Europ. J. Combinatorics (1996) **17,** 379–390

Strong Convergence and a Game of Numbers

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S. Mozes investigated a certain solitary game played on a weighted graph. Numbers are placed on the nodes of the graph, and a move consists of changing the sign of a negative number and changing the numbers on the neighboring nodes according to the weights on the edges. Mozes proved that the game has a strong convergence property when the edges have certain positive integer weights. However, his approach would give no information in the case of other weights. In this paper we first prove that strong convergence is equivalent to the fact that the game has as a certain 'polygon property'. We can then, in a rather elementary way, characterize the assignments of weights that imply the polygon property, and hence strong convergence. Finally, we make a natural generalization of the game, where we also have weights on the nodes. The conditions for strong convergence generalize nicely to this game.

1. Introduction

The numbers game on a general graph was introduced by Mozes [8], but has also been studied by Björner [2] and Eriksson [4]. The original definition was as follows.

Let G be a loop-free, undirected, connected graph of N nodes. Place a real number on each node. A move now consists of first picking a node i with a negative number f_i, then adding the number f_i to the number at each neighbor j of i, and finally reversing the sign at node i. If no move is possible, the game has terminated. Mozes observed that if the position of the game is represented by the point in $\mathbb{R}^N$ with coordinates $(f_1, f_2, \ldots, f_N)$, then each move is equivalent to a reflection in some hyperplane.

The original numbers game has an unexpected convergence property. In any starting position where some play sequence leads to a terminal position, every way of playing will lead to the same terminal position in the same number of moves. In other words, the length and result of the game is independent of what choices are made. This property is called *strong convergence,* and it is known also from other combinatorial problems. In Section 2 we give a characterization of strongly convergent games as games having a certain *polygon property.*

Björner (see [2] or [4]) found an elementary inductive proof of strong convergence of the original numbers game, while Mozes [8], using properties of the action of a Weyl group of a Kac–Moody algebra associated with G on its apartment, was able to extend the set of graphs with the convergence property to some edge-weighted graphs. To each edge (i, j) he assigned a pair of positive integer weights, k_{ij} and k_{ji}. When playing node i, f_i is multiplied by k_{ij} to obtain the term that is added to f_j.

Mozes studied only those integer weight assignments for which the weight matrix $K = (k_{ij})$ can be made symmetric by multiplication with a non-degenerate diagonal matrix. In Section 3 of this paper we will show that this is not a necessary condition for strong convergence. In fact, by examining which weights give the polygon property, we can state both necessary and sufficient conditions on the weights.

Furthermore, we will examine the game that we get if we let the 'reflections' carry the reflected point 'too far' or 'too close', i.e. if playing i means multiplying f_i by some

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arbitrary negative constant. We may denote this constant by $(1 + k_{ii})$ in order to obtain a very compact form of describing the move i:

$$f_j := f_j + k_{ij} f_i \qquad \text{for } j = 1, 2, \ldots, N.$$

Thus, in Mozes's edge-weighted game we had $k_{ii} = -2$ for every node i. The node-weighted game is treated in Section 4, by solving the recursions that describe the numbers arising during alternating play sequences. Once again, we obtain necessary and sufficient conditions for strong convergence.

Of course, the result in Section 3 is just a special case of the result in Section 4. The reasons for giving the purely edge-weighted game a separate treatment is its importance, and the use of an appealing and simple reflection model, which is not practicable in the general case.

2. Strong Convergence

We are interested in one-player games, such that in any position of the game there are zero or more *legal moves* leading to other positions. A position in which no move is possible is called a *terminal position*.

Definition. A game is said to have the *strong convergence property* if, given any start position from which there exists a terminating play sequence, every way of playing from this start position will lead to the same terminal position, and in the same number of moves.

Observe that this definition allows start positions from which one can go on playing forever, in which case there is no way of reaching a terminal position.

The strong convergence property of weighted and unweighted versions of the game of numbers has been studied by Mozes [8], Björner [2] and Eriksson [4], and, in a less general setting, by Alon, Krasikov and Peres [1]. The property is owned also by several other combinatorial algorithms. A related game played on a graph, the chips game, was shown to be strongly convergent by Björner, Lovasz and Shor [3].

It is clear that a strongly convergent game must have the following property.

Defnition. A game is said to have the *polygon property* if, given any position in which two different moves, x and y, are legal, either there are two play sequences of the same length and beginning with x and y respectively, that result in the same position, or there are two such play sequences which can be continued forever.

We will now prove that the strongly convergent games can indeed be characterized as the games having the polygon property. This characterization immediately proves strong convergence of the chips game (where all moves commute, i.e. $xy \equiv yx$) and the original numbers game (where $xy \equiv yx$ if x and y are nodes which are not neighbors, and $xyx \equiv yxy$ if x and y are neighbors). In the later sections we will use the polygon property of edge-weighted and node-weighted versions of the numbers game.

Theorem 2.1. *A game has the strong convergence property iff it has the polygon property.*

Proof. ($\Rightarrow$) Suppose that we have a strongly convergent game and a position in which two different moves, x and y, are legal. By strong convergence, either all play sequences lead to the same terminal position after the same number of moves, or all play sequences can be continued indefinitely. In particular, this holds for play sequences beginning with x or y thus verifying the polygon property.

($\Leftarrow$) Suppose that we have a game which has the polygon property. We shall prove that if there is a play sequence from a position p to some terminal position t, then all play sequences from p eventually reach t, and in the same number of moves. Suppose not. Then there is some *shortest possible* pair of equally long sequences, say of length k, such that they start from the same position p, that one sequence begins with, say, x, and leads to some terminal position t, while the other sequence begins with, say, y, and does not lead to t. Since we have chosen the shortest possible such pair, no sequences from p beginning with y can lead to t in k moves, while all play sequences beginning with x must lead to t in k moves. This obviously contradicts the polygon property. □

3. The Edge-weighted Game

Definition. The *edge-weighted game* is defined as follows. Let G be a loop-free, undirected graph of N nodes, which are numbered $1, 2, \ldots, N$. To each edge (i, j) is assigned a pair of strictly positive weights, k_{ij} and k_{ji}. Place a real number on each node. Let f_j denote the number on node j. The position can be represented by a column vector of the numbers $f = (f_1, f_2, \ldots, f_N)^t$. A move consists of three steps. First choose a node i such that the number on it is negative. Then, to the number on each node j connected to i by an edge (i, j), add the number on i multiplied with the weight k_{ij}. Finally, change the sign of the number on node i. If no number is negative, then no move is possible and we have what we will call a terminal position.

If we set $k_{ii} = -2$ for every node i, and $k_{ij} = 0$ whenever i and j are not neighbors, we may simply define the move of playing i as $f_j := f_j + k_{ij} f_i$ for every node j in the graph. If we collect the weights in a square matrix $K = (k_{ij})$, we can express the move as a linear transformation of the position vector p. If the ith unit vector is denoted by e_i, we obtain $f := f + K^t e_i e_i^t f$.

Note that we may have $k_{ij} \neq k_{ji}$, with the interpretation that we may have different weights in the two directions of an edge. Every possible assignment of weights to the edges gives a different game. We want to determine which games behave well in the sense of having the strong convergence property.

Mozes proved that for any selection of weights from the positive integers, such that the matrix K is symmetrizable by left multiplication with a diagonal matrix D, the game will have the strong convergence property. However, he gave no necessary conditions. The following theorem characterizes in full the possible assignments of weights that make the edge-weighted game strongly convergent.

Theorem 3.1. *The edge-weighted game on a graph G with edge weights $\{k_{ij}\}$ is strongly convergent iff for each edge (i, j) of G the corresponding weight product $k_{ji}k_{ij}$ is either in the countable set* $\{4\cos^2(\pi/n) \mid n = 3, 4, \ldots\}$ *or in the interval* $[4, \infty)$.

Remark 3.2. An appealing assignment of weights is one such that we obtain a 'law of conservation', where the sum of the numbers on all nodes remains constant during the game. This kind of assignment will in general *not* give a strongly convergent game. For example, take G to be the tree with three leaves, numbered 1 to 3, all connected to one center node, numbered 4. If we set $k_{i4} = 2$ and $k_{4i} = 2/3$ for $i = 1, 2, 3$, the game

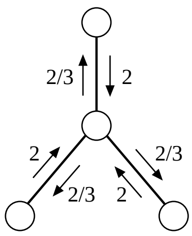

FIGURE 1. A weighted graph on which the game is conservative but not strongly convergent.

will be conservative, but for any edge the weight product is 4/3, which is not equal to $4\cos^2(\pi/n)$ for any integer n, so the game cannot be strongly convergent: see Figure 1.

REMARK 3.3. For the reader who is familiar with the standard geometric representation of Coxeter groups, the spectrum of admissible weight products in Theorem 3.1 may ring a bell (cf. [7, p. 109]). Indeed, from the theory evolved in this paper, one can deduce that the moves of a strongly convergent edge-weighted numbers game generator a Coxeter group. This is done in [5].

The proof of Theorem 3.1 uses Theorem 2.1, the characterization of strongly convergent games as those having the polygon property. The proof calls for some non-trivial results, which for the most part will follow from study of a simple reflection model.

We will write $\alpha \equiv_f \beta$ if α and β are two play sequences that result in the same position when played from position f. If this is true for any position we just drop the index: $\alpha \equiv \beta$. The following observations are fundamental.

OBSERVATION 3.4. xy and yx are both legal play sequences iff both x and y are legal moves. If x and y are not neighbors, then $xy \equiv yx$. This follows immediately from the definition of the game.

We are now going to investigate when two alternating play sequences consisting of more than two moves are equal.

DEFINITION. Let $(xyx\cdots)_n$ denote the alternating play sequence $xyxy\cdots$ of length n. Note that if n is odd then $(xyx\cdots)_n$ will both begin and end with the move x.

LEMMA 3.5. *Suppose that $k_{xy}k_{yx} = 4\cos^2\alpha$, where $0 < \alpha < \pi/2$. Suppose further that both numbers f_x and f_y are negative in the current position f. Then there is a positive integer $n \leqslant \lceil \pi/\alpha \rceil$ such that $(xyx\cdots)_n$ is a legal play sequence while $(xyx\cdots)_{n+1}$ is not a legal play sequence, i.e. the last move is not legal.*

PROOF. We first show that the numbers on x and y during the alternating play sequence can be represented by a point that travels in the plane due to repeated reflections. Let L_x and L_y be two lines in $\mathbb{R}^2$ with angle α. Introduce oblique coordinate axes, orthogonal to L_x and L_y respectively, and scaled so that one unit on the 'x-axis' equals $k_{xy}/2\cos\alpha$ units on the 'y-axis'. Reflection in L_x will take a point with coordinates (f_x, f_y) to $(-f_x, f_y + f_{xy}f_x)$. Similarly, reflection in L_y takes (f_x, f_y) to $(f_x + k_{yx}f_y, -f_y)$. Thus, during a play sequence consisting of only x and y moves, in

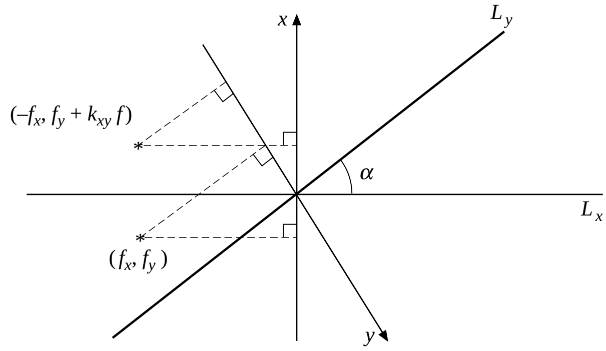

FIGURE 2. An example of a reflection in line L_x.

particular an alternating sequence, the numbers on x and y equal the coordinates of the point in the coordinating reflection sequence: see Figure 2.

If one bears in mind that consecutive reflections in first L_x, then L_y, equal a rotation by an angle 2α, it should be quite clear that a point with both coordinates negative will be reflected to a point with both coordinates non-negative in at most $\lceil \pi/\alpha \rceil$ moves, where $\lceil \cdot \rceil$ denotes the integer ceiling function. Since non-negative coordinates mean non-negative numbers on x and y, the play sequence cannot be continued. □

Let us now compare the positions resulting from two alternating sequences, $(xyx\cdots)_n$ and $(yxy\cdots)_n$. Denote the empty play sequence by 1.

LEMMA 3.6. *Suppose that $k_{xy}k_{yx} = 4\cos^2\alpha$, where $0 < \alpha \leqslant \pi/2$. Let f be a position such that both f_x and f_y are negative. Then $(xyx\cdots)_n \equiv_f (yxy\cdots)_n$, with both sequences legal, iff $\alpha = \pi/n$. In fact, in this case even $(xyx\cdots)_n \equiv (yxy\cdots)_n$ holds.*

PROOF. Observe that if we do not pay attention to legality, we have $xx \equiv 1$; that is, we get back to the original position if we play the same node twice. Thus $(xyx\cdots)_n \equiv_f (yxy\cdots)_n$ iff $(xyx\cdots)_{2n} \equiv_f 1$. This implies, in the reflection model, that n consecutive rotations by 2α should equal 2π, and that $\alpha = \pi/n$ is the only solution that makes the play sequences legal. (To verify that this is really so, draw a simple sketch!)

We shall now prove that, if $\alpha = \pi/n$, then the numbers on all nodes other than x and y are also identical in the two resulting positions. For any node u, during the (illegal) play sequence $(xyx\cdots)_{2n}$ the number f_u will change by n terms that come from playing x and n terms that come from playing y. The sum of the x terms is k_{xu} times the x-coordinate of the plane vector that is the sum of the initial vector (f_x, f_y) and $n-1$ consecutive rotations of it by 2α: see Figure 3. Clearly, this vector sum is zero, so in

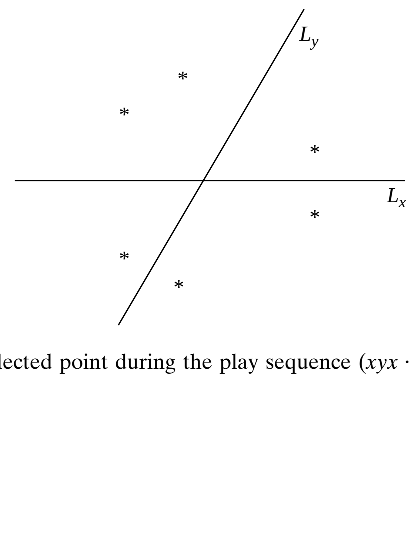

FIGURE 3. The reflected point during the play sequence $(xyx\cdots)_6$, when $\alpha = \pi/3$.

particular the x-coordinate is zero. The sum of the y terms is analogously seen to be zero, so the resulting value of f_u will be the same as the initial value. Observe that here we need not assume any particular property of the position f. Thus $(xyx\cdots)_{2n}\equiv 1$ and hence $(xyx\cdots)_n\equiv(yxy\cdots)_n$. □

Finally, we have to analyze the situation in which the weight product $k_{xy}k_{yx}\geqslant 4$. We use a recursive technique, which will be put to further use in Section 4.

LEMMA 3.7. *If both x and y are legal moves and $k_{xy}k_{yx}\geqslant 4$, then the alternating sequences $xyxy\cdots$ and $yxyx\cdots$ can be legally continued forever.*

PROOF. Assume that $k_{xy}k_{yx}\geqslant 4$, and that both x and y are legal moves, i.e. $X_0=f_x<0$ and $Y_0=f_y<0$. Suppose, without loss of generality, that the first move is y. Let X_n and Y_n be the values at x and y after the sequence $(yxy\cdots)_{2n}$. Play out the moves yx to verify that

$$\begin{cases} Y_1=k_{xy}X_0+(k_{xy}k_{yz}-1)Y_0, \\ X_1=-X_0-k_{yx}Y_0, \end{cases}$$

and similarly for higher indices. Thus we obtain the following system of coupled recursions:

$$\begin{cases} Y_n=k_{xy}X_{n-1}+(k_{xy}k_{yx}-1)Y_{n-1}, \\ X_n=-X_{n-1}-k_{yx}Y_{n-1}. \end{cases}$$

If we eliminate all X_n we obtain a second order recursion for Y_n;

$$Y_n=(k_{xy}k_{yx}-2)Y_{n-1}Y_{n-2}.$$

First note that we have $Y_1<Y_0<0$. Since $k_{xy}k_{yx}-2\geqslant 2$ by assumption, the recursion gives $Y_2\leqslant 2Y_1-Y_0<Y_1$. Further application of the recursion formula generally gives $Y_n<Y_{n-1}$. Thus, every Y_n is negative and so playing y is always legal after $(yxy\cdots)_{2n}$. A similar argument shows that playing x will always be legal after $(yxy\cdots)_{2n+1}$, so the alternating play sequence can be legally continued forever. □

We now have the means to prove the theorem that characterizes the strongly convergent edge-weighted games.

PROOF OF THEOREM 3.1. (⇒) Suppose that for some edge (x, y) the weight product $k_{xy}k_{yx}$ is less than 4 but does not equal $4\cos^2(\pi/n)$ for any integer $n\geqslant 3$. We want to show that there exists an initial position from which we can play in different ways which do not lead to the same terminal position, at least not after the same number of moves. Choose as starting position $f_x=f_y=-1$ and f_u very large for all other nodes u. Then, by Lemma 3.5, there are alternating sequences $(xyx\cdots)_n$ and $(yxy\cdots)_m$ which cannot be legally continued; thus they result in terminal positions. From Lemma 3.6 we deduce that if these two terminal positions should be equal, then the two sequences must be of different lengths. Thus the game is not strongly convergent.

($\Leftarrow$) Suppose that for every edge (x, y) the weight product satisfies either $k_{xy}k_{yx} \geqslant 4$ or $k_{xy}k_{yx} = 4\cos^2(\pi/n_{xy})$ for some integer $n_{xy} \geqslant 3$. By Theorem 2.1 it is enough to verify the polygon property. Suppose that two different moves, x and y, are legal in some position. Then the polygon property follows from Observation 3.1 if x and y are not neighbors, from Lemma 3.6 if $k_{xy}k_{yx} = 4\cos^2(\pi/n_{xy})$, and from Lemma 3.7 if $k_{xy}k_{yx} \geqslant 4$. □

4. The Node-weighted Game

Consider once again a move in the edge-weighted game. Playing node i results in $f_i := -f_i$ and $f_j := f_j + k_{ij}f_i$ for any other node j. Suppose that the effect on the number on the played node was instead $f_i := -w_i f_i$, where $w_i > 0$ should be interpreted as a *node weight.*

Definition. The *node-weighted game* is played on a simple graph G where each edge (i, j) has a couple of positive weights, k_{ij} and k_{ji}, and each node i has a node weight $w_i > 0$. The rules are the same as in the edge-weighted game, with the exception that the number on the played node is now multiplied by the weight of the node after the change of sign.

We are going to prove the following characterization of strongly convergent games.

Theorem 4.1. *The node weighted game on a graph G with edge weights $\{k_{ij}\}$ is strongly convergent iff for each edge (i, j) of G the corresponding weight product satisfies either $k_{ji}k_{ij} \geqslant 2\sqrt{w_i w_j} + w_i + w_j$ or $k_{ji}k_{ij} = 2\sqrt{w_i w_j}\cos(2\pi/n) + w_i + w_j$ for some integer $n \geqslant 3$, and $w_i = w_j$ if n is odd.*

Note that if all node weights are equal to 1, then Theorem 4.1 is equivalent to Theorem 3.1. The two theorems will also have the same proof, as soon as we have shown the appropriate lemmas in the node-weighted case. From our analysis of the edge-weighted game we know that it is crucial for the strong convergence whether or not two alternating legal sequences $(xyx\cdots)_n$ and $(yxy\cdots)_n$ give the same result. As before, non-neighbors pose no problem.

Observation 4.2. xy and yx are both legal play sequences iff both x and y are legal moves. If x and y are not neighbors, then $xy \equiv yx$. This follows from the definition of the node-weighted game.

Suppose in the following that we have a couple of neighbors, x and y, with numbers $f_x = X$ and $f_y = Y$, where both X and Y are negative, so both nodes may be legally played. In order to avoid cumbersome notation, set $k_{xy} = p$ and $k_{yx} = q$. Then w_x, w_y, p and q are all strictly greater than zero.

We are interested in what happens with the numbers on the other nodes. For symmetry reasons, it will be enough to study a node, say u, which is a neighbor of x but not of y. If we play out a few moves of the sequence $xyx\cdots$ we obtain Table 1. Playing in the other order, $yxy\cdots$, we obtain Table 2.

TABLE 1
The numbers on x, y and u when playing $xyx\cdots$

	f_x		f_y		f_u	
	$\cdot X$	$\cdot Y$	$\cdot X$	$\cdot Y$	$\cdot X$	$\cdot Y$
	1	0	0	1	0	0
Play x	$-w_x$	0	p	1	k_{xu}	0
Play y	$-w_x+pq$	q	$-w_yp$	$-w_y$	k_{xu}	0
Play x	$w_x^2-w_xpq$	$-w_xq$	$-w_xp-w_yp+p^2q$	$-w_y+pq$	$k_{xu}(1-w_x+pq)$	$k_{xu}q$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Clearly, the value of for example the coefficient before X in the expression for f_x on a certain row is determined by the two coefficients before X on the row two rows up. Thus, the coefficients are described by coupled recursions. One can then solve for one of the columns and, as in Lemma 3.7, one verifies that for each one of the first four columns in Tables 1 and 2, the rows satisfy the recursion

$$C_n=(pq-w_x-w_y)C_{n-2}-w_xw_yC_{n-4}, \tag{1}$$

where C_n is the value of the nth row. The solution of this recursion is obtained on a nice form after the substitution

$$\cos 2\alpha=\frac{pq-w_x-w_y}{2\sqrt{w_xw_y}} \tag{2}$$

if the right-hand side is strictly between -1 and 1. (Otherwise, leap to Lemmas 4.6 and 4.7.) As usual solve the characteristic equation of the recursion. Then the general, real, solution can be written

$$C_n=\begin{cases}K_e\cdot(w_xw_y)^{n/4}\sin(n\alpha+\phi_e) & \text{if } n \text{ is even;}\\ K_o\cdot(w_xw_y)^{n/4}\sin(n\alpha+\phi_o) & \text{if } n \text{ is odd.}\end{cases} \tag{3}$$

We choose ϕ_e and ϕ_o to be in the interval $[-\pi/2,\pi/2)$. Then the arbitrary constants K_e and ϕ_e are determined by the two initial values C_0 and C_2, while K_o and ϕ_o are determined by C_1 and C_3.

LEMMA 4.3. *Suppose that* $k_{xy}k_{yx}=2\sqrt{w_xw_y}\cos 2\alpha+w_x+w_y$, *where* $0<\alpha<\pi/2$. *Suppose further that both* f_x *and* f_y *are negative in the current position. Then there exists*

TABLE 2
The numbers on x, y and u when playing $yxy\cdots$

	f_x		f_y		f_u	
	$\cdot X$	$\cdot Y$	$\cdot X$	$\cdot Y$	$\cdot X$	$\cdot Y$
	1	0	0	1	0	0
Play y	1	q	0	$-w_y$	0	0
Play x	$-w_x$	$-w_xq$	p	$-w_y+pq$	k_{xu}	$k_{xu}q$
Play y	$-w_x+pq$	$-w_xq-w_yq+pq^2$	$-w_yp$	$w_y^2-w_ypq$	k_{xu}	$k_{xu}q$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

some positive integer $n \leq \lceil \pi/\alpha \rceil + 1$ *such that* $(xyx \cdots)_n$ *is a legal play sequence while* $(xyx \cdots)_{n+1}$ *is not a legal play sequence. In particular, if* $\alpha = \pi/m$ *for some positive integer* m, *and* $w_x = w_y$ *if* m *is odd, then* $(xyx \cdots)_m$ *is legal.*

PROOF. If is clear from the explicit solution (3) of the recursion that C_n will have the opposite sign of C_0 for some even $n \leq \lceil \pi/\alpha \rceil + 1$. If we take C_0 to be f_x and C_n to be the number on x after $(xyx \cdots)_n$, this means that playing x will not be legal after $(xyx \cdots)_n$. (Possibly, some y move was illegal earlier, in which case the longest legal sequence is even shorter.)

Now suppose that $\alpha = \pi/m$ for some positive even m, and that $f_x = X$ and $f_y = Y$ are negative in the current position. Let $C_0 = X < 0$. In order to show that $(xyx \cdots)_m$ is legal, the trick is to observe that if we play two moves backwards, $y^{-1}x^{-1}$, we obtain a positive number on x, $C_{-2} = -w_y^{-1}X - qw_x^{-1}w_y^{-1}Y > 0$. From (3) we deduce that $0 \leq \phi_e < 2\pi/m$, so for even n between 0 and $m-2$ we will have $C_n < 0$, since $\sin(n\pi/m + \phi_e)$ is positive. In other words, x is a legal move every time it is played during $(xyx \cdots)_n$. To show that the y moves are also legal, let $C_1 < 0$ be the number on y after the first move x, observe that playing backwards gives $C_{-1} > 0$, and draw the desired conclusion as above.

If m is odd we may use the same argument again, if only we can show the sharper inequality $0 \leq \phi_e < \alpha$, so that $\sin(n\pi/m + \phi_e)$ has the same sign for all even n between 0 and $m-1$. Suppose also that $w_x = w_y$ and denote this by w for short. Then (3) gives

$$\frac{\sin \phi_e}{-\sin(\phi_e - 2\alpha)} = \frac{C_0}{-wC_{-2}} = \frac{X}{X + (q/w)Y} < 1.$$

Thus $\sin \phi_e < -\sin(\phi_e - 2\alpha)$, and we have chosen ϕ_e to be in $[-\pi/2, \pi/2)$, so we must have $0 \leq \phi_e < \alpha$. (Note that this computation depends heavily on the assumption that $w_x = w_y$.) □

We want to know when two legal play sequences, $(xyx \cdots)_n$ and $(yxy \cdots)_n$, give the same result. The sequences are equivalent if the position described by the nth row in Table 1 is equal to that of the nth row in Table 2. To simplify matters, we introduce Table 3, in which is tabulated the difference between each value in Table 1 and the corresponding value in Table 2. Thus the sequences $(xyx \cdots)_n$ and $(yxy \cdots)_n$ are equal iff the nth row of Table 3 represents zero values on nodes x, y and u. At first sight, it would seem that these values could well be zero without the individual columns being zero, by clever choice of the initial numbers X and Y. However, we will see that this is not so. Note that this will mean that if the two play sequences result in the same position when played from an initial position f in which both X and Y are negative (so

TABLE 3
The difference between Tables 1 and 2

f_x		f_y		f_u	
$\cdot X$	$\cdot Y$	$\cdot X$	$\cdot Y$	$\cdot X$	$\cdot Y$
0	0	0	0	0	0
$-(w_x+1)$	$-q$	p	$1+w_y$	k_{xu}	0
pq	$q(1+w_x)$	$-p(1+w_y)$	$-pq$	0	$-k_{xu}q$
$-(1+w_x)(-w_x+pq)$	$-q(-w_y+pq)$	$p(pq-w_x)$	$(1+w_y)(pq-w_y)$	$k_{xu}(pq-w_x)$	0
⋮	⋮	⋮	⋮	⋮	⋮

that $(xyx\cdots)_n \equiv_f (yxy\cdots)_n)$, then they will result in equal positions from all such initial positions (that is, $(xyx\cdots)_n \equiv (yxy\cdots)_n$).

Since the first four columns in Tables 1 and 2 obey the same linear recursion (1), so does every linear combination of these columns; in particular, the first four columns in Table 3. More surprisingly, the fifth and sixth columns in Table 3 also obey (1). This is seen as follows. A glance at Tables 1 and 2 will confirm that corresponding columns are really equal; the columns of the first table are simply shifted one row up or down with respect to the other table. Therefore the accumulated numbers that u receives in $(xyx\cdots)$, and respectively in $(yxy\cdots)$, differ only by the term from the last move. The result is that in the fifth column of Table 3, the even rows will be zero, while the odd rows will be k_{xu} times the value one row earlier in the first column of Table 1—and analogously for the sixth column.

Since all columns in Table 3 obey the recursion (1), we may treat complete rows, viewed as sixtuples, as the same time. The recursion can be used backwards to compute 'rows' with negative index.

LEMMA 4.4. *Let C_n denote the nth row of Table 3. If $w_x = w_y = w$, then $w^n C_{-n} = -C_n$, for odd n.*

PROOF. Let $w_x = w_y = w$. The recursion (1) used backwards takes the shape

$$w^2 C_{n-4} = (pq - 2w)C_{n-2} - C_n$$

and it is directly verified that in Table 3 we obtain $wC_{-1} = -C_1$ and $w^3 C_{-3} = -C_3$. The statement then follows by induction. □

With Lemma 4.4 in hand, we can deal with the case in which $(xyx\cdots)_n \equiv (yxy\cdots)_n$ and n is odd.

LEMMA 4.5. *Suppose that $k_{xy}k_{yx} = 2\sqrt{w_x w_y}\cos 2\alpha + w_x + w_y$, where $0 < \alpha < \pi/2$, and suppose that both f_x and f_y are negative. Then $(xyx\cdots)_n \equiv_f (yxy\cdots)_n$, with both sequences legal, iff $\alpha = \pi/n$, and, if n is odd, $w_x = w_y$.*

PROOF. We know that $(xyx\cdots)_n \equiv (yxy\cdots)_n$ exactly if if nth row of Table 3, which tabulates the difference between the positions arising from the two alternating play sequences, gives zero values on nodes x, y and u. First, suppose that $\alpha = \pi/n$, where n is even. Then, clearly, the formula (3) with $C_0 = 0$ implies $\phi_e = 0$, and thus $\sin(n\alpha + \phi_e) = \sin\pi = 0$, so $C_n = 0$. In words, the nth row in the table is all zeros: thus $(xyx\cdots)_n \equiv (yxy\cdots)_n$ and these play sequences are legal by Lemma 4.3.

Now suppose that $\alpha = \pi/n$, where n is odd, and suppose further that $w_x = w_y = w$. Then Lemma 4.4 says that $w^n C_{-n} = -C_n$. On the other hand, since $n\alpha = \pi$ we have $\sin(-n\alpha + \phi_o) = \sin(n\alpha + \phi_o)$, so by (3) C_{-n} and C_n must have the same sign. This is only possible if $C_n = 0$, and we can argue as above.

To prove the converse, suppose that row n looks like $(v_1, v_2, v_3, v_4, v_5, v_6)$. We want

$$v_1 X + v_2 Y = 0 = v_3 X + v_4 Y. \tag{4}$$

Since X and Y are strictly negative, this implies that

$$v_2 v_3 - v_1 v_4 = 0.$$

We begin with the case of even n. From rows 0 and 2 in Table 3 we see that $(1 + w_x)v_4 = -pv_2$ and $(1 + w_y)v_1 = -qv_2$, so in this case the equation is reduced to

$$[(1 + w_x)(1 + w_y) - pq] \cdot v_2 v_3 = 0.$$

By the inequality of the arithmetic and geometric means, we have

$$(1+w_x)(1+w_y)-pq \geqslant 2\sqrt{w_x w_y}+w_x+w_y-pq > 2\cos 2\alpha\sqrt{w_x w_y}+w_x+w_y-pq=0.$$

Thus we obtain $v_2 v_3 = 0$. But then $v_1 = v_2 = v_3 = v_4 = 0$ from equation (4) and the relations between the columns. We also have $v_5 = 0$ and $v_6 = -k_{xu}v_1/p = 0$. The explicit solution (3), with $C_0 = C_n = 0$, gives immediately that α is some multiple of π/n, and then Lemma 4.3 states that $\alpha = \pi/n$ is the only legal solution.

Finally, we treat n odd. Rows 1 and 3 in Table 3 tell us that $(1+w_x)v_3 = -pv_1$ and $(1+w_y)v_2 = -qv_4$, and by the above argument we obtain $v_1 = v_2 = v_3 = v_4 = v_5 = v_6 = 0$. Then the phase ϕ_o must be equal for all the columns, and thus all columns must be equal except for some constant factors. Compare, for example, the second and third columns to deduce that this implies $w_x = w_y$. By Lemma 4.4, $C_n = 0$ now implies that $C_{-n} = 0$, and then (3) forces α to be some multiple of $\pi/2n$, i.e. for some positive integer c we have $\alpha = c\pi/2n$. Since we know that there is a change of sign between C_{-1} and C_1, c must be at least two. Then Lemma 4.3 states that $\alpha = \pi/n$ is the only legal solution. □

Lemma 3.7 says that the edge-weighted game goes on forever if x and y are both legal moves, and the edge-weight product $k_{xy}k_{yx}$ is larger than or equal to 4. That result also generalizes to the node-weight game.

Lemma 4.6. *If both x and y are legal moves and* $k_{xy}k_{yx} \geqslant 2\sqrt{w_x w_y}+w_x+w_y$, *then the alternating play sequences* $xyxy\cdots$ *and* $yxyx\cdots$ *can be legally continued forever.*

Proof. The argument in Lemma 3.7 is easily modified to this situation. □

Finally, we must take care of the case in which the substitution (2) is impossible because of the right-hand side being less than or equal to -1. This case never arises in the edge-weighted game.

Lemma 4.7. *If x and y are neighbors, and* $k_{xy}k_{yx} \leqslant -2\sqrt{w_x w_y}+w_x+wy$, *then there exists a starting position of the node-weighted game from which there is one play sequence which terminates in two moves, and another two move sequence which does not result in the same position. Hence, the game is not strongly convergent.*

Proof. Let $p>0$ and $q>0$ be as in the tables. Suppose that $pq \leqslant -2\sqrt{w_x w_y}+w_x+w_y$. Then at least one of the inequalities $pq < w_y$ and $pq < w_x$ must hold. Assume, without loss of generality, that the latter holds. Take $f_x = -q$, $f_y = -w_x + pq$ and very large positive numbers on all other nodes. Since $pq < w_x$, both nodes may be legally played. After the two moves xy the resulting position has $f_x = 0$ and $f_y = w_x w_y > 0$, so it is terminal. On the other hand, after the two moves yx, the number on x is $f_x = w_x q(w_x - pq + 1) > 0$, so the resulting position from yx is not the same as from xy. □

At last, all our work will be rewarded.

Proof of Theorem 4.1. The proof of Theorem 3.1, modified in the obvious way, goes through. □

Remark 4.8. A natural question to ask about any strongly convergent game is: From which start positions does the game terminate? For the strongly convergent edge-weighted games, the answer is the positions that belong to the *Tits cone* in the corresponding geometric representation of a Coxeter group. This was shown by Mozes for the cases that he considered, and was then shown in general in the author's Ph.D. thesis [6]. For general node-weighted games, the question has not yet been answered.

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Received 23 July 1991 and accepted in revised form 31 May 1995

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