformly, there is no loss of generality in assuming there exists a $d > 0$ such that for all n and all $x \in X_{+1} \cup X_{-1}$,

$$d \leqslant \frac{\partial W[x, y_n(x)]}{\partial y}.$$

Since $q_n \to q^*$ uniformly, there exists a $\delta > 0$ such that for all n and all $x \in X$, $q_n(x) \geqslant \delta$. By a straightforward argument, using Lemma 1 and (10), it follows that there exists a $c > 0$ such that for all n,

$$c \leqslant \max_{x \in X_{+1} \cup X_{-1}} \frac{\sigma(x)\,[r^*(x)\,q_n(x) - p_n(x)]}{\|r^*\,q_n - p_n\|}.$$

Using the above results in (6), we conclude

$$\begin{aligned}\alpha_n \|E(f^* - r^*) - E(f^* - r_n)\| &\geqslant dc\|r^* q_n - p_n\| \\ &\geqslant dc\delta\|r^* - r_n\|.\end{aligned}$$

An application of the mean value theorem to this inequality gives the existence of an $m > 0$ such that

$$m\alpha_n\|r_n - r^*\| \geqslant dc\delta\|r^* - r_n\|.$$

Since $r_n \not\equiv r^*$ and $\alpha_n \to 0$, this yields the desired contradiction and completes the proof.

We now focus our attention on the continuity of τ at a normal point f^*. Let

$$F = \{f \in C(X) : l \leqslant f - \tau f^* \leqslant u\}. \qquad (11)$$

For each $f \in F$, we consider the question of finding a solution to the problem of minimizing $\|E(f - r)\|$ for $r \in R$.

THEOREM 3. *Let f^* be a normal point of $C(X)$. Then there exists an $\alpha > 0$ such that $f_0 \in F$ and $\|f^* - f_0\| < \alpha$ imply that f_0 has at least one best approximation. Moreover, there exists a constant $\beta > 0$ such that for any best approximation r_0 to f_0,*

$$\|E(f^* - \tau f^*) - E(f_0 - r_0)\| \leqslant \beta\|(f^* - f_0)\|. \qquad (12)$$

Proof. Let r^* be the best approximation to f^*. The search for a best approximation to f_0 may be confined to those $r_0 \in R$ for which

$$\|E(f_0 - r_0)\| \leqslant \|E(f_0 - r^*)\|.$$

Such r_0 satisfy (using the triangle inequality)

$$\|E(f^* - r^*) - E(f_0 - r_0)\| \leqslant \|E(f^* - r^*) - E(f^* - r_0)\| + \|E(f^* - r_0) - E(f_0 - r_0)\|.$$

Using Theorem 2 and then the triangle inequality and other manipulations, it follows that the above is

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0)\| - \|E(f^* - r^*)\|] + \|E(f^* - r_0) - E(f_0 - r_0)\|$$

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0) - E(f_0 - r_0)\| + \|E(f_0 - r_0)\| - \|E(f^* - r^*)\|] + \|E(f^* - r_0) - E(f_0 - r_0)\|$$

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0) - E(f_0 - r_0)\| + \|E(f_0 - r^*)\| - \|E(f^* - r^*)\|] + \|E(f^* - r_0) - E(f_0 - r_0)\|$$

$$\leqslant \frac{1}{\alpha}[\|E(f^* - r_0) - E(f_0 - r_0)\| + \|E(f_0 - r^*) - E(f^* - r^*)\|] + \|E(f^* - r_0) - E(f_0 - r_0)\|.$$

Application of the mean value theorem to each of the three "normed" quantities above, leads to the result (12). The proof is then completed by use of the methods in [*5*], p. 168, and [*6*].

It is worth noting that many generalized weight function approximations which do not have the restricted range condition can be considered to have it. For example, suppose $W(x, y)$ satisfies:

(a) $\operatorname{sgn} W(x, y) = \operatorname{sgn} y$;

(b) $W(x, y)$ and $\partial W(x, y)/\partial y$ are continuous;

(c) $\partial W(x, y)/\partial y > 0$ when $y \neq 0$, and $\lim_{|y| \to \infty} |W(x, y)| = \infty$.

This allows us to select $u(x)$ sufficiently large, and $l(x)$ sufficiently small, so that $X_{+2} = \varnothing$ and $X_{-2} = \varnothing$. Then the results of Theorems 2 and 3 hold. These results are, thus, important if one is considering the computational aspects of this problem.

Next we consider the case where $P + (\tau f^*)\, Q$ is a Haar subspace but f^* is not necessarily a normal point of $C(X)$.

THEOREM 4. *Let $\{f_n\} \subset F$ and $\{r_n\} \subset R$ be two sequences such that*

$$f_n \to f^*$$

and

$$\|E(f_n - r_n)\| \to \|E(f^* - r^*)\|.$$

Here $r^* = \tau f^*$. *If* r_n *is written in the normalized form* $r_n = p_n/q_n$, *with* $\|p_n\| + \|q_n\| = 1$, *then the sequence* $\{(p_n, q_n)\}$ *converges to the subspace*

$$M \equiv \{(p,q): p \in P, q \in Q, -p + r^* q \equiv 0\};$$

that is,

$$\text{distance}\,[M, (p_n, q_n)] \to 0.$$

Proof. If $r^* \equiv f^*$ we find $E(f_n - r_n) \to 0$, and hence by the properties of the weight function, $f_n - r_n \to 0$. Thus *a fortiori* we obtain the desired result.

If $f^* \not\equiv r^*$ and the result is false then there exist subsequences of $\{f_n\}$ and $\{r_n\}$ which we do not relabel satisfying

(a) there exist an $\epsilon > 0$ such that distance $[M, (p_n, q_n)] \geqslant \epsilon$ for all n;

(b) $p_n \to p$, $q_n \to q$ where $\|p\| + \|q\| = 1$.

For $x \in X_{+1} \cup X_{-1}$,

$$\|E(f_n - r_n)\| - \|E(f^* - r^*)\| \geqslant \sigma_{r^*}(x)\,[E(f_n - r_n)(x) - E(f^* - r^*)(x)].$$

Using the same techniques as were employed in the proof of Theorem 2, one can verify that

$$0 \geqslant \sigma_{r^*}(x)\,[r^*(x)q(x) - p(x)]. \tag{13}$$

Since inequality (13) also holds for $x \in X_{+2} \cup X_{-2}$, it follows by Lemma 1 that

$$r^* q - p \equiv 0.$$

This contradicts the assumption that

$$\text{distance}\,[M, (p_n, q_n)] \geqslant \epsilon \qquad \text{for all } n$$

and completes the proof.

For the remainder of this section we specialize to the situation where $X = [a,b]$. We make the assumption that for each nonzero $q \in Q$, the set of zeros of q is of measure zero.

THEOREM 5. *If* $\{r_n\} \subset R$ *and* $\{f_n\} \subset F$ *are such that* $r_n = p_n/q_n$, $\|p_n\| + \|q_n\| = 1$, $(p_n, q_n) \to M$, *and* $f_n \to f^*$, *then* $E(f_n - r_n) \to E(f^* - r^*)$ *in measure. Here* $M = \{(p,q): p \in P, q \in Q, -p + r^* q \equiv 0\}$.

Proof. Assume the contrary. We can then find subsequences of $\{r_n\}$ and $\{f_n\}$, which we do not relabel, such that

(a) There exist an $\epsilon > 0$ and a positive integer k such that if

$$B_n \equiv \{x: |E(f_n - r_n)(x) - E(f^* - r^*)(x)| > 1/k\}$$

then the measure of B_n is greater than ϵ for all n;

(b) $p_n \to p$, $q_n \to q$ where $\|p\| + \|q\| = 1$.

Since $\|p\| + \|q\| = 1$ and $-p + r^*q \equiv 0$, we conclude that $q \not\equiv 0$. Let

$$X_0 = \{x : q(x) \neq 0\}.$$

By hypothesis the measure of X_0 is $b - a$. Choose a closed set $X_1 \subset X_0$ such that the measure of X_1 is $b - a$. On X_1, $E(f_n - r_n) \to E(f^* - r^*)$ uniformly. Thus for large n, $B_n \cap X_1 = \varnothing$, which implies that B_n has measure zero. This is a contradiction.

The following result is then clear.

THEOREM 6. *If r^* is a best approximation to f^* and $P + r^* Q$ is a Haar subspace, then for every pair of sequences $\{r_n\} \subset R$ and $\{f_n\} \subset F$ such that $f_n \to f$ and $\|E(f_n - r_n)\| \to \|E(f^* - r^*)\|$, $E(f_n - r_n) \to E(f^* - r^*)$ in measure.*

3. Rational Approximation with Interpolation

We turn now to a different sort of restricted range approximation. Using the ordinary uniform norm as a measure of error we are interested in finding a best rational approximation which interpolates $f(x)$ on a prescribed point set. To be more specific, let $\{x_1, \ldots, x_k\} \subset X$, where $k \leqslant$ dimension P, be a given set of points. For $f \in C(X)$ let

$$R_f = \{r \equiv p/q : p \in P; q \in Q; q > 0; r(x_i) = f(x_i), \quad i = 1, \ldots, k\}$$

Then we call $r^* \in R_f$ a best approximation to f from R_f iff

$$\text{distance}(R_f, f) = \|f - r^*\|.$$

For each $r \in R_f$ define

$$S_r = \{-p + rq : p \in P; q \in Q; (-p + rq)(x_i) = 0, \quad i = 1, \ldots, k\}.$$

DEFINITION. S_r is called an interpolating Haar subspace iff every nonzero element in S_r has at most $d(r) - 1$ zeros distinct from $\{x_1, \ldots, x_k\}$. $d(r)$ is the dimension of the subspace S_r.

Clearly if $P + rQ$ is a Haar subspace, then S_r is an interpolating Haar subspace. The following theorem and lemma are given in [8].

THEOREM 7. *r is a best approximation to f from R_f* iff

$$0 \in H\{\sigma(x)\hat{x} : x \in X_r\}$$

where

$$\sigma(x) = \operatorname{sgn}[f(x) - r(x)], \; X_r = \{x \in X : |f(x) - r(x)| = \|f - r\|\},$$

Here $\hat{x} \equiv (g_1(x), g_2(x), \ldots, g_n(x))$, where $g_1, g_2, \ldots, g_n$ is a basis of S_r.

LEMMA 2. *If r is a best approximation to f from R_f, where $r \not\equiv f$ and S_r is an interpolating Haar subspace, then $h \in S_r$ and $\sigma(x)h(x) \geqslant 0$ for all $x \in X_r$ imply $h \equiv 0$.*

In [8], under the assumption that the dimension of the interpolating Haar subspace S_r is (dimension P + dimension $Q - 1 - k$), the Lipschitz continuity of the best approximation operator at f was demonstrated. In general we will show that only convergence in measure can be expected.

THEOREM 8. *Let r be a best approximation to f from R_f and assume S_r is an interpolating Haar subspace. Let $\{r_n\}$ and $\{f_n\}$ be two sequences with the properties:*

(a) $r_n \in R_{f_n}$, *where* $r_n = p_n/q_n$ *and* $\|p_n\| + \|q_n\| = 1$;

(b) $f_n \to f$;

(c) $\|f_n - r_n\| \to \|f - r\|$.

Define $M = \{(p,q) \in P \times Q\colon -p + rq \equiv 0\}$. *Then*

$$\text{distance } [(p_n, q_n), M] \to 0$$

Proof. For the case $r \equiv f$, the result is clear. If $r \not\equiv f$, assume that the result is false. Then (by taking subsequences if necessary) there exists an $\epsilon > 0$ such that

$$\text{distance } ((p_n, q_n), M) \geqslant \epsilon \tag{14}$$

for all n. By taking further subsequences we can secure that $p_n \to p$ and $q_n \to q$. Now, for each interpolating point x_i,

$$-p_n(x_i) + q_n(x_i) f_n(x_i) = 0.$$

Since $f_n(x_i) = r_n(x_i)$, one finds by taking the limit,

$$-p(x_i) + q(x_i)\, r(x_i) = 0.$$

Hence $-p + rq \in S_r$. By the same argument used in Theorem 2,

$$-p(x) + q(x)\, r(x) = 0$$

for each $x \in X_r$. Hence by Lemma 2

$$-p + rq \equiv 0.$$

This contradicts (14).

THEOREM 9. *If $r^* \in R_f$, and $P + r^* Q$ is a Haar subspace, then there exists a $\gamma > 0$ such that $\|f - g\| < \gamma$ implies that R_g is nonempty. Furthermore, if $f_n \to f$ and $\|f - f_n\| < \gamma$, there exist $r_n \in R_{f_n}$ such that $r_n \to r^*$.*

Proof. Consider the system of equations for p and q

$$-p(x_i) + g(x_i)q(x_i) = 0 \qquad i = 1, \ldots, k.$$

By hypothesis, this system can be solved in a neighborhood of $p = p^*$, $q = q^*$ and $g = f$ for a p and q such that if $r = p/q$, $r \in R_g$ and r is close to r^*.

COROLLARY. *Under the same hypotheses as in the previous theorem, $f_n \to f$ implies distance $(R_{f_n}, f_n) \to$ distance (R_f, f).*

Now if we specialize to the case where $X = [a, b]$ and assume for each nonzero $q \in Q$, that the set of zeros of q has measure zero, we find, pursuing the same ideas as in the restricted range case, that:

THEOREM 10. *Assume r is a best approximation to f from R_f and S_r is an interpolating Haar subspace. Then if $\{r_n\}$ and $\{f_n\}$ are two sequences such that $f_n \to f$, $r_n \in R_{f_n}$ and $\| f_n - r_n \| \to \| f - r \|$, then $r_n \to r$ in measure.*

REFERENCES

1. H. L. LOEB, D. G. MOURSUND, AND G. D. TAYLOR. Uniform rational weighted approximations having restricted ranges. *J. Approx. Theory.* To appear.
2. D. G. MOURSUND, Chebyshev approximations using a generalized weight function. *J. SIAM Numer. Anal.* 3 (1966), 435–450.
3. D. G. MOURSUND AND G. D. TAYLOR, Optimal starting values for the Newton-Raphson calculation of inverses of certain functions. *J. SIAM Numer. Anal.* **5** (1968), 138–150.
4. D. G. MOURSUND, Optimal starting values for Newton-Raphson calculation of $\sqrt{x}$. *Comm. ACM* **10** (1967), 430–432.
5. E. W. CHENEY, "Introduction to Approximation Theory." McGraw Hill, Inc., N.Y., 1966.
6. E. W. CHENEY AND H. L. LOEB, Generalized rational approximations. *J. SIAM Numer. Anal.* Ser. B, **1** (1964), 11–25.
7. H. L. LOEB, Approximation by generalized rationals. *J. SIAM Numer. Anal.* 3 (1966), 34–55.
8. C. GILORMINI, Approximation par fractions généralisées dont les coefficients des relations linéaires. *C. R. Acad. Sc. Paris A.* **264** (1967), 795–798.