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Seeing The Möbius Disc-Transformation Group Like Never Before

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Abstract—The introduction of the gyration notion into nonassociative algebra, hyperbolic geometry, and relativity physics is motivated in this article by the emergence of the gyrogroup notion in the theory of the Möbius transformation group of the complex open unit disc. It suggests the prefix "gyro" that we use to emphasize analogies. Thus, for instance, gyrogroups are classified into gyrocommutative and nongyrocommutative gyrogroups in full analogy with the classification of groups into commutative and noncommutative groups. The road from the Thomas precession of the special theory of relativity to the Thomas gyration as well as the resulting new theory is presented in the author's book: *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces* [1]. The main result of this article is a theorem that allows the validity of some gyration identities to be extended from gyrocommutative gyrogroups into gyrogroups that need not be gyrocommutative. To set the stage for the main result, the Möbius disc-transformation group is studied in a novel way that suggests the notion of the gyrogroup and its gyrations.

1. INTRODUCTION

More than 150 years have passed since August Ferdinand Möbius first studied the transformations that now bear his name. Yet, the rich structure he thereby exposed is still far from being exhausted. The Möbius transformation group of the complex open disc gives rise to a binary operation in the disc that turns out to be a gyrocommutative gyrogroup operation. Like groups, gyrogroups are classified into gyrocommutative and nongyrocommutative gyrogroups. Two examples of a gyrocommutative gyrogroup, studied in [1–4], are the following.

(1) The Einstein relativity gyrogroup $(\mathbb{V}_c = \mathbb{R}^3_c, \oplus_{\mathrm{E}})$, where $\mathbb{V}_c = \mathbb{R}^3_c$ is the open c-ball (that is, a ball of radius $c > 0$) of the Euclidean three-space $\mathbb{V} = \mathbb{R}^3$ of all relativistically admissible velocities, and $\oplus_{\mathrm{E}}$ is the Einstein velocity addition of relativistically admissible velocities in $\mathbb{R}^3_c$. More generally, $\mathbb{V}_c$ is the open c-ball of any real inner product space $\mathbb{V}$. Einstein gyrogroups $(\mathbb{V}_c, \oplus_{\mathrm{E}})$ admit scalar multiplication, turning them into Einstein gyrovector spaces $(\mathbb{V}_c, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$. These, in turn, form the setting for the Beltrami (also known as the Klein) ball model of hyperbolic geometry [1]. The hyperbolic trigonometry of the hyperbolic geometry of Einstein gyrovector spaces $(\mathbb{V}_c, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$, and hence, of the Beltrami ball model of hyperbolic geometry, is studied in [5].

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(2) The Möbius gyrogroup $(\mathbb{V}_c, \oplus_{\mathrm{M}})$, where $\mathbb{V}_c$ is the open c-ball of any real inner product space $\mathbb{V}$, and $\oplus_{\mathrm{M}}$ is the Möbius addition in the ball. Möbius gyrogroups $(\mathbb{V}_c, \oplus_{\mathrm{M}})$ admit scalar multiplication, turning them into Möbius gyrovector spaces $(\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$. These, in turn, form the setting for the Poincaré ball model of hyperbolic geometry [1]. The hyperbolic trigonometry of the hyperbolic geometry of Möbius gyrovector spaces $(\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$, and hence, of the Poincaré ball model of hyperbolic geometry, is studied in [6].

It is the Einstein relativity gyrogroup that suggests the prefix "gyro" that we extensively use to emphasize analogies with classical counterparts. More specifically, Einstein's addition is regulated by the Thomas precession of the special theory of relativity. The latter, in turn, is extended by abstraction to the so-called Thomas gyration, suggesting the prefix "gyro" [1].

The gyrogroup prehistory is described in [7, p. 142]. Following increased interest in the study of gyrogroups in hyperbolic geometry and relativity physics [1,8], in group theory [9–11], in loop theory [12–14], and in analysis [4], we present in this article a theorem that allows the validity of some gyration identities to be extended from gyrocommutative gyrogroups to gyrogroups that need not be gyrocommutative. Gyration identities, in turn, satisfy interesting functional equations like the gyration loop properties in Section 2 and identities in Section 6.

To present our motivational approach to gyrations of gyrogroups we begin with the seemingly accidental striking coincidences that the Möbius self-transformations of the disc generate in terms of a peculiar rotation of the disc, called a *gyration*. Since coincidences in mathematics are not accidental, they necessarily lead to a new theory. Indeed, following the presentation in Section 2 of the striking coincidences to which the Möbius gyrations give rise, the resulting new theory that explains the coincidences, called *gyrogroup theory* [1], will be developed in Sections 3–8. The observations and the results in Sections 2–8 set the stage for our main result, in Theorem 19 of Section 9, on the *gyrocommutative protection principle*.

2. THE MÖBIUS DISC-TRANSFORMATIONS GROUP

The Möbius transformation group is a source of inspiration in complex analysis [15] and geometry [16–21]. The most general Möbius self-transformation of the complex open unit disc $\mathbb{D}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, of the complex plane $\mathbb{C}$ can be written as [22–26]

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z},$$

which we write as

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z), \tag{1}$$

$z_0 \in \mathbb{D}$, $\theta \in \mathbb{R}$. In fact, each bijective analytic function that maps the disc $\mathbb{D}$ onto itself is a Möbius transformation of the form (1) [23, p. 211]. The polar decomposition of the Möbius transformation group of the disc is a most beautiful result that has a great appeal to the Poincaré disc model of hyperbolic geometry, as we will see in this article. Suggestively, we define the Möbius addition $\oplus$ in the disc, allowing the generic Möbius transformation of the disc to be viewed as a Möbius *left gyrotranslation*

$$z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}, \tag{2}$$

$z_0 \in \mathbb{D}$, followed by a rotation of the disc by angle θ,

$$z \mapsto e^{i\theta} z, \tag{3}$$

$\theta \in \mathbb{R}$. The term "gyrotranslation" emphasizes the result that the Möbius addition $\oplus$ that we define in (2) is a gyrogroup operation [1], as we will explain in the sequel. A generalization of (2) to the ball of any real inner product space is found in [3,4,27], and a generalization of (2) from complex numbers to quaternions with applications to nonassociative geometry is found in [28].

The representation (1) of the Möbius transformation of the disc provides an example of the gyrodecomposition $\Gamma = GH$ that we will study in Section 3. It is the gyrodecomposition $M = \mathbb{D}R$ of the group $\Gamma = M$ of all Möbius self-transformations of the disc into the subgroup $H = R$ of M of all rotations of the disc,

$$R = \{e^{i\theta} : \theta \in \mathbb{R}\} \tag{4}$$

and the subset $G = \mathbb{D}$ of M of all left gyrotranslations of the disc, which we identify with points of the disc. The point $z_0 \in \mathbb{D}$ in (2) is identified with its corresponding left gyrotranslation (2) of the disc $\mathbb{D}$ by z_0.

The prefix "gyro" that we use to emphasize analogies with classical notions, stems from the Thomas gyration, which will soon become clear. The resulting Möbius addition in the disc $\mathbb{D}$ is neither commutative nor associative. To 'repair' the breakdown of commutativity in the Möbius addition we associate it with the *gyration* (or, rotation) generated by $a, b \in \mathbb{D}$,

$$\mathrm{gyr}\,[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b} \in R$$

giving rise to the gyrocommutative law of Möbius addition,

$$a \oplus b = \mathrm{gyr}\,[a,b](b \oplus a), \qquad \text{gyrocommutative law.}$$

Following the gyration definition, the gyrocommutative law is not terribly surprising, but we are not finished.

Coincidentally, the gyration that repairs the breakdown of commutativity in the Möbius addition repairs the breakdown of associativity as well, giving rise to identities that capture analogies. Thus, for instance, for all $a, b, c \in \mathbb{D}$ we have

$$\begin{aligned}
&a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a,b]c, && \text{left gyroassociative law,}\\
&(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}\,[b,a]c), && \text{right gyroassociative law,}\\
&\mathrm{gyr}\,[a,b] = \mathrm{gyr}\,[a \oplus b, b], && \text{left loop property,}\\
&\mathrm{gyr}\,[a,b] = \mathrm{gyr}\,[a, b \oplus a], && \text{right loop property.}
\end{aligned}$$

The Möbius addition is clearly regulated by its associated gyration and, accordingly, the Möbius addition and its associated gyration are inextricably linked. Furthermore, Möbius addition $\oplus$ comes with a dual addition, called the Möbius coaddition $\boxplus$, with which it shares duality symmetries like

$$\begin{aligned}
a \boxplus b &= a \oplus \mathrm{gyr}\,[a, \ominus b]b,\\
a \oplus b &= a \boxplus \mathrm{gyr}\,[a,b]b.
\end{aligned} \tag{5}$$

and which allows us to capture analogies, like the following left and right cancellation laws in the disc $\mathbb{D}$:

$$\begin{aligned}
x \oplus (\ominus x \oplus a) &= a,\\
(a \ominus x) \boxplus x &= a,
\end{aligned} \tag{6}$$

and the ones shown in Table 1.

Where there are coincidences there is significance. The emerging coincidences to which the gyration gives rise uncover an interesting algebraic structure that merits extension by abstraction, leading to the grouplike structure called a gyrogroup.

We will find that gyrogroups are generalized groups that share remarkable analogies with groups. In full analogy with the following groups.

(1) Gyrogroups are classified into gyrocommutative gyrogroups and nongyrocommutative gyrogroups.

Table 1. Euclidean-hyperbolic analogies for the Möbius addition $\oplus$ and coaddition $\boxplus$ in gyrovector spaces, studied in [1].

Notion	Euclidean	Hyperbolic
Addition	$+$	$\oplus$ and $\boxplus$
Commutativity	$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	$\mathbf{a}\oplus\mathbf{b}=\mathrm{gyr}[\mathbf{a},\mathbf{b}](\mathbf{b}\oplus\mathbf{a})$ $\mathbf{a}\boxplus\mathbf{b}=\mathbf{b}\boxplus\mathbf{a}$
Associativity	$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$	$\mathbf{a}\oplus(\mathbf{b}\oplus\mathbf{c})=(\mathbf{a}\oplus\mathbf{b})\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{c}$ $(\mathbf{a}\oplus\mathbf{b})\oplus\mathbf{c}=\mathbf{a}\oplus(\mathbf{b}\oplus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{c})$
Distance	$\Vert\mathbf{a}-\mathbf{b}\Vert$	$\Vert\mathbf{a}\ominus\mathbf{b}\Vert$
Pythagorean Theorem	$\Vert\mathbf{A}\Vert^2+\Vert\mathbf{B}\Vert^2=\Vert\mathbf{C}\Vert^2$	$\Vert\mathbf{A}\Vert^2\oplus\Vert\mathbf{B}\Vert^2=\Vert\mathbf{C}\Vert^2$ in the Poincaré unit ball model
Gyrovector Transport	$-\mathbf{a}_1+\mathbf{b}_1=-\mathbf{a}_0+\mathbf{b}_0$	$\ominus\mathbf{a}_1\oplus\mathbf{b}_1=\ominus\mathbf{a}_0\oplus\mathbf{b}_0$
Parallel Transport	$-\mathbf{a}_1+\mathbf{b}_1=-\mathbf{a}_0+\mathbf{b}_0$	$\ominus\mathbf{a}_1\oplus\mathbf{b}_1=\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)$
Triangle Inequality	$\Vert-\mathbf{a}+\mathbf{b}\Vert-\Vert-\mathbf{b}+\mathbf{c}\Vert$ $\le\Vert-\mathbf{a}+\mathbf{c}\Vert$ $\le\Vert-\mathbf{a}+\mathbf{b}\Vert+\Vert-\mathbf{b}+\mathbf{c}\Vert$	$\Vert\ominus\mathbf{a}\oplus\mathbf{b}\Vert\ominus\Vert\ominus\mathbf{b}\oplus\mathbf{c}\Vert$ $\le\Vert\ominus\mathbf{a}\oplus\mathbf{c}\Vert$ $\le\Vert\ominus\mathbf{a}\oplus\mathbf{b}\Vert\oplus\Vert\ominus\mathbf{b}\oplus\mathbf{c}\Vert$
Geodesics	$\mathbf{a}+(-\mathbf{a}+\mathbf{b})t$ $-\infty<t<\infty$	$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t$ $-\infty<t<\infty$
Midpoint $(\mathbf{a},\mathbf{b})$	$\mathbf{a}+\frac{1}{2}(-\mathbf{a}+\mathbf{b})=\frac{1}{2}(\mathbf{a}+\mathbf{b})$	$\mathbf{a}\oplus\frac{1}{2}\otimes(\ominus\mathbf{a}\oplus\mathbf{b})=\frac{1}{2}\otimes(\mathbf{a}\boxplus\mathbf{b})$
Angle cosines between rays	$\cos\alpha=\frac{-\mathbf{a}+\mathbf{b}}{\Vert-\mathbf{a}+\mathbf{b}\Vert}\cdot\frac{-\mathbf{a}+\mathbf{c}}{\Vert-\mathbf{a}+\mathbf{c}\Vert}$	$\cos\alpha=\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\Vert\ominus\mathbf{a}\oplus\mathbf{b}\Vert}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\Vert\ominus\mathbf{a}\oplus\mathbf{c}\Vert}$
Equations	$\mathbf{a}+\mathbf{x}=\mathbf{b}$ $\mathbf{x}=-\mathbf{a}+\mathbf{b}$ $\mathbf{y}+\mathbf{a}=\mathbf{b}$ $\mathbf{y}=\mathbf{b}-\mathbf{a}$	$\mathbf{a}\oplus\mathbf{x}=\mathbf{b}$ $\mathbf{x}=\ominus\mathbf{a}\oplus\mathbf{b}$ $\mathbf{y}\oplus\mathbf{a}=\mathbf{b}$ $\mathbf{y}=\mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a}=\mathbf{b}\boxminus\mathbf{a}$
Cancellation Laws	$\mathbf{x}+(-\mathbf{x}+\mathbf{a})=\mathbf{a}$ $(\mathbf{a}-\mathbf{x})+\mathbf{x}=\mathbf{a}$	$\mathbf{x}\oplus(\ominus\mathbf{x}\oplus\mathbf{a})=\mathbf{a}$ $(\mathbf{a}\ominus\mathbf{x})\boxplus\mathbf{x}=\mathbf{a}$ $(\mathbf{a}\boxminus\mathbf{x})\oplus\mathbf{x}=\mathbf{a}$

(2) Some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces.
(3) Gyrovector spaces, in turn, provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified.
(4) Moreover, the resulting analogies shared by the motions of Euclidean geometry (that is, a commutative group of translations and a group of rotations) and the motions of hyperbolic geometry (that is, a gyrocommutative gyrogroup of left gyrotranslations and a group of rotations) induce analogies shared by the Galilei transformation and the Lorentz transformation. These analogies, in turn, enable Lorentz transformation problems, hitherto poorly understood, to be straightforwardly solved in full analogy with the respective solutions of their Galilean counterparts. A point in case is, for instance, the determination of the frozen shape (as opposed to the apparent shape) of relativistically moving objects in [1].

Our way of seeing the Möbius disc-transformation group like never before is illustrated by Figures 1 and 2. To demonstrate the ability of gyroformalism to capture analogies:

(1) we present graphically in Figure 1 the hyperbolic Pythagorean theorem in the Möbius gyrovector plane $(\mathbb{D},\oplus,\otimes)$, studied in [1,6,29]; and

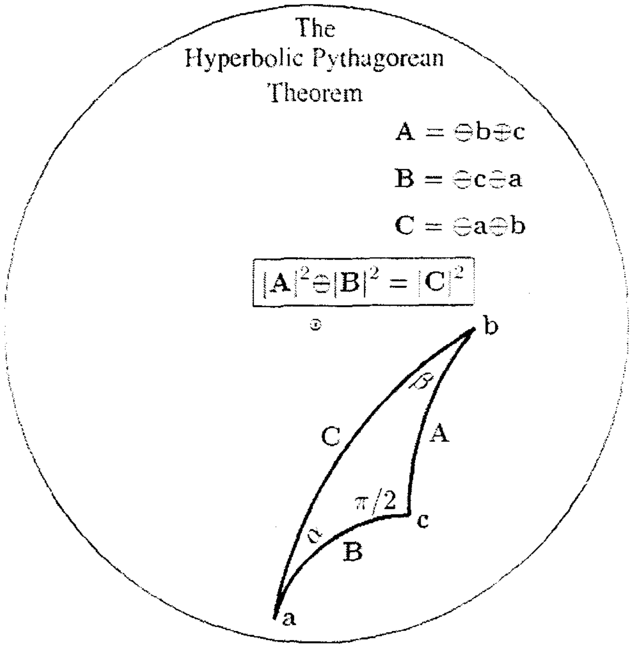

Figure 1. Artful application of the Möbius transformation in hyperbolic geometry. I. The hyperbolic Pythagorean theorem for Möbius right angled hyperbolic triangles in the complex open unit disc $\mathbb{D}$ in a form visually analogous to its Euclidean counterpart [29].

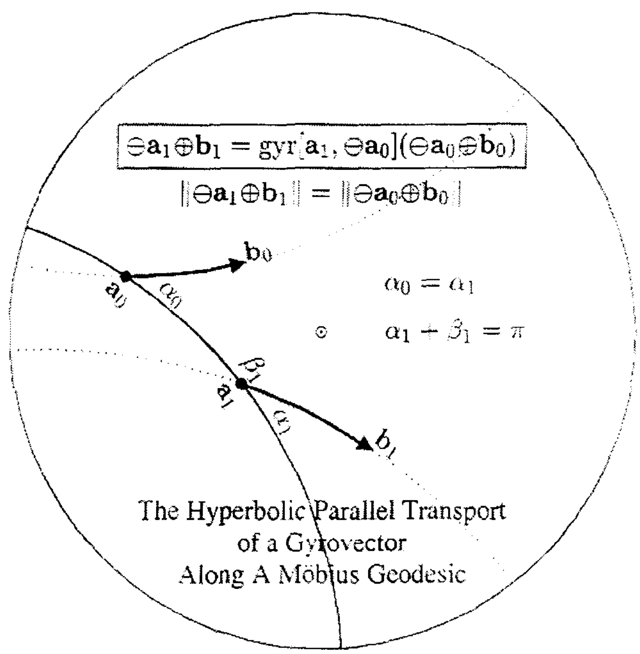

Figure 2. Artful application of the Möbius transformation in hyperbolic geometry. II. The Poincaré n-dimensional ball model of hyperbolic geometry turns out to be the n-dimensional Möbius gyrovector space. We find in [1], that in gyroformalism the nonassociative algebra of the hyperbolic parallel transport of a gyrovector $(-\mathbf{a}_0 \oplus \mathbf{b}_0)$ rooted at $\mathbf{a}_0$ to the gyrovector $(-\mathbf{a}_1 \oplus \mathbf{b}_1)$ rooted at $\mathbf{a}_1$ along the Möbius geodisc that links $\mathbf{a}_0$ and $\mathbf{a}_1$ in a Möbius gyrovector space $(\mathbb{R}^2_c, \oplus, \otimes)$ is fully analogous to the algebra of its Euclidean counterpart. The special case of $n = 2$ is shown here graphically.

(2) we present graphically in Figure 2 the algebra of the hyperbolic parallel transport along geodesics in the Möbius gyrovector plane $(\mathbb{R}_c^2, \oplus, \otimes)$, studied in [1].

Typically, to the study of hyperbolic geometry as the geometry of gyrovector spaces, the hyperbolic geometry concepts shown in Figures 1 and 2 turn out to be fully analogous to their Euclidean counterparts. In contrast, in pregyrovector space theory:

(1) the hyperbolic Pythagorean theorem appeared in the literature in a form which shares no visual analogies with its Euclidean counterpart;
(2) parallel transport in classical hyperbolic geometry is achieved by methods of differential geometry rather than by methods of nonassociative algebra.

An interesting application of the left loop property emerges when we solve the equation

$$x \oplus a = b \tag{7}$$

in the Möbius gyrogroup $(\mathbb{D}, \oplus)$ for given $a, b \in \mathbb{D}$.

If $x \in \mathbb{D}$ is a solution of (7), then by the left gyroassociative law, the left loop property, and (7), we have

$$\begin{aligned} x &= x \oplus 0 \\ &= x \oplus (a \oplus (\ominus a)) \\ &= (x \oplus a) \oplus \operatorname{gyr}[x, a](\ominus a) \\ &= (x \oplus a) \oplus (\ominus \operatorname{gyr}[x, a]a) \\ &= (x \oplus a) \ominus \operatorname{gyr}[x, a]a \\ &= b \ominus \operatorname{gyr}[x, a]a \\ &= b \ominus \operatorname{gyr}[x \oplus a, a]a \\ &= b \ominus \operatorname{gyr}[b, a]a \\ &= b \boxminus a, \end{aligned} \tag{8}$$

where we abbreviate: $a \ominus b = a \oplus (-b) = a \oplus (\ominus b)$ and $b \boxminus a = b \boxplus (-a)$. Clearly, it is the left loop property that makes the left gyroassociative law effective in solving (7) in (8).

Thus, if a solution of $x \oplus a = b$ exists, it must have the form

$$x = b \ominus \operatorname{gyr}[b, a]a = b \boxminus a. \tag{9}$$

Conversely, one must show that $x = b \ominus \operatorname{gyr}[b, a]a$ is indeed a solution of (7). This is a delicate matter, verified in [1]. Substituting the solution (9) in its equation (7), we obtain the dual right cancellation law

$$(b \boxminus a) \oplus a = b, \tag{10}$$

which shares duality symmetries with the right cancellation law in (6). The role which analogies with classical results play in the study of the Möbius disc-transformation group is enhanced by the hyperbolic trigonometry of the disc, both in its Beltrami and Poincaré models of hyperbolic geometry, studied in [5,6], where it is extended to the ball.

A natural extension of the Möbius disc gyrogroup $(\mathbb{D}, \oplus_{\mathrm{M}})$, $\oplus_{\mathrm{M}} = \oplus$, into the Möbius ball gyrogroups $(\mathbb{V}_c, \oplus_{\mathrm{M}})$, $\mathbb{V}_c$ being the ball of radius $c > 0$ of any real inner product space $\mathbb{V}$, is described in [1,3,4]. The latter, in turn, are isomorphic to the corresponding Einstein ball gyrogroups $(\mathbb{V}_c, \oplus_{\mathrm{E}})$. Here, the gyrocommutative gyrogroup operation $\oplus_{\mathrm{E}}$ is the abstract extension of the Einstein velocity addition of special relativity theory. Interestingly, when the ball $\mathbb{V}_c$ of the abstract real inner product space $\mathbb{V}$ is realized by the ball $\mathbb{R}_c^3$ of the Euclidean three-space $\mathbb{R}^3$, the gyrogroup gyration specializes to the Thomas precession of Einstein's special relativity theory. Accordingly, the Thomas gyration is the extension by abstraction of the relativistic Thomas precession [1,2].

3. GYROGROUPS

We will now extend by abstraction the observations we made in Section 2 about the seemingly accidental coincidences that the Möbius self-transformations of the disc generate. The extension by abstraction will enable us to understand why the gyration coincidences encountered in Section 2 are not accidental.

A groupoid $(G, \odot)$ is a nonempty set G with a binary operation $\odot$. An automorphism of a groupoid $(G, \odot)$ is a bijective self-map of G that respects its binary operation $\odot$. The set of all automorphisms of a groupoid $(G, \odot)$ forms a group, denoted $\mathrm{Aut}\,(G, \odot)$, with group operation given by bijection composition.

In order to reproduce the Möbius addition and its gyration on the abstract level, we take the key features of the polar decomposition (1) of the Möbius transformation group of the disc, obtaining the definition of the gyrodecomposition in Definition 1 below. We will then show that, as anticipated, the gyrodecomposition allows us to reproduce the Möbius addition and its associated gyration, along with some of the coincidences observed in Section 2, on the abstract level. Surprisingly, this extension by abstraction of the Möbius addition and its gyration enables the Möbius transformation group of the disc and the coincidences observed in Section 2 to be clearly explained. Understanding the Möbius groupoid $(\mathbb{D}, \oplus_{\mathrm{M}})$ on the abstract level, in turn, leads us to the discovery of the gyrogroup, a most natural extension of the group notion. Owing to analogies that the gyrogroup notion shares with the notion of the group, gyrogroup theory is developed along analogies it shares with group theory in [1].

In the following definition, we thus extend the polar decomposition in (1) by abstraction, suggestively calling it a gyrodecomposition [9].

DEFINITION 1. GYRODECOMPOSITION. *Let Γ be a group possessing the unique decomposition*

$$\Gamma = GH \tag{11}$$

in the sense that every element $\gamma \in \Gamma$ can be written uniquely as

$$\gamma = gh, \tag{12}$$

where $g \in G$ and $h \in H$. The decomposition (11) is said to be a gyrodecomposition of the group Γ if

(i) *H, $H < \Gamma$, is a subgroup of Γ,*
(ii) *G, $G \subset \Gamma$, is a subset of Γ normalized by H, that is, $H \subseteq N_\Gamma(G)$, where $N_\Gamma(G) = \{\gamma \in \Gamma : \gamma G \gamma^{-1} = G\}$ is the normalizer of G in Γ,*
(iii) *$1_\Gamma \in G$, 1_Γ being the identity element of Γ, and*
(iv) *G is closed under inversion in Γ, $G = G^{-1}$.*

In the same way that the Möbius addition $\oplus$ and its associated *gyrooperation* gyr were extracted in (2) from the polar decomposition (1) of the Möbius disc-transformation group, we now extract a binary operation $\odot$ in G and its associated gyrooperation gyr from the gyrodecomposition (11).

The gyrodecomposition (11) induces a binary operation $\odot$ in G, turning it into a groupoid. Let $g_1, g_2 \in G$. The gyrodecomposition of $g_1 g_2 \in \Gamma$ gives the unique decomposition

$$g_1 g_2 = (g_1 \odot g_2) h(g_1, g_2), \tag{13}$$

where $g_1 \odot g_2 \in G$ and $h(g_1, g_2) \in H$. We say that $g_1 \odot g_2$ is the composition of g_1 and g_2 in the groupoid $(G, \odot)$, and that $h(g_1, g_2)$ is the element of H generated by g_1 and g_2.

Equipped with the binary operation $\odot$ in G, that the unique decomposition (11) of G determines by (13), the resulting groupoid $(G, \odot)$ possesses the *gyration operation*

$$\mathrm{gyr} : G \times G \to \mathrm{Aut}\,(G, \odot), \tag{14}$$

where the *gyrations* $\mathrm{gyr}\,[g_1, g_2] \in \mathrm{Aut}\,(G, \odot)$ of the groupoid $(G, \odot)$ are the self-maps of G given by the equation

$$\mathrm{gyr}\,[g_1, g_2]g = h(g_1, g_2)gh^{-1}(g_1, g_2), \tag{15}$$

$g_1, g_2, g \in G$, where we use the notation $h^{-1}(g_1, g_2) = (h(g_1, g_2))^{-1}$. Clearly,

$$\mathrm{gyr}\,[g_1, g_2]g \in G, \tag{16}$$

for any $g, g_1, g_2 \in G$ since G is normalized by H. Moreover, according to Theorem 2.11 in [9], the gyration $\mathrm{gyr}\,[g_1, g_2]$ is an automorphism of the groupoid $(G, \odot)$ for any $g_1, g_2 \in G$, so that

$$\mathrm{gyr}\,[g_1, g_2] \in \mathrm{Aut}\,(G, \odot) \tag{17}$$

as anticipated in (14). The gyrations of G are therefore also called *gyroautomorphisms* of G indicating that there might be automorphisms of G that are not gyroautomorphisms of G. A *gyroautomorphism group*, $\mathrm{Aut}_0(G, \odot)$, of the groupoid $(G, \odot)$ is any subgroup of $\mathrm{Aut}\,(G, \odot)$ (not necessarily the smallest one) that contains all the gyroautomorphisms of G. In general, the gyroautomorphisms of a gyrogroup $(G, \odot)$ do not form a group.

Decomposition (11) thus gives rise to a groupoid, $(G, \odot)$, equipped with a *gyrooperation* gyr that generates gyroautomorphisms $\mathrm{gyr}\,[g_1, g_2]$, $g_1, g_2 \in G$. This groupoid turns out to be a *left gyrogroup*, defined in Definition 3 below, following Definition 2 of the gyrogroup.

DEFINITION 2. GYROGROUPS. *A groupoid $(G, \odot)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 1, called a left identity, satisfying*

$$1 \odot a = a, \qquad \textit{left identity}, \tag{G1}$$

for all $a \in G$. There is an element $1 \in G$ satisfying axiom (G1) such that for each a in G there is an element a^{-1} in G, called a left inverse of a, satisfying

$$a^{-1} \odot a = 1, \qquad \textit{left inverse.} \tag{G2}$$

Moreover, for any $a, b, z \in G$ there exists a unique element $\mathrm{gyr}\,[a, b]z \in G$ such that

$$a \odot (b \odot z) = (a \odot b) \odot \mathrm{gyr}\,[a, b]z, \qquad \textit{left gyroassociative law.} \tag{G3}$$

If $gyr[a, b]$ denotes the map $gyr[a, b] : G \to G$ given by $z \mapsto \mathrm{gyr}\,[a, b]z$ then

$$\mathrm{gyr}\,[a, b] \in \mathrm{Aut}\,(G, \odot), \qquad \textit{gyroautomorphism}, \tag{G4}$$

and $gyr[a, b]$ is called the Thomas gyration, or the gyroautomorphism of G, generated by $a, b \in G$. Finally, the gyroautomorphism $gyr[a, b]$ generated by any $a, b \in G$ satisfies

$$\mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[a \odot b, b], \qquad \textit{left loop property.} \tag{G5}$$

Relaxing the left loop property into the *weak left loop property* we obtain the *left gyrogroup*.

DEFINITION 3. LEFT GYROGROUPS. *A groupoid $(G, \odot)$ satisfying the gyrogroup axioms (G1)–(G4), Definition 2, in which the left loop property (G5) of Definition 2 is replaced by the weaker property,*

$$\mathrm{gyr}\,[a, a^{-1}] = id, \qquad \textit{weak left loop property}, \tag{G5$'$}$$

is called a left gyrogroup, id being the identity map.

DEFINITION 4. GYROCOMMUTATIVE GYROGROUPS. *A gyrogroup $(G, \odot)$ is gyrocommutative if for all $a, b \in G$*

$$a \odot b = \operatorname{gyr}[a,b](b \odot a), \qquad \textit{gyrocommutative law.} \tag{G6}$$

An example of a gyrocommutative gyrogroup is provided by the Möbius disc-gyrogroup $(\mathbb{D}, \oplus_{\mathrm{M}})$ studied in Section 2, which turns out to be gyrocommutative.

It follows from Definition 3 that a left gyrogroup that possesses the left loop property is a gyrogroup.

Gyrogroups possess the *left cancellation law*, (6), enabling the left gyroassociative law (G3) to be solved for $\operatorname{gyr}[a,b]z$, thus obtaining the *gyrogroup gyration-operation identity*

$$\operatorname{gyr}[a,b]z = (a \odot b)^{-1} \odot \{a \odot (b \odot z)\} \tag{18}$$

that expresses the effects of a gyrogroup gyrooperation, gyr, in terms of the gyrogroup operation, $\odot$, and inversion.

4. THE LEFT GYROGROUP REPRESENTATION THEOREM

The pair of groups (Γ, H) in the gyrodecomposition (11) is called the *enveloping pair* that generates, by means of the construction presented in Section 3, the left gyrogroup G in (11). It has been verified in [2] that every left gyrogroup G possesses an enveloping pair (Γ, H) from which it can be generated, called a generating enveloping pair of G. In fact, the notion of the left gyrogroup is not defined in [2], but the verification in [2] for gyrogroups is valid for left gyrogroups as well since it makes no use of the left loop property.

The gyrations $\operatorname{gyr}[g_1, g_2]$ of a left gyrogroup G generated by an enveloping pair (Γ, H) are described in Section 3. They are given in terms of elements $h \in H$ by (15). In general, gyrations do not correspond injectively to elements of H since two distinct elements of H may generate, by means of (15), the same gyration. To exclude such a possibility, insuring that gyrations correspond by (15) to elements of H injectively, we *reduce* the enveloping pair (Γ, H) that generates a left gyrogroup G by means of the gyrodecomposition (11),

$$\Gamma = GH \tag{19}$$

into the *reduced enveloping pair*

$$(\Gamma_G, H_G) = \left(\frac{\Gamma}{C_H(G)}, \frac{H}{C_H(G)}\right), \tag{20}$$

where

$$C_H(G) = \left\{h \in H : hgh^{-1} = g, \text{ for all } g \in G\right\} \tag{21}$$

the centralizer of G in H, turns out to be a normal subgroup of Γ.

In (20), we thus 'mod out' the centralizer $C_H(G)$ of G in H to reduce the enveloping pair (Γ, H), that generates a left gyrogroup G, into the reduced enveloping pair (Γ_G, H_G), that generates the same left gyrogroup G.

The reduced enveloping pair (Γ_G, H_G) in (20) generates the left gyrogroup G in (19) by means of the gyrodecomposition (Lemma 3.2 in [9])

$$\Gamma_G = GH_G. \tag{22}$$

As opposed to the general case of a left gyrogroup generated by an enveloping pair, described in Section 3, the gyrations of the left gyrogroup G in (22), generated by the reduced enveloping pair (Γ_G, H_G) correspond, by (15), injectively to elements of H_G.

Thus, every left gyrogroup G is generated by some enveloping pair (Γ, H). Furthermore, we can assume without loss of generality that the enveloping pair (Γ, H) is reduced, so that the gyrations of G correspond bijectively, by (15), to the elements of a subset of H. We formalize this result in the following.

THEOREM 5. THE LEFT GYROGROUP REPRESENTATION THEOREM. *Let $(G, \odot)$ be a left gyrogroup. There exists an enveloping pair (Γ, H) that generates the left gyrogroup $(G, \odot)$*

$$\Gamma = GH \tag{23}$$

such that each gyroautomorphism

$$\mathrm{gyr}\,[g_1, g_2] \in \mathrm{Aut}\,(G, \odot) \tag{24}$$

of G, $g_1, g_2 \in G$, has a unique $h = h(g_1, g_2) \in H$ satisfying

$$\mathrm{gyr}\,[g_1, g_2]g = hgh^{-1}. \tag{25}$$

The subset G of Γ in (23) is called, in group theory, a transversal to the subgroup H in the group Γ. Following Theorem 5, we can study left gyrogroups in group theory by representing them in terms of transversals to subgroups in groups. Accordingly, a transversal that arises from a gyrodecomposition is called a *gyrotransversal* [9] and, by Theorem 5, every gyrotransversal is a left gyrogroup.

5. MORE ABOUT GYROGROUPS

Gyrogroups, both gyrocommutative and nongyrocommutative, abound in group theory [9,10]. Several gyrocommutative gyrogroups that possess important applications in relativity physics and hyperbolic geometry are studied in [1]. An example of a nongyrocommutative gyrogroup arose naturally in [30] in the extension of the gyrocommutative gyrogroup of Lorentz boosts (a Lorentz boost is a Lorentz transformation without rotation) with real time into the nongyrocommutative gyrogroup of Lorentz boosts with complex time. The usefulness of complex time in quantum mechanics, in turn, is well known [31].

The validity of the identity

$$\mathrm{gyr}\,[z, a \odot b] = \mathrm{gyr}\,[z, \mathrm{gyr}\,[a, b](b \odot a)], \tag{26}$$

for any three elements $a, b, z \in G$ in any gyrocommutative gyrogroup $(G, \odot)$ follows from the gyrocommutative law (G6) in Definition 4. Surprisingly, however, identity (26) remains valid in any nongyrocommutative gyrogroup as well, regardless of whether the gyrocommutative law (G6) is satisfied or not. It seems that the Thomas gyration allows gyrogroup expressions that form its arguments to be treated as if they obeyed the gyrocommutative law. To understand the observation in (26), we will generalize it and verify it in Theorem 19.

When the abstract gyrogroup $(G, \odot)$ is realized by the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus_{\mathrm{E}})$, where

(1) $\mathbb{R}^3_c$ is the set of all relativistically admissible velocities in the Euclidean three-space $\mathbb{R}^3$, $\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\}$, c being the vacuum speed of light, and where
(2) the binary operation $\oplus_{\mathrm{E}}$ in $\mathbb{R}^3_c$ is the Einstein velocity addition of special relativity theory,

the Thomas gyration reduces to the relativistic peculiar rotation known as the Thomas precession [1,2]. In this sense, the Thomas gyration is the generalization by abstraction of the Thomas precession of the special theory of relativity.

The usefulness of the realizations of the Thomas gyration in several models of hyperbolic geometry is also well known [1]. Understanding the peculiar gyrogroup automorphisms, the Thomas gyrations, is therefore important in hyperbolic geometry, relativity physics, and nonassociative algebra.

Examples of gyrogroups, finite and infinite, gyrocommutative, and nongyrocommutative are found in [1,9,10]. Some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces. These, in turn, form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry [2,32]. An elegant result that demonstrates the unification of Euclidean and hyperbolic geometry that gyrovector spaces allow emerged from the discovery of the truly hyperbolic Pythagorean theorem in [29,32] that, unexpectedly [1], shares visual analogies with its Euclidean counterpart.

A gyrogroup $(G, \odot)$ is naturally equipped with a secondary binary operation, called the co-operation. The gyrogroup co-operation definition as well as some of its elementary properties, studied in [1,2], are presented below.

DEFINITION 6. THE GYROGROUP CO-OPERATIONS. *Let $(G, \odot)$ be a gyrogroup. A binary co-operation $\boxdot$ in G, that coexists with the gyrogroup binary operation $\odot$ in G, is defined by the equation*

$$a \boxdot b = a \odot \operatorname{gyr}\left[a, b^{-1}\right] b, \qquad \textit{gyrogroup cooperation.} \tag{G7}$$

The operation $\odot$ and the cooperation $\boxdot$ of a gyrogroup $(G, \odot)$ are collectively called the dual operations of the gyrogroup. The cooperation is also called the dual operation.

THEOREM 7. *(See [1, Theorem 2.31, p. 58].) Let $(G, \odot)$ be a gyrogroup. Its operation $\odot$ is given in terms of its cooperation $\boxdot$ by the identity*

$$a \odot b = a \boxdot \operatorname{gyr}[a, b]b, \qquad \textit{cooperation.} \tag{G8}$$

THEOREM 8. *(See [1, Theorem 2.39, p. 62].) A gyrogroup $(G, \odot)$ is gyrocommutative if and only if*

$$(a \odot b)^{-1} = a^{-1} \odot b^{-1}, \qquad \textit{automorphic inverse.} \tag{G9}$$

Identities (G7) and (G8) herald the emergence of the duality symmetries to which the two dual binary operations of a gyrogroup give rise.

THEOREM 9. *(See [1, Theorem 2.45, p. 65].) A gyrogroup $(G, \odot)$ is gyrocommutative if and only if its associated dual loop $(G, \boxdot)$ is commutative,*

$$a \boxdot b = b \boxdot a, \qquad \textit{commutative cooperation.} \tag{G10}$$

Gyrogroup theory is presented in [1,2]. The dual binary operations in a gyrogroup uncover useful duality symmetries in gyrogroups and in gyrovector spaces, as well as in hyperbolic geometry. In particular, they give rise to the *left cancellation law*

$$a \odot \left(a^{-1} \odot b\right) = b \tag{27}$$

and to the two *dual right cancellation laws*

$$\begin{aligned} \left(b \odot a^{-1}\right) \boxdot a &= a, \\ \left(b \boxdot a^{-1}\right) \odot a &= a. \end{aligned} \tag{28}$$

The left and the right cancellation laws indicate that in order to capture analogies that gyrogroups share with groups, the two dual binary operations in a gyrogroup are needed.

Gyrogroup operations in the arguments of Thomas gyrations are useful in gyrogroup theory, as evidenced, for instance, from the loop property (G5). An example of a gyrogroup identity that exhibits the usefulness of gyrogroup operations in the arguments of Thomas gyrations is presented in the following theorem, the proof of which is found in Theorem 5.11 of [2].

THEOREM 10. *Any three elements a, b, c of a gyrogroup $(G, \odot)$ satisfy the identity*

$$\operatorname{gyr}[a, b \odot c] \operatorname{gyr}[b, c] = \operatorname{gyr}[a, b] \operatorname{gyr}\left[\left(b^{-1} \odot a^{-1}\right)^{-1}, c\right]. \tag{29}$$

If the gyrogroup $(G, \odot)$ is gyrocommutative, then

$$\operatorname{gyr}[a, b \odot c] \operatorname{gyr}[b, c] = \operatorname{gyr}[a, b] \operatorname{gyr}[b \odot a, c]. \tag{30}$$

While the validity of identity (30) was verified in [2] only for gyrocommutative gyrogroups (by applying the automorphic inverse law (G9) to (29)), it was observed that it is valid in some nongyrocommutative gyrogroups as well, and no counter example was found; see the paragraph below the proof of Theorem 5.11 in [2]. It is therefore conjectured that Thomas gyrations allow gyrocommutative operations in their arguments even in nongyrocommutative gyrogroups. It is this conjecture that we formalize and verify in Theorem 19.

Another observation that supports the conjecture is provided by the gyrogroup cooperation $\boxdot$ which, according to Theorem 9, is

(i) commutative in a gyrocommutative gyrogroup, and
(ii) noncommutative in a nongyrocommutative gyrogroup.

Accordingly, one would expect to have the gyrogroup identity

$$\operatorname{gyr}[a, b \boxdot c] = \operatorname{gyr}[a, c \boxdot b], \tag{31}$$

$a, b, c \in G$, valid only if the gyrogroup $G = (G, \odot)$ is gyrocommutative. Surprisingly, however, observations indicate that identity (31) remains valid in any nongyrocommutative gyrogroup as well. It thus seems that gyrocommutative manipulations in arguments of gyrations, as in (31), are allowed even in gyrations of nongyrocommutative gyrogroups. Theorem 19 will show that this is indeed the case.

6. GYRATION IDENTITIES

The left loop property of gyrations, presented in Section 2, is one of the elegant, useful gyration identities of gyrogroup theory. Several other gyration identities, taken from [1], are listed in the following theorem.

THEOREM 11. *Let $(G, +)$ be a gyrogroup. Then, for all $a, b, c \in G$ we have the following identities.*

$$\operatorname{gyr}[a, b + c] \operatorname{gyr}[b, c] = \operatorname{gyr}[a + b, \operatorname{gyr}[a, b]c] \operatorname{gyr}[a, b] \tag{32}$$

(by [1, Theorem 2.15]).

$$\operatorname{gyr}[\operatorname{gyr}[a, b]a, \operatorname{gyr}[a, b]b] = \operatorname{gyr}[a, b] \tag{33}$$

[1, p. 46].

$$\operatorname{gyr}[a, b] = \operatorname{gyr}[-a, -b], \tag{34}$$

$$\operatorname{gyr}^{-1}[a, b] = \operatorname{gyr}[b, a], \tag{35}$$

$$\operatorname{gyr}[a, b] = \operatorname{gyr}[b, -\operatorname{gyr}[b, a]a], \tag{36}$$

(by [1, Theorem 2.30]).

$$\operatorname{gyr}[-a + b, -(-a + c)] = \operatorname{gyr}[a, -b] \operatorname{gyr}[b, -c] \operatorname{gyr}[c, -a] \tag{37}$$

(by [1, Theorem 2.41]). The two equation

$$\begin{aligned} y &= -\operatorname{gyr}[a, x]x, \\ x &= -\operatorname{gyr}[a, y]y, \end{aligned} \tag{38}$$

in a gyrogroup $(G,+)$ are equivalent for any $a, x, y \in G$ (by Theorem 2.41 in [1]), so that each of them turns out to be the solution of the other one for the unknown x or y.

$$\mathrm{gyr}\,[a, b+c]\mathrm{gyr}\,[b, c] = \mathrm{gyr}\,[a, b]\mathrm{gyr}\,[-(-b-a), c] \tag{39}$$

so that if the gyrogroup $(G,+)$ is gyrocommutative then

$$\mathrm{gyr}\,[a, b+c]\mathrm{gyr}\,[b, c] = \mathrm{gyr}\,[a, b]\mathrm{gyr}\,[b+a, c] \tag{40}$$

(by [1, Theorem 2.46]).

Furthermore, each of the two composite gyrations J_1 and J_2,

$$J_1 = \mathrm{gyr}\,[a, x]\mathrm{gyr}\,[-(x+a), x+b]\mathrm{gyr}\,[x, b], \tag{41}$$

$$J_2 = \mathrm{gyr}\,[a, x]\mathrm{gyr}\,[-\mathrm{gyr}\,[x, a](a-b), x+b]\mathrm{gyr}\,[x, b], \tag{42}$$

is independent of $x \in G$ (by [1, Theorems 2.47 and 2.48]).

One can readily check the validity of the identities in Theorem 11 for the special case when the abstract gyrogroup $(G,+)$ is realized by the Möbius disc gyrogroup $(\mathbb{D}, \oplus_{\mathrm{M}})$, presented in Section 2. These identities follow from the gyrogroup axioms in Definition 2, as shown in [1].

Identity (40) in Theorem 11 is verified in [1] only for gyrocommutative gyrogroups. It follows from (39) owing to the *automorphic inverse property* ((G9) in Theorem 8)

$$-(-b-a) = b+a \tag{43}$$

in a gyrogroup $(G,+)$, which is valid for all $a, b \in G$ if and only if the gyrogroup G is gyrocommutative [1, Theorem 2.39].

However, observations in various nongyrocommutative gyrogroups suggest the conjecture that (40) remains valid in nongyrocommutative gyrogroups as well. Indeed, the main goal of this article is to establish a general theorem, Theorem 19, from which the validity of (40) is extended to nongyrocommutative gyrogroups as well.

It thus seems that gyrogroup expressions that form the arguments of gyrations in nongyrocommutative gyrogroups are "protected" from violations of the gyrocommutative law. Specifically for the present special observation, the expression $-(-b-a)$ that forms an argument of a gyration in (39) is "protected" from violations of the gyrocommutative law in the sense that it can undergo a gyrocommutative manipulation into $(b+a)$ in the gyration argument even in a nongyrocommutative gyrogroup, where the gyrocommutative law is violated, and hence, where (43) is invalid. This "protection" from violations of the gyrocommutative law in nongyrocommutative gyrogroups became known as the *gyrocommutative protection principle* in the author's book *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces* [1].

7. NONGYROCOMMUTATIVE GYROGROUPS

Gyrogroups, both gyrocommutative and nongyrocommutative, abound in group theory, as shown in [10]. In particular, the Möbius disc-gyrogroup $(\mathbb{D}, \oplus_{\mathrm{M}})$ that we studied in Section 2 turns out to be gyrocommutative. A table of a finite nongyrocommutative gyrogroup of order 16 is presented in [1,5]. An interesting example of nongyrocommutative gyrogroups is provided by Lorentz boosts with complex time. A Lorentz boost in the special theory of relativity is a Lorentz transformation without rotation. The set of all Lorentz boosts does not form a subgroup of the Lorentz group. Rather, it forms a gyrocommutative gyrogroup that sits inside the Lorentz group as a subset [33]. The extension of this gyrocommutative gyrogroup of boosts from real time to complex time gives a nongyrocommutative gyrogroup [30].

An interesting example of a nongroup, nongyrocommutative matrix gyrogroup, studied in [10], is provided by the set of all 4×4 real or complex upper triangular matrices with diagonal elements 1,

$$M(x) = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{44}$$

with gyrogroup operation $\odot$ given by the equation

$$M(x) \odot M(y) = (M(x))^2 M(y)(M(x))^{-1}. \tag{45}$$

Nongyrocommutative gyrogroups are also studied in [14]. It has been observed in several examples that gyrogroup expressions that form the arguments of gyrations of nongyrocommutative gyrogroups exhibit the behavior of their counterparts in gyrocommutative gyrogroups, as explained in the paragraph containing (31). It is this observation that we will formalize and verify in Theorem 19.

8. GYROCOMMUTATIVE-EQUALITY BETWEEN GYROGROUP EXPRESSIONS

DEFINITION 12. *Let $(G, \odot)$ be an abstract gyrogroup. A gyrogroup expression E (in n gyrogroup variables), $g = E(g_1, \dots, g_n)$, is a rule that assigns a single element $g \in G$ to any n elements $g_1, \dots, g_n \in G$ in terms of gyrogroup compositions and inversions.*

(1) *Two gyrogroup expressions E_1 and E_2 are equal, $E_1 = E_2$, if their values are identically equal, $E_1(g_1, \dots, g_n) = E_2(g_1, \dots, g_n)$, for all $g_1, \dots, g_n \in G$.*
(2) *Two gyrogroup expressions E_1 and E_2 are gyrocommutative-equal, $E_1 \sim E_2$, if they are equal under the assumption that the abstract gyrogroup $(G, \odot)$ is gyrocommutative.*

Clearly, two equal gyrogroup expressions are gyrocommutative-equal. Thus, $E_1 = E_2 \Rightarrow E_1 \sim E_2$. Several examples illustrating Definition 12 follow.

EXAMPLE 13. Gyration effects, $\mathrm{gyr}\,[a, b]z$, in a gyrogroup G, $a, b, z \in G$, are gyrogroup expressions since they can be described in terms of gyrogroup compositions and inversions, as evidenced from the gyrogroup gyration-operation identity (18).

EXAMPLE 14. The two gyrogroup expressions E_1 and E_2,

$$E_1(a, b, c, z) = \mathrm{gyr}\,[a, b \odot c]\mathrm{gyr}\,[b, c]z, \tag{46}$$
$$E_2(a, b, c, z) = \mathrm{gyr}\,[a \odot b, \mathrm{gyr}\,[a, b]c]\mathrm{gyr}\,[a, b]z, \tag{47}$$

are equal since, by (32), $E_1(a, b, c, z) = E_2(a, b, c, z)$ in any gyrogroup $(G, \odot)$ for all $a, b, c, z \in G$. Clearly, E_1 and E_2 are, in particular, gyrocommutative-equal.

EXAMPLE 15. The two gyrogroup expressions

$$\begin{aligned} E_1(a, b) &= a + b, \\ E_2(a, b) &= \mathrm{gyr}\,[a, b](b + a), \end{aligned} \tag{48}$$

are gyrocommutative-equal since $E_1(a, b) = E_2(a, b)$ in any gyrocommutative gyrogroup $(G, +)$. They are, however, not equal since, in general, $E_1(a, b) \neq E_2(a, b)$ in a gyrogroup $(G, +)$. Thus, in any gyrogroup $a + b \sim \mathrm{gyr}\,[a, b](b + a)$ while, in general, $a + b \neq \mathrm{gyr}\,[a, b](b + a)$.

EXAMPLE 16. The two gyrogroup expressions

$$\begin{aligned} E_1(a, b) &= a + b, \\ E_2(a, b) &= -(-a - b), \end{aligned} \tag{49}$$

in a gyrogroup $(G, +)$ are not equal, but are gyrocommutative-equal according to Theorem 8. Thus, in any gyrogroup $a + b \sim -(-a - b)$ but, in general, $a + b \neq -(-a - b)$.

EXAMPLE 17. The two gyrogroup expressions

$$\begin{aligned} E_1(a, b) &= a \boxdot b, \\ E_2(a, b) &= b \boxdot a, \end{aligned} \tag{50}$$

in any gyrogroup $(G, \odot)$ with cooperation $\boxdot$ are gyrocommutative-equal according to Theorem 9. Thus, $a \boxdot b \sim b \boxdot a$, but $E_1 \neq E_2$ since, in general, $a \boxdot b \neq b \boxdot a$.

9. THE GYROCOMMUTATIVE PROTECTION PRINCIPLE

DEFINITION 18. GYRATION NESTED EXPRESSIONS. *A gyrogroup expression E in a gyrogroup G is nested in a gyration of G if it forms one of the two arguments of the gyration.*

We are now in a position to state and prove the main result of this article, which is the *gyrocommutative protection principle*. This principle has already been in use in the gyrogroup theory book [1], but with no proof (for being out of the scope of the book).

THEOREM 19. THE GYROCOMMUTATIVE PROTECTION PRINCIPLE. *Let E_k and E'_k be two gyrocommutative-equal expressions in a gyrogroup G, $k = 1, 2$,*

$$\begin{aligned} E_1 &\sim E'_1, \\ E_2 &\sim E'_2. \end{aligned} \tag{51}$$

Then, in G,

$$\operatorname{gyr}[E_1, E_2] = \operatorname{gyr}[E'_1, E'_2]. \tag{52}$$

The proof of Theorem 19 is given below, following the presentation of some relevant results.

Identity (52) of Theorem 19 asserts that a gyrogroup expression nested in a gyration of a gyrogroup G is protected from violations of the gyrocommutative law that take place in G if G is noncommutative. Thus, for instance, the gyrocommutative protection principle, Theorem 19, implies that identities (31) and (40), which are valid in any gyrocommutative gyrogroup, are valid in any gyrogroup (that need not be gyrocommutative) as well.

EXAMPLE 20. As an example illustrating the use of Theorem 19 we establish the validity of the identity

$$\operatorname{gyr}[a, b \oplus (c \boxplus d)] = \operatorname{gyr}[a, b \oplus (d \boxplus c)] \tag{53}$$

in any gyrogroup $(G, \oplus)$ with gyrogroup operation $\oplus$ and gyrogroup cooperation $\boxplus$.

Let

$$\begin{aligned} E_1 &= b \oplus (c \boxplus d), \\ E_2 &= b \oplus (d \boxplus c). \end{aligned} \tag{54}$$

Clearly, the gyrogroup expressions E_1 and E_2 are not equal, $E_1 \neq E_2$ since, their values are equal for all $b, c, d \in G$ if and only if the gyrogroup $(G, \oplus)$ is gyrocommutative. However, for that reason E_1 and E_2 are gyrocommutative-equal, $E_1 \sim E_2$. Hence, by Theorem 19, $\operatorname{gyr}[a, E_1] = \operatorname{gyr}[a, E_2]$ in G, as desired.

EXAMPLE 21. As another example illustrating the use of Theorem 19 we show that

$$\begin{aligned} \operatorname{gyr}[[u, v], c] &= id, \\ \operatorname{gyr}[\operatorname{gyr}[a, b][u, v], c] &= id, \end{aligned} \tag{55}$$

for any $a, b, c, u, v \in G$, where id is the identity automorphism of a gyrogroup $(G, \odot)$ and where $[u, v]$ is the commutator

$$[u, v] = (u \odot v) \odot \left(u^{-1} \odot v^{-1}\right). \tag{56}$$

The commutator $[u, v]$ satisfies $[u, v] = 1_G$ for all $u, v \in G$ if and only if G is gyrocommutative, 1_G being the identity element of G. Hence, in a gyrocommutative gyrogroup $(G, \odot)$ we have

$$\operatorname{gyr}[[u, v], c] = \operatorname{gyr}[\operatorname{gyr}[a, b][u, v], c] = \operatorname{gyr}[1_G, c] = id, \tag{57}$$

for all $a, b, c, u, v \in G$.

Finally, it follows from the gyrocommutative protection principle that (57) remains valid in any gyrogroup, regardless of whether it is gyrocommutative on nongyrocommutative.

To verify Theorem 19 we need the following two definitions, a lemma, and a theorem from [9], which are presented below.

DEFINITION 22. *(See [9, Definition 4.7].) A nonempty subset X of a gyrogroup $(P, \odot)$ is a subgroup (of a gyrogroup) if it is a group under the restriction of $\odot$ to X.*

DEFINITION 23. *(See [9, Definition 4.8].) A subgroup X of a gyrogroup P is normal in P if*

(1) $\operatorname{gyr}[a, x] = id$ *for all $x \in X$ and $a \in P$, id being the identity map.*
(2) $\operatorname{gyr}[a, b]X \subseteq X$ *for all $a, b \in P$.*
(3) $a \odot X = X \odot a$ *for all $a \in P$.*

Note that Conditions (2) and (3) in Definition 23 assert that for any $x \in X$ there exist $x_1, x_2 \in X$ such that $\operatorname{gyr}[a, b]x = x_1$ and $a \odot x = x_2 \odot a$.

LEMMA 24. *(See [9, Lemma 4.9].) If X is a normal subgroup of a gyrogroup P, then P/X forms a factor gyrogroup.*

THEOREM 25. *(See [9, Theorem 4.11].) Any gyrogroup $(P, \odot)$ has a normal subgroup Θ such that P/Θ is a gyrocommutative gyrogroup. (Note that Θ is a group, with group operation given by the restriction of $\odot$ to Θ.)*

The proof of Theorem 25, presented in [9], is based on an important result recently discovered by Aschbacher [34]. Having Theorem 25 in hand, we can now verify Theorem 19 on the gyrocommutative protection principle.

PROOF OF THEOREM 19. Let $(G, \odot)$ be a gyrogroup. Following Theorem 5, the left gyrogroup representation theorem, let (Γ, H) be a reduced enveloping pair that generates the gyrogroup $(G, \odot)$, so that

$$\Gamma = GH. \tag{58}$$

By Theorem 25, there exists a normal subgroup Θ of the gyrogroup G such that the factor gyrogroup G/Θ is gyrocommutative. Gyrogroup expressions in G induce gyrogroup expressions in G/Θ since, by Definition 12, a gyrogroup expression is totally determined by gyrogroup compositions and inversions. Thus, if E is a gyrogroup expression in the gyrogroup G then $E\Theta$ (a coset of Θ) is a gyrogroup expression in the gyrocommutative factor gyrogroup G/Θ. Moreover, two gyrocommutative-equal expressions E and F, $E \sim F$, in G induce two gyrocommutative-equal expressions $E\Theta$ and $F\Theta$, $E\Theta \sim F\Theta$, in G/Θ. But the gyrogroup G/Θ is gyrocommutative so that $E\Theta \sim F\Theta \Rightarrow E\Theta = F\Theta$. Hence, in particular, it follows from (51) that

$$\begin{aligned} E_1\Theta &= E_1'\Theta, \\ E_2\Theta &= E_2'\Theta, \end{aligned} \tag{59}$$

so that

$$\begin{aligned} E_1 &= E_1'\theta_1, \\ E_2 &= E_2'\theta_2, \end{aligned} \tag{60}$$

for some $\theta_1, \theta_2 \in \Theta$.

Having values in G, the expressions E_1 and E_2 in G have the composition E_1E_2 in Γ, which possesses the gyrodecomposition (58), resulting in the identity

$$E_1E_2 = (E_1 \odot E_2)h(E_1, E_2) \tag{61}$$

according to (13). We also have from (60)

$$\begin{aligned} E_1E_2 &= E_1'\theta_1 E_2'\theta_2 \\ &= \theta E_1' E_2', \end{aligned} \tag{62}$$

for some $\theta \in \Theta$. Hence, the gyrodecomposition in (11) and in (58) applied to (62) gives

$$\begin{aligned} E_1E_2 &= \theta E_1'E_2' \\ &= \theta \,(E_1' \odot E_2')\, h\,(E_1', E_2') \\ &= \{\theta \odot (E_1' \odot E_2')\}\, h\,(\theta, E_1' \odot E_2')\, h\,(E_1', E_2') \\ &= \{\theta \odot (E_1' \odot E_2')\}\, h\,(E_1', E_2'), \end{aligned} \tag{63}$$

since, by condition (1) in Definition 23, $\operatorname{gyr}[\theta, g] = id$ so that $h(\theta, g) = 1_H$ is the identity element of H for all $\theta \in \Theta$ and $g \in G$ in the decomposition (58).

Comparing the two unique decompositions of E_1E_2 in (61) and (63) we have

$$h(E_1, E_2) = h\,(E_1', E_2'). \tag{64}$$

Finally, by means of the correspondence (15) between gyrations $\operatorname{gyr}[g_1, g_2] \in \operatorname{Aut}(G, \odot)$ and elements $h \in H$ in the reduced enveloping pair (Γ, H) that generates the gyrogroup $(G, \odot)$, Identity (64) is equivalent to the desired Identity (52). The proof of Theorem 19 is thus complete, as well as the task we faced in this article.

Readers who want to know more about gyrogroup theory and, subsequently, about gyrovector space theory will find it rewarding to read the author's book *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces* [1]. Furthermore, readers who want to know more on a higher abstract level will find it rewarding to read Sabinin's book [35] on smooth quasigroups and loops in nonassociative algebra and differential geometry.

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