
On the table and the chair

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ABSTRACT

We find topological models for the tiling dynamical systems corresponding to the chair and table rep-tiles.

1. INTRODUCTION

A *rep-tile* is a polygon that can be tiled by a finite number of smaller, congruent copies of itself. Two well known examples are shown in Figure 1.

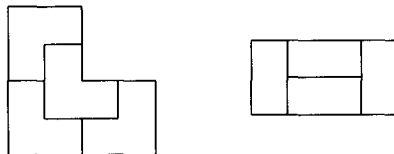


Figure 1. The *chair* and *table* rep-tiles.

We call these the *chair* and the *table*. Both of these rep-tiles are also *polyominoes* (cf. [2]), meaning they are edge to edge unions of squares.

Given a rep-tile τ , there is a corresponding set X of *self-similar* tilings of the plane. To get this, we decompose τ into its small copies, obtaining a τ -shaped *patch*. Then we expand the small tiles in this patch back to their original size. Iterating this process, we obtain a sequence $x_1, x_2, x_3 \dots$ of larger and larger patches.

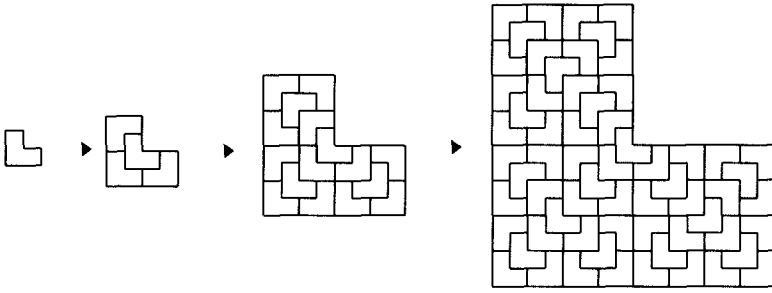


Figure 2. The patches x_1, x_2, x_3, x_4 of chairs.

We define X to be the set of all tilings x of \mathbb{R}^2 such that, up to a translation, any patch in x is also a patch in x_n for some n .

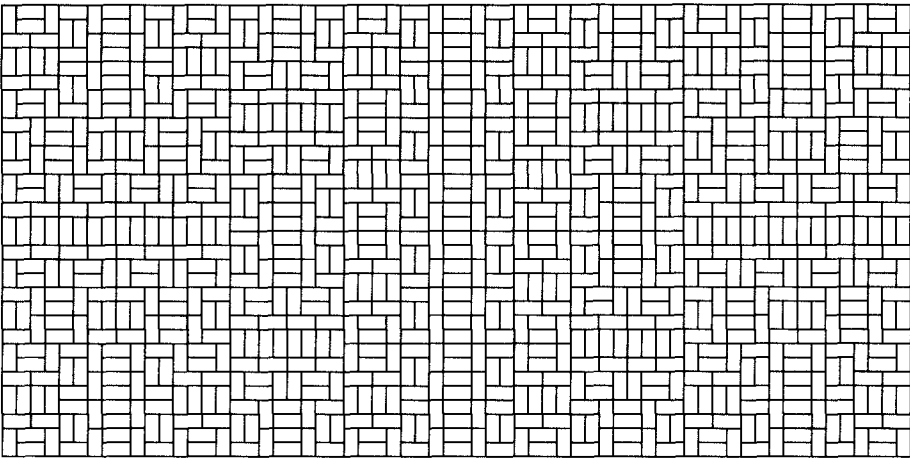


Figure 3. Part of a table tiling.

We denote the sets of self-similar *chair tilings* and *table tilings* by X_c and X_t respectively.

Sets X of tilings of \mathbb{R}^2 can be equipped with a natural compact metric topology in which the translation action T of \mathbb{R}^2 on X is continuous. One can study this action, called a *tiling dynamical system*, using the methods of ergodic theory and topological dynamics (cf. e.g. [11], [10], [14], [9]). For the table and chair tilings, these tiling dynamical systems were first studied by Solomyak [14]. He showed that the chair tiling dynamical system is strictly ergodic, and has *pure discrete spectrum*, with ‘eigenvalues’ equal to the dyadic rational points in \mathbb{R}^2 . By the Halmos von Neumann Theorem (cf. [16]), this shows that the chair tiling dynamical system is metrically isomorphic to a *Kronecker system* (an action by rotations on a compact abelian group). Solomyak [14] showed that the table tiling system has the same eigenvalues as the chair tiling system, but that it also has a continuous component in its spectrum. Hence the table is not metrically isomorphic to the chair.

For the well known *Penrose tilings* a more detailed result is known. Like the chair, the Penrose tiling dynamical system has pure discrete spectrum [10]. It is metrically isomorphic to a Kronecker system with \mathbb{R}^2 acting on \mathbb{T}^4 , but this metric isomorphism is not a topological conjugacy. Instead, it is realized by a continuous almost 1 to 1 factor mapping [10]. We say \mathbb{T}^4 is a *topological model* for the space of Penrose tilings, and that the Penrose tiling dynamical system is *almost topologically conjugate* to a Kronecker system on \mathbb{T}^4 . It turns out that the points where the factor mapping fails to be 1 to 1 are precisely the ‘exceptional’ Penrose tilings, namely those with infinite *worms* and the *cartwheel Penrose tiling* (cf. [10], [4]). Thus the exceptional Penrose tilings have dynamical significance.

The purpose of this paper is strengthen Solomyak’s results by obtaining topological models for the table and the chair tiling dynamical systems. For the chair, our result is directly analogous to the Penrose case. We show that the chair tiling system is an almost 1 to 1 extension of a Kronecker system. For the table, we show that the corresponding dynamical system is an almost 4 to 1 extension of the same Kronecker system. Then it is a relatively easy corollary that the table system has continuous component in its spectrum. As in the Penrose case, the points where the factor mappings fail to be 1 to 1 (or 4 to 1) correspond to interesting tilings. In particular, we display structures in the chair tilings that are analogous to the worm and cartwheel Penrose tilings.

While our results for tables and chairs resemble the results in the Penrose case, the proofs for these tilings are completely different. In particular, we use simple ideas from symbolic dynamics. These methods are quite general and apply to all polyomino rep-tiles.

The author wishes to thank Mike Boyle, Natalie Priebe, and Boris Solomyak for helpful conversations.

2. DYNAMICS BACKGROUND

For a locally compact abelian group (or semigroup) \mathbb{G} , a \mathbb{G} -*dynamical system* (X, \mathbb{G}, T) is a continuous action T of \mathbb{G} on a compact metric space X . For simplicity, we sometimes denote this dynamical system by X .

A Borel set $E \subseteq X$ is *invariant* if $T^{\mathbf{g}}E = E$ for all $\mathbf{g} \in \mathbb{G}$. The dynamical system (X, \mathbb{G}, T) is *minimal* if there are no proper closed invariant sets. A point $x \in X$ is called *almost periodic* if for any $U \subseteq X$ open, $\{\mathbf{g} : T^{\mathbf{g}}x \in U\} \subseteq \mathbb{G}$ is relatively dense (cf. [16]). By a well known theorem of Gottschalk [3], X is minimal if and only if every point is almost periodic and some point has a dense orbit.

An *invariant measure* for (X, \mathbb{G}, T) is a Borel probability measure μ on X such that $\mu(T^{-\mathbf{g}}E) = \mu(E)$ for all Borel $E \subseteq X$ and all $\mathbf{g} \in \mathbb{G}$. If the invariant measure is unique, then the dynamical system is called *uniquely ergodic* (cf. [16]). Minimality together with unique ergodicity is called *strict ergodicity*.

Given (X, \mathbb{G}, T) and (Y, \mathbb{G}, S) , suppose $\varphi : X \rightarrow Y$ is a continuous surjection such that $\varphi(T^{\mathbf{g}}x) = S^{\mathbf{g}}(\varphi(x))$. We say X is an *extension* of Y via the *factor mapping* φ . If φ is also 1 to 1 then it is called a *topological conjugacy*. If X and Y

are strictly ergodic and there exists $y \in Y$ so that $|\varphi^{-1}(y)| = 1$, then $|\varphi^{-1}(y)| = 1$ for all y in some invariant dense G_δ set $Y_0 \subseteq Y$, and we say φ is an *almost 1 to 1* extension. If $|\varphi^{-1}(y)| = K$ for all y in a dense G_δ set Y_0 , we say φ is *almost K to 1*. If $|\varphi^{-1}(y)| \leq L < \infty$ we say φ is *uniformly bounded to 1*. If φ is almost 1 to 1 and $\nu(Y_0) = 1$, where ν is the unique invariant measure for Y , we say X and Y are *almost topologically conjugate*. This implies *metric isomorphism* (cf. [16]).

Now let $\mathbb{G} = \mathbb{R}^2$ or \mathbb{Z}^2 , and let (X, \mathbb{G}, T) be strictly ergodic with unique invariant measure μ . The *dual group* $\widehat{\mathbb{G}}$ of \mathbb{G} is \mathbb{R}^2 if $\mathbb{G} = \mathbb{R}^2$ and is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ if $\mathbb{G} = \mathbb{Z}^2$. A complex Borel function f on X is called an *eigenfunction* for the ‘eigenvalue’ $\mathbf{w} \in \widehat{\mathbb{G}}$ if for μ a.e. $x \in X$,

$$(2.1) \quad f(T^{\mathbf{g}}x) = e^{2\pi i \langle \mathbf{g}, \mathbf{w} \rangle} f(x),$$

where $\langle \mathbf{g}, \mathbf{w} \rangle$ denotes the ‘dot’ product. The set Σ of eigenvalues is always a countable discrete subgroup of $\widehat{\mathbb{G}}$ ([16]). The zero element $0 \in \Sigma$ corresponds to the constant functions, and all eigenvalues are simple (cf. [16]). If $\Sigma \setminus \{0\} \neq \emptyset$ then X is said to have *nontrivial discrete spectrum*. If the eigenfunctions have dense span in $L^2(X, \mu)$, then X is said to have *pure discrete spectrum*. Otherwise, X is said to have a *continuous component in its spectrum*. The extreme case, $\Sigma = \{0\}$, called *weak mixing*, does not occur for the examples considered in this paper.

A *Kronecker system* (X, \mathbb{G}, T) is a strictly ergodic action T of \mathbb{G} by rotations on a compact abelian group X . Kronecker systems always have pure discrete spectrum, and for any countable subgroup $\Sigma \subseteq \widehat{\mathbb{G}}$, there is a unique Kronecker system with eigenvalue group Σ (cf. [16]). By the *Halmos–von Neuman theorem*, a \mathbb{G} dynamical system X with pure discrete spectrum, having eigenvalue group Σ , is metrically isomorphic to this canonical Kronecker system (cf. [16]).

For a \mathbb{Z}^2 dynamical system (X', \mathbb{Z}^2, S) we construct an \mathbb{R}^2 dynamical system (X, \mathbb{R}^2, T) , called the *suspension*, as follows. Let $X = X' \times [0, 1)^2$. Given $\mathbf{t} + \mathbf{r} \in \mathbb{R}^2$, write $\mathbf{t} + \mathbf{r} = \mathbf{n} + \mathbf{s}$ uniquely where $\mathbf{n} \in \mathbb{Z}^2$ and $\mathbf{s} \in [0, 1)^2$. We define $T^{\mathbf{t}}(x, \mathbf{r}) = (S^{\mathbf{n}}x, \mathbf{s})$. Most of the properties that we are interested in here (e.g. strict ergodicity, pure discrete spectrum, having a continuous component in the spectrum, being an almost 1 to 1 extension, etc.) are preserved by this construction. In particular, a \mathbb{Z}^2 system is Kronecker if and only if its suspension is an \mathbb{R}^2 Kronecker system. The suspension of a \mathbb{Z}^2 dynamical system X' , with eigenvalue group $\Sigma \subseteq \mathbb{T}^2$, is an \mathbb{R}^2 system with eigenvalue group $\widetilde{\Sigma} \subseteq \mathbb{R}^2$, where $\widetilde{\Sigma}$ is the lift of Σ to \mathbb{R}^2 . We write $\Sigma = e^{2\pi i \widetilde{\Sigma}}$.

3. TILING DYNAMICAL SYSTEMS

A *tile* (in this paper) is a connected polygon in \mathbb{R}^2 . We say two tiles are *equivalent* if they differ by a translation. A finite collection π of inequivalent tiles is called a *prototile set*. There are four chair prototiles (four rotations of the chair) and two table prototiles (two rotations of the table).

Let X be the set of all tilings of \mathbb{R}^2 by tiles equivalent to tiles in π . We assume $X \neq \emptyset$. A *patch* is a finite set of tiles with a connected union. Two patches are

equivalent if they differ by a translation. We always assume X satisfies the *finite local pictures* condition: for all tilings $x \in X$, and for all $k \in \mathbb{N}$, there are finitely many equivalence classes of k tile patches. We call X a *full π -shift*. For tilings by chair or table prototiles, we will satisfy the finite local pictures condition by requiring that the squares in the prototiles fit ‘edge to edge’. Let $T^t x$ denote the translation of $x \in X$ by $t \in \mathbb{R}^2$. Note that the full π -shift X is T -invariant.

The *tiling topology* on a full π -shift X is defined as follows. Two tilings $x, y \in X$ are “ ϵ -close” if for some $t \in \mathbb{R}^2$ with $\|t\| < \epsilon$, $T^t x$ and y agree in a $\frac{1}{\epsilon} \times \frac{1}{\epsilon}$ square around the origin. A complete metric d is defined by putting $d(x, y)$ equal to the infimum of all such ϵ (cf. [12]). Note that the finite local pictures condition is equivalent to X being totally bounded. It follows that X is compact (cf. [12]). Clearly the action T of \mathbb{R}^2 by translation is continuous. A *tiling dynamical system* is a triple (X_0, \mathbb{R}^2, T) where $X_0 \subseteq X$ is closed and T -invariant. We call X_0 a *tiling space*.

4. REP-TILE SETS

The sets X_c and X_t of chair and table tilings are closed and T -invariant; they define the *chair* and *table* tiling dynamical systems. Let us now describe their construction a bit more precisely.

Let π be a set of prototiles in \mathbb{R}^2 and let $\lambda > 1$. Suppose for each tile $\tau \in \pi$, $\sigma(\tau)$ is a tiling of $\lambda\tau$ by translates of the tiles in π . We call (π, σ) a *rep-tile set*. A single rep-tile τ defines a rep-tile set provided the group of rotations generated by the rotations of the copies τ in $\sigma(\tau)$ is finite¹. We define the *substitution matrix* M to be the $|\pi| \times |\pi|$ matrix with entries $m_{\tau, \nu}$ equal to the number of copies of $\nu \in \pi$ appearing in $\sigma(\tau)$. We assume M is *primitive*, i.e., $M^k > 0$ for some $k \geq 1$. We define the *inflation mapping* $\sigma : X \rightarrow X$ on the full π -shift as follows: First we expand x linearly by λ , fixing the location of $\mathbf{0} \in \mathbb{R}^2$, and then we subdivide each $\tau \in x \in X$ according to the rep-tile relation σ .

Our next goal is to define $X_\sigma \subseteq X$. By induction we define a sequence of patches x_n , putting $x_1 = \{\tau\}$ for some $\tau \in \pi$, and $x_{n+1} = \sigma(x_n)$. We define $X_\sigma \subseteq X$ to be the set of all tilings $x \in X$ of \mathbb{R}^2 such that any patch in x is equivalent to a patch in x_n for some n . A simple compactness argument (cf. [13]) shows $X_\sigma \neq \emptyset$. Since $M^k > 0$, X_σ is independent of the initial tile τ . Clearly X_σ is closed and T -invariant, so $(X_\sigma, T, \mathbb{R}^2)$ is a tiling dynamical system.

It is easy to see that any $x \in X_\sigma$ is an almost periodic point. Geometrically, this means that any patch that occurs in x occurs again within a bounded distance from an arbitrary point in x . We call such a tiling an *almost periodic tiling* (cf. [11], [10], [14]). Up to equivalence, any two tilings in X_σ have the same patches, so $(X_\sigma, T, \mathbb{R}^2)$ is always minimal. One can also show, using an argument based on the Perron-Frobenius Theorem, that $(X_\sigma, T, \mathbb{R}^2)$ is uniquely ergodic (cf. [14]). We denote the unique invariant measure by μ . In general, we want to avoid the possibility that $x \in X_\sigma$ is *periodic*, in which case T is a tran-

¹ The *pinwheel tiling* [8] is an example without this property.

sitive action of \mathbb{R}^2 on $X_\sigma = \mathbb{T}^2$, (cf. [11]). Aperiodicity is equivalent to the bijectivity of the inflation mapping $\sigma : X_\sigma \rightarrow X_\sigma$ [15]. In specific examples, *ad hoc* arguments are needed establish this. However, it is easy to see that invertibility holds for σ in the case of the table and chair rep-tiles. Hence all table and chair tilings are aperiodic.

5. DISCRETIZATION

Let X be a full π -shift for a set π of polyomino prototiles. Consider the decomposition of the tiles in π into their underlying squares. Let ℓ be a map that assigns a label from a finite alphabet \mathcal{A} to each such square. If ℓ is bijective we say the labeling is *maximal*. For any $x \in X$, there is $\mathbf{t} \in [0, 1]^2$ such that the squares in x induce the partition $\{([0, 1]^2 + \mathbf{t} + \mathbf{v}) : \mathbf{v} \in \mathbb{Z}^2\}$ of \mathbb{R}^2 . We assign each of these squares a label $\ell([0, 1]^2 + \mathbf{t} + \mathbf{v})$ according to how it is tiled in x by π . We define $\delta : X \rightarrow \mathcal{A}^{\mathbb{Z}^2}$ by $\delta(x)_\mathbf{v} = \ell([0, 1]^2 + \mathbf{t} + \mathbf{v})$. Letting S denote the \mathbb{Z}^2 shift on $\mathcal{A}^{\mathbb{Z}^2}$, we have that $X' = \delta(X) \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is S -invariant, so (X', \mathbb{Z}^2, S) is a \mathbb{Z}^2 dynamical system. We call ℓ *faithful* if (X, \mathbb{R}^2, T) is the suspension of (X', \mathbb{Z}^2, S) . Any maximal labeling is faithful, but a faithful labeling need not be maximal, as we will see below. Figure 4 shows a maximal labeling for the tables, with $\mathcal{A} = \{p, q, r, s\}$.

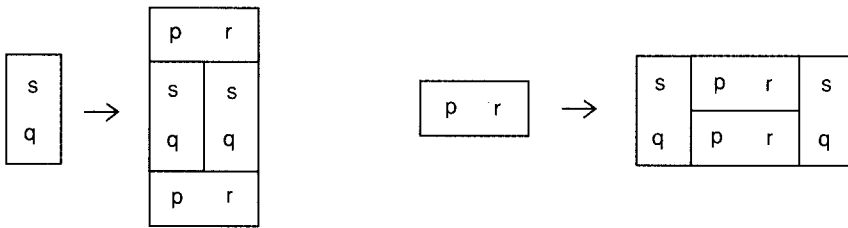


Figure 4. A maximal labeling for the tables and the derivation of σ_t .

Given a rep-tile set (σ, π) and a faithful labeling ℓ , there is an alternative way to construct $X'_\sigma = \delta(X)$: as a \mathbb{Z}^2 -substitution space. A 2×2 substitution is a mapping $\sigma : \mathcal{A} \rightarrow \mathcal{A}^{\{0,1\}^2}$, where \mathcal{A} is a finite alphabet. Figure 4 shows how to obtain the substitution σ_t associated with the table rep-tile. It is given by

$$(5.1) \quad p \mapsto \begin{matrix} s & p \\ q & p \end{matrix} \quad q \mapsto \begin{matrix} q & q \\ p & r \end{matrix} \quad r \mapsto \begin{matrix} r & s \\ r & q \end{matrix} \quad s \mapsto \begin{matrix} p & r \\ s & s \end{matrix}.$$

Now we describe how to construct a substitution space $X'_\sigma \subseteq \mathcal{A}^{\mathbb{Z}^2}$ from a substitution σ . A mapping $x : B \rightarrow \mathcal{A}$, where $B \subseteq \mathbb{Z}^2$ is finite, is called a *block*, and B is called the *locus* of the block. Blocks that are translates of each other are called *equivalent*. We define $X'_\sigma \subseteq X$ to be the set of all points x so that each block in x is equivalent to a block in $\sigma^n(a)$ for some $n \in \mathbb{N}$ and $a \in \mathcal{A}$. This is just the discrete version of the construction of X_σ . When σ comes from a rep-tile set (σ, π) , we have $X'_\sigma = \delta(X_\sigma)$. In particular, for the table we obtain $X'_t = \delta(X_t) = X'_{\sigma_t}$. We call (X'_t, \mathbb{Z}^2, S) the *table substitution system*.

For chair tilings the maximal labeling ℓ results in an inconvenient 12 letter alphabet. To avoid this we use the ‘arrow’ labels shown in Figure 5.

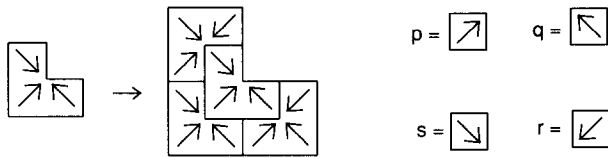


Figure 5. The arrow labels, their replication, and the letters that replace them.

This labeling is faithful since a vertex point in a chair tiling is the ‘convex’ vertex of a chair tile if and only if it has three arrows pointing directly at it. Any arrow participates in exactly one such triple. We call this the *three arrow rule*. After replacing arrows with letters, as shown in Figure 5, we obtain the chair substitution σ_c

$$(5.2) \quad p \mapsto \begin{matrix} s & p \\ p & q \end{matrix} \quad q \mapsto \begin{matrix} q & r \\ p & q \end{matrix} \quad r \mapsto \begin{matrix} s & r \\ r & q \end{matrix} \quad s \mapsto \begin{matrix} s & r \\ p & s \end{matrix}.$$

Note the similarity between (5.2) and (5.1); only the positions of the letters are different. As with the table, we obtain $X'_c = \delta(X_c) = X'_{\sigma_c}$. We call (X'_c, \mathbb{Z}^2, S) the *chair substitution system*.

6. GRAPHS OF A SUBSTITUTION

Given a \mathbb{Z}^2 substitution σ over \mathcal{A} , we define two associated directed graphs and the corresponding 1-dimensional shifts of finite type. The *forward substitution graph* G_σ^+ is defined as follows: The vertex set is \mathcal{A} , and a directed edge goes from a to each symbol in $\sigma(a)$. The edges are labeled according to the position of the target symbol in $\sigma(a)$. We label these positions $1, \dots, 4$ going from left to right, top to bottom. The graph G_σ^+ , called *table forward*, corresponds to the table substitution.

A second graph, called the *reverse substitution graph*, is obtained from the forward substitution graph by reversing the arrows.

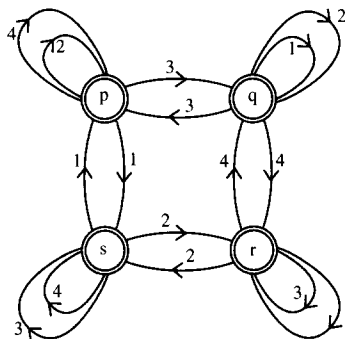


Figure 6. The *Table forward* = *table reverse* graph.

Lemma 6.1. $G_t^+ = G_t^-$, i.e. table forward = table reverse.

One sided infinite paths through the table reverse graph can be labeled in two ways: a *vertex labeling* $Y_t \subseteq \{p, q, r, s\}^{\mathbb{N}}$, or an *edge labeling* $Z \subseteq \{1, 2, 3, 4\}^{\mathbb{N}}$. Since each vertex has an out-edge with each label, $Z = \{1, 2, 3, 4\}^{\mathbb{N}}$ is the full 1-sided 4-shift. However, Y_t is a proper subshift of $\{p, q, r, s\}^{\mathbb{N}}$. It is a subshift of finite type (cf. [6], [5]) and we call it the *table subshift*. Let R denote the left shift on Z and on Y_t , and let $\psi : Y_t \rightarrow Z$ denote the factor mapping that reads the edge labels off a given vertex path.

Lemma 6.2. For any $z \in Z$ and $a \in \{p, q, r, s\}$, there exists unique $y \in Y_t$ starting at a and following the edges in z . In particular, ψ is everywhere 4 to 1.

Since each vertex has an out-edge with each label, the proof is clear.

Now we consider the *chair forward* and the *chair reverse* graphs, G_c^+ and G_c^- . These are not the same! In particular, the vertices in *chair reverse* do not have out-edges with each possible label.

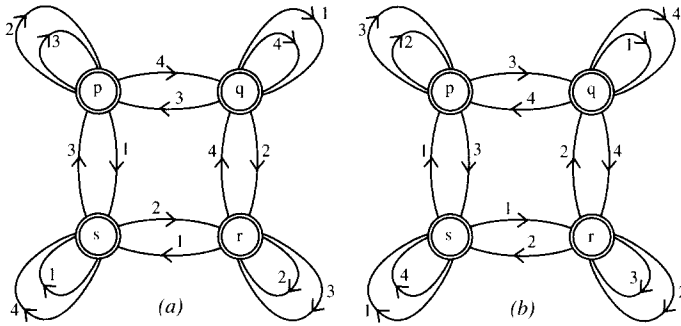


Figure 7. The *chair forward* (a) and *chair reverse* (b) graphs.

Here we let $Y_c \subseteq \{p, q, r, s\}^{\mathbb{Z}}$ denote the *chair-reverse shift* and, as above, let $\psi : Y_c \rightarrow Z$ be the edge label reading map.

We call the blocks 12, 21, 14, 41, 34, 43, 13 and 31 *good blocks*. We say $z \in Z$ is a *good point* if it has infinitely many good blocks, and we denote the set of good points by Z_1 . Any *transitive point* (i.e., a point with a dense orbit in (Z, \mathbb{N}, R)) is good, so Z_1 contains a dense G_δ set in the product topology on Z .

Proposition 6.3. The mapping $\psi : Y_c \rightarrow Z$ a factor map (i.e. it is onto), which is almost 1 to 1. In particular, $|\psi^{-1}(z)| = 1$ if and only if $z \in Z_1$. If $z \notin Z_1$ then $|\psi^{-1}(z)| \in \{1, 2\}$.

Proof. If we reverse a right-infinite path through chair reverse we get a left-infinite path through chair forward. It can be labeled either by vertices or by edges. Let z^{-1} be the reversal of a good point $z \in Z_1$. Since the good blocks are reversible, z^{-1} also has infinitely many good blocks.

Consider any good block, say 12, which occurs at some time $-j < 0$ in z^- . Note that following the path 12 in G_c^+ always leads to the vertex r . It follows that if y^- is a vertex path corresponding to the edge path z^- , then y^- is completely determined from time $-j + 1$ up through time 0. This is because each vertex in *chair forward* has exactly one out-edge with each label. All of the other good blocks have this same ‘synchronization’ property, although they lead to different vertices.

Now since $z \in Z_1$, we have a sequence $-j_i \rightarrow -\infty$ such that there is a good block at time $-j_i$ in z^- . For each i , y^- is determined from $-j_i + 1$ to 0. In the limit, this determines a unique y^- , and its reverse y satisfies $\psi(y) = z$.

Next, suppose $z \in Z$ has only finitely many good blocks. Without loss of generality, by applying R^n , we can assume z has no good blocks. Then z has only the blocks 11, 44, 14, and 41, or only the blocks 22, 33, 23, and 32 (we cannot switch without creating a good block). In the first case we have $y = .qqqqq \dots$ or $y = .sssss \dots$ and in the second $y = .ppppp \dots$ or $y = .rrrrr \dots$. \square

7. THE ALGEBRAIC MODEL: THE ADDING MACHINE

Let $\mathbb{D} = \{0, 1\}^{\mathbb{N}} = \{.d_1 d_2 d_3 \dots\}$ provided with the product topology and the operation $+$ defined as coordinate-wise addition with *right carry*. Then \mathbb{D} is a compact abelian group called the *dyadic integers*. For $b = .100000 \dots$ we define $A : \mathbb{D} \rightarrow \mathbb{D}$ by $A(d) = b + d$. Then $(\mathbb{D}, \mathbb{Z}, A)$ is a strictly ergodic Kronecker system called the *Kakutani-von Neumann adding machine*. We denote the eigenvalue group $e^{2\pi i \mathbb{Z}[\frac{1}{2}]} = \{e^{2\pi i k/2^n} : k \in \mathbb{Z}, n \in \mathbb{N}\}$. Here $\mathbb{Z}[\frac{1}{2}] = \{j/2^n : j \in \mathbb{Z}, n \in \mathbb{N}\}$ is the group of *dyadic rationals*.

We use $d \in \mathbb{D}$ to code *block structures* on \mathbb{Z} . A block structure is a collection of partitions of \mathbb{Z} into sets called *loci*. A 1-locus consists of a pair of adjacent points in \mathbb{Z} . There are two choices: if 0 is on the left side of the 1-locus containing it, then $d_1 = 0$. Otherwise we put $d_1 = 1$. Given a partition of \mathbb{Z} into $(n - 1)$ -loci, we define the n -loci to each consist of two adjacent $(n - 1)$ -loci. The n -locus containing 0 is called the *principal n -locus*. If the principal $(n - 1)$ -locus is on the left side of the principal n -locus we put $d_n = 0$, and otherwise we put $d_n = 1$. A key observation is the following.

Lemma 7.1. *The action of A on \mathbb{D} implements a left shift of the block structure.*

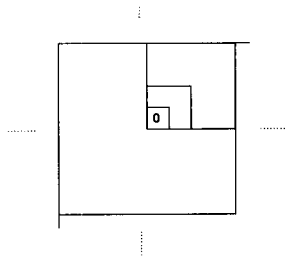


Figure 8. Part of the block structure for $z = .3321 \dots$ in the notation of (7.1).

Now we consider the product group

$$\mathbb{D}^2 = \mathbb{D} \times \mathbb{D} = \{.(0, 0)(0, 1)(1, 0) \dots\} = (\{0, 1\}^2)^{\mathbb{N}}$$

and define $A_1, A_2 : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ by

$$A_1 d = d + .(1, 0)(0, 0)(0, 0) \dots$$

$$A_2 d = d + .(0, 1)(0, 0)(0, 0) \dots$$

Then A_1 and A_2 generate a \mathbb{Z}^2 action A on \mathbb{D}^2 by $A^{\mathbf{n}} = A_1^{n_1} A_2^{n_2}$ where $\mathbf{n} = (n_1, n_2)$. We call the strictly ergodic Kronecker dynamical system $(\mathbb{D}^2, \mathbb{Z}^2, A)$ the \mathbb{Z}^2 adding machine. It has eigenvalue group

$$e^{2\pi i \mathbb{Z}[\frac{1}{2}]^2} = \{(e^{2\pi i k/2^m}, e^{2\pi i j/2^n}) : j, k \in \mathbb{Z}, m, n \in \mathbb{N}\} \subseteq \mathbb{T}^2.$$

It will be convenient to identify \mathbb{D}^2 with the full 4-shift Z via the code

$$(7.1) \quad (0, 1) \leftrightarrow 1 \quad (1, 1) \leftrightarrow 2 \quad (0, 0) \leftrightarrow 3 \quad (1, 0) \leftrightarrow 4.$$

From here on, we denote the \mathbb{Z}^2 adding machine by (Z, \mathbb{Z}^2, A) . Note that this is a different action on Z than the full 4-shift (Z, \mathbb{N}, R) .

As in the 1-dimensional case, we think of a point in $z \in Z$ as a \mathbb{Z}^2 block structure. Here, an n -locus consists of a $2^n \times 2^n$ square in \mathbb{Z}^2 . Each $(n + 1)$ -locus consists of four n -loci. The principal n -locus, which is the one containing $\mathbf{0} \in \mathbb{Z}^2$, will be denoted $B_n(z)$. For $z \in Z$, the entry z_1 tells which element of the 1-locus is $\mathbf{0}$, and z_n tells which of the four $(n - 1)$ -loci in the principal n -locus is the principal $(n - 1)$ -locus. Part of a block structure is illustrated in Figure 8. By the discussion of 1-dimensional block structures above, it follows that the mappings A_1 and A_2 correspond to the left-shift and the down-shift respectively.

8. MAIN RESULTS

In this section we state our main results for the chair and table dynamical systems. We begin with the discrete (i.e., substitution) cases.

Theorem 8.1. *The chair substitution system (X'_c, \mathbb{Z}^2, S) is a strictly ergodic almost 1 to 1, uniformly bounded to 1, extension of the \mathbb{Z}^2 adding machine (Z, \mathbb{Z}^2, A) . In particular, (X'_c, \mathbb{Z}^2, S) is almost topologically conjugate to the \mathbb{Z}^2 adding machine, and thus has almost topological pure discrete spectrum with eigenvalue group $e^{2\pi i \mathbb{Z}[\frac{1}{2}]^2}$. Moreover, for any $z \in Z$,*

$$(8.1) \quad |\varphi^{-1}(z)| \in \{1, 2, 5\},$$

where φ denotes the factor mapping.

Theorem 8.2. *The table substitution system (X'_t, \mathbb{Z}^2, S) is a strictly ergodic almost 4 to 1, uniformly bounded to 1 extension of the \mathbb{Z}^2 adding machine (Z, \mathbb{Z}^2, A) . In particular, (X'_t, \mathbb{Z}^2, S) has eigenvalue group $e^{2\pi i \mathbb{Z}[\frac{1}{2}]^2}$, but it also has a continuous component in its spectrum.*

Next we state the corresponding results for table and chair tiling dynamical systems. These involve the idea of the \mathbb{R}^2 -adding machine dynamical system, which we denote by $(\tilde{Z}, \mathbb{R}^2, A)$. We will not describe this dynamical system explicitly, but we note that it is uniquely defined in two equivalent ways: (i) as the suspension of the \mathbb{Z}^2 adding machine, and (ii) as the unique \mathbb{R}^2 Kronecker system with eigenvalue group $\mathbb{Z}[\frac{1}{2}]^2 \subseteq \mathbb{R}^2$.

Theorem 8.3. *The chair tiling system (X_c, \mathbb{R}^2, T) is a strictly ergodic almost 1 to 1, uniformly bounded to 1 extension of the \mathbb{R}^2 adding machine $(\tilde{Z}, \mathbb{R}^2, A)$. In particular, (X_c, \mathbb{R}^2, T) is almost topologically conjugate to the \mathbb{R}^2 adding machine, and thus has almost topological pure discrete spectrum with eigenvalues $\mathbb{Z}[\frac{1}{2}]^2$. Moreover, for any $z \in \tilde{Z}$, $|\varphi^{-1}(z)| \in \{1, 2, 5\}$, where φ denotes the factor mapping.*

Theorem 8.4. *The table tiling system (X_t, \mathbb{R}^2, T) is a strictly ergodic almost 4 to 1, uniformly bounded to 1, extension of the \mathbb{R}^2 adding machine $(\tilde{Z}, \mathbb{R}^2, A)$. In particular, (X_t, \mathbb{R}^2, S) has eigenvalue group $\mathbb{Z}[\frac{1}{2}]^2$, but it also has continuous component in its spectrum.*

Theorems 8.3 and 8.4 follow directly from Theorems 8.1 and 8.2 using the remarks at the end of Section 2. The proofs of Theorems 8.1 and 8.2 occupy the next four sections.

Comment. The underlying group in the \mathbb{R}^2 adding machine dynamical system is $\tilde{Z} = \tilde{\mathbb{D}} \times \tilde{\mathbb{D}}$, where $\tilde{\mathbb{D}}$ is the group known as the *solenoid* (it is the suspension of \mathbb{D}). One can show that the inflation mapping σ_c on the space X_c of chair tilings is an almost 1 to 1 extension, via φ , of an ergodic group automorphism $B \times B$ of $\tilde{\mathbb{D}} \times \tilde{\mathbb{D}}$. Here B is the *inverse limit* (or equivalently the *natural extension*) of the mapping $z \mapsto z^2$ on the unit circle in \mathbb{C} .

9. COMPLETE BLOCK STRUCTURES

The first step in the proofs of Theorems 8.2 and 8.1 is to describe the process of filling in block structures (in the tiling literature, this construction is called ‘up down generation of tilings’, cf. [13]).

A 2×2 substitution σ on \mathcal{A} induces a mapping, also denoted σ , on $\mathcal{A}^{\mathbb{Z}^2}$. This is obtained by applying σ to each symbol in $x \in \mathcal{A}^{\mathbb{Z}^2}$ and ‘concatenating’ together a new sequence $\sigma(x) \in \mathcal{A}^{\mathbb{Z}^2}$. To be well defined, we must say where to put the origin. We put it at the lower left symbol (i.e., in position 3) of $\sigma(x_0)$. Note that $\sigma(X'_\sigma) = X'_\sigma \subseteq \mathcal{A}^{\mathbb{Z}^2}$.

Lemma 9.1. *Let σ be a 2×2 substitution on \mathcal{A} with X'_σ the corresponding substitution space. Assume X'_σ is aperiodic. Then there exists a factor mapping $\varphi : (X'_\sigma, \mathbb{Z}^2, S) \rightarrow (Z, \mathbb{Z}^2, A)$.*

Proof. It will be convenient to work with the suspension X_σ of X'_σ , letting σ also denote the inflation mapping corresponding to the substitution σ . In par-

ticular, given $x' \in X'_\sigma$ we construct $x \in X_\sigma$ out of squares $[i, i + 1) \times [j, j + 1)$ labeled $x'_{(i,j)}$ (we think of these as labeled tiles). Since X'_σ is aperiodic, so is X_σ , and it follows from [15] that σ restricted to X_σ is invertible. The 1-loci in x are the σ images of the tiles in $\sigma^{-1}(x)$. Each 1-locus consists of a 2×2 arrangement of four tiles. Similarly, the n -loci in x are the σ^n images of the tiles in $\sigma^{-n}(x)$. Knowing the n -loci for each n , allows us to find the principal n -locus for each n . This gives a block structure on x , which restricts to a block structure on x' in the obvious way. We denote this block structure by $\varphi(x') \in Z$. Clearly the mapping φ is continuous, and it is equivariant since A implements the shift action on Z . The fact that φ is onto follows from the minimality of (Z, \mathbb{Z}^2, A) . \square

Given a block structure $z \in Z$, we define

$$(9.1) \quad B(z) = \bigcup_{n=1}^{\infty} B_n(z).$$

We call a block structure $z \in Z$ *complete* if $B(z) = \mathbb{Z}^2$. We denote the set of complete block structures by Z_0 . This set is dense G_δ .

Now we describe how to fill in a complete block structure. For $z \in Z$, let $y \in Y$ be such that $\psi(y) = z$. We construct a sequence $x_n(y)$ as follows: We put $x_0(y) = y_1$, which we locate at the origin in \mathbb{Z}^2 . Then we define $x_n(y)$ to be the block equivalent to $\sigma^n(y_{n+1})$ with locus $B_n(z)$.

Lemma 9.2. *The restriction of $x_{n+1}(y)$ to $B_n(z)$ is $x_n(y)$.*

Proof. Let us denote the four n -loci in the principal $(n + 1)$ -locus $B_{n+1}(z)$ by $B_n^1, B_n^2, B_n^3, B_n^4$. Note that one of these, namely $B_n^{\bar{n}}$, is the principal n -locus.

We have

$$(9.2) \quad x_n(y) \sim \sigma^n(y_{n+1}) = \left| \begin{array}{cc} \sigma^{n-1}(\sigma(y_{n+1})_1) & \sigma^{n-1}(\sigma(y_{n+1})_2) \\ \sigma^{n-1}(\sigma(y_{n+1})_3) & \sigma^{n-1}(\sigma(y_{n+1})_4) \end{array} \right|.$$

The edge in G_σ^- from y_n to y_{n+1} is labeled z_n so

$$\sigma(y_{n+1})_{z_n} = y_n,$$

which means that the matrix (9.2) has the block

$$\sigma^{n-1}(\sigma(y_{n+1})_{z_n}) = \sigma^{n-1}(y_n) \sim x_{n-1}(y)$$

in the position corresponding to z_n , namely in the the locus $B_n(z)$. \square

Proposition 9.3. *Let $z \in Z_0$. Then for each $y \in \psi^{-1}(z)$ there exists $x(y) \in X'_\sigma$ so that $\varphi(x) = z$.*

Proof. We define $x(y) \in \mathcal{A}^{\mathbb{Z}^2}$ by putting $x(y)_m = x_n(y)_m$ for any n large enough that $m \in B_n(z)$. This is well defined by Lemma 9.2. Now $x(y) \in X'_\sigma$, since all its blocks are blocks in $x_n(y) \sim \sigma^n(y_{n+1})$ for some n . Clearly we have $\varphi(x) = z$. \square

Corollary 9.4. For the chair substitution system, the factor mapping $\varphi : X'_c \rightarrow Z$ is almost 1 to 1.

Proof. The set $Z_0 \cap Z_1$ is dense G_δ , and for $z \in Z_0 \cap Z_1$, $y = \psi^{-1}(z)$ is unique and $x(y) = \varphi^{-1}(z)$ is also unique. \square

Corollary 9.5. For the table substitution system, the factor mapping $\varphi : X'_t \rightarrow Z$ is almost 4 to 1.

Proof. The set Z_0 is dense G_δ , and each $z \in Z_0$ has four preimages $y \in \phi^{-1}(z)$. Each of these gives a different $x(y) \in \varphi^{-1}(z)$. \square

10. THE CASE $z \in Z_0 \setminus Z_1$

This case occurs only for the chair. It corresponds to those $z \in Z$ that are complete, but where ψ is not 1 to 1. Let us suppose z has only the blocks 22, 33, 23 and 32, so that $\psi^{-1}(z) = \{y^1, y^2\}$ where $y^1 = .pppp\dots$ and $y^2 = .rrrr\dots$. We obtain two points $x(y^1), x(y^2) \in \varphi^{-1}(z)$ that differ only along their diagonal. These points correspond to tilings part of which are shown in Figure 9 (a) and (b).

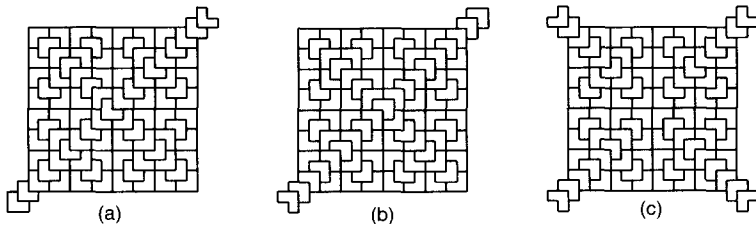


Figure 9. Flipping an infinite worm: (a) and (b). The Ferris wheel tiling: (c).

It follows that for these cases, $|\varphi^{-1}(z)| = |\{x(y_1), x(y_2)\}| = 2$. Note that the infinite stack of chair tiles along the diagonal in Figure 9 (a) and (b) can be ‘flipped’ leaving all the other tiles in the tiling intact. We call this configuration an *infinite worm* by analogy with the similar structures found in some Penrose tilings (cf. [4]). Any two such tilings x_1 and x_2 have the property that $d(T^{t_n}x_1, T^{t_n}x_2) \rightarrow 0$ as $n \rightarrow \infty$, provided $\|t_n\| \rightarrow \infty$ and the distance from t_n to the worm is unbounded. Such pairs of points are called *proximal* or *homoclinic*.

11. INCOMPLETE BLOCK STRUCTURES

There are eight kinds of incomplete block structures $z \in Z \setminus Z_0$. We characterize them as either $\frac{1}{4}$ -plane types or $\frac{1}{2}$ -plane types, according to $B(z)$. It suffices to consider these block structures up to equivalence (i.e., translation). The $\frac{1}{4}$ -plane types – those where $B(z)$ is one of the four quadrants of \mathbb{Z}^2 – correspond

to $z^k = .kkkk\dots$ for $k = 1, 2, 3, 4$. Up to equivalence, each of them is unique. The four $\frac{1}{2}$ -plane types correspond to z having only entries k and ℓ , and infinitely many of each, where (k, ℓ) is one of the pairs $(1, 2)$, $(3, 4)$, $(1, 3)$ or $(2, 4)$. There are infinitely many equivalence classes of each of these types, and each class has another class as its *reflection*. For example, the reflection of a $(1, 2)$ upper $\frac{1}{2}$ -plane type is the lower $\frac{1}{2}$ -plane $(3, 4)$ type that is obtained via the transformation $1 \leftrightarrow 3, 2 \leftrightarrow 4$. We denote the reflection of $z \in Z \setminus Z_0$ by z^* .

If $z \in Z \setminus Z_0$ and $y \in \psi^{-1}(z)$, we construct $x(y) \in \mathcal{A}^{B(z)}$ in the same way that we constructed whole plane points for complete z . A point $x(y) \in \mathcal{A}^{B(z)}$ can be extended to $x \in \mathcal{A}^{\mathbb{Z}^2}$ by pasting it together with other points of the same type (i.e., two $\frac{1}{2}$ -plane points or four $\frac{1}{4}$ -plane points).

We consider the $\frac{1}{4}$ -plane case first, and for concreteness we assume $z = z^1 = .111\dots$. For each $a = 1, 2, 3, 4$, let $y^a \in \psi^{-1}(z^a)$. To get x , we paste together the points $x(y_1)$, $x(y_2)$, $x(y_3)$, and $x(y_4)$ (we need to translate $x(y_2)$, $x(y_3)$, and $x(y_4)$ first). *A priori*, there are $|\psi^{-1}(z)|^4$ possible versions of x , but as we will see below, not all these points belong to X'_σ . However, whenever such an x does belong to X'_σ , we have $\varphi(x) = z$. It follows that

$$(11.1) \quad |\varphi^{-1}(z)| \leq |\psi^{-1}(z)|^4.$$

By Lemma 6.2, the right hand side of (11.1) for the table is $\leq 4^4 = 256$, and by Proposition 6.3, it is $\leq 2^4 = 16$ for the chair.

The argument for the $\frac{1}{2}$ -plane cases is nearly the same. Given z , we let $y \in \psi^{-1}(z)$ and let $y^* \in \psi^{-1}(z^*)$. We construct x by pasting together $x(y)$ and (a translate of) $x(y^*)$. Again, not all such points x are in X'_σ , but we have

$$(11.2) \quad |\varphi^{-1}(z)| \leq |\psi^{-1}(z)|^2.$$

Combining (11.1) and (11.2), we have the following.

Proposition 11.1. *For any aperiodic substitution X'_σ the factor mapping $\varphi : X'_\sigma \rightarrow Z$ is uniformly bounded to 1.*

Now we want to improve the estimates (11.1) and (11.2). We start with the chair.

The $\frac{1}{2}$ -plane cases all belong to Z_1 , so these points all have unique preimages.

For the $\frac{1}{4}$ -cases, we note that it suffices to specify a single symbol at each corner, since then we can apply σ_c^n , $n = 1, 2, \dots$. There are *a priori* 4^4 choices. Without loss of generality, we can take the corners to be $z^k = .kkkk\dots$, $k = 1, \dots, 4$. Looking at vertex paths in G_c^- that correspond to the corners $z^k = .kkkk\dots$, just 16 possibilities remain:

$$\begin{vmatrix} q \text{ or } s & p \text{ or } r \\ p \text{ or } r & q \text{ or } s \end{vmatrix}$$

However, only five of these actually occur, since the others violate the *three arrow rule*. The corresponding tiling patches around the origin are shown in Figure 10.

It follows that for the $\frac{1}{4}$ -plane block structures, in the case of the the chair, we

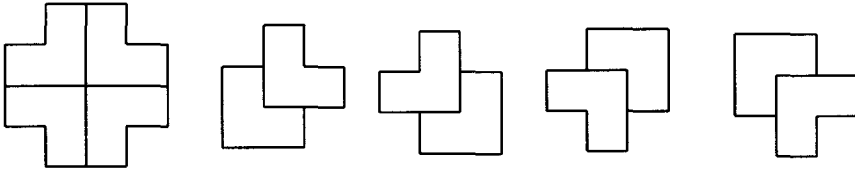


Figure 10. Allowable $\frac{1}{4}$ -plane types for the chair.

have $|\varphi^{-1}(z)| = 5$. Four of these tilings are congruent via a rotation. We refer to this (congruence class) as the *cartwheel chair tiling*, by analogy with the cartwheel Penrose tiling (cf. [4]). Near the origin these look the same as Figure 9 (a) and (b). In particular, such tilings have an infinite worm, but unlike the tilings discussed in Section 10, they also have two ‘half-infinite worms’. The fifth tiling, which is not congruent to the others, has no direct Penrose analogue. We refer to this tiling, shown in Figure 9 (c), as the *Ferris wheel*. It has 4-fold rotational symmetry and four half-infinite worms. Note that all five of these tilings are proximal, and they can be interconverted by ‘flipping’ the infinite X-shaped configuration of chair tiles along the diagonal and the reverse-diagonal. We have now proved (8.1).

Now we turn to the case of table tilings. We will prove the following.

Theorem 11.2. *For the table tilings and table substitution, $|\varphi^{-1}(z)| \in \{4, 10, 24\}$.*

Proof. By Proposition 9.3, we know that for complete block structures $|\varphi^{-1}(z)| = 4$.

Let us consider a pair left and right $\frac{1}{2}$ -plane types, z and z^* pasted together along the y -axis, and the pair of adjacent symbols on either side of the y -axis at $\mathbf{0}$. Note that a p on the left must be paired with an r on the right, but all of the 3^2 combinations of other symbol pairs are allowed. Thus we have $|\varphi^{-1}(z)| = 1 + 3^2 = 10$. The upper and lower $\frac{1}{2}$ -plane cases are the same.

For the $\frac{1}{4}$ -plane cases, we consider the allowed vertex types at the origin. All possible vertex types in the full table shift (modulo rotation and reflection) are shown in Figure 11. However, the types (g) and (h) do not occur in any tiling $x \in X_t$ (i.e., in any table tiling) since they are not blocks in any $\sigma_t^n(\tau)$.

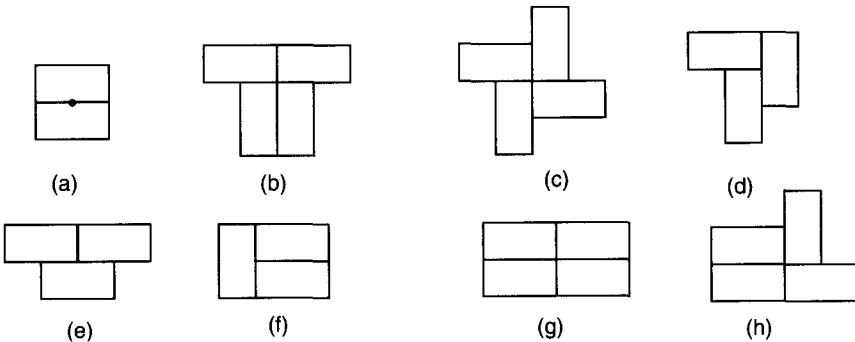


Figure 11. Vertex types for the full table shift.

We can enumerate the $\frac{1}{4}$ -planes cases by counting the rotations and reflections of each allowed picture: (a) 2, (b) 4, (c) 2, (d) 8, (e) 4, and (f) 4, for a total of 24. Thus in the $\frac{1}{4}$ -plane table case, $|\varphi^{-1}(z)| = 24$. \square

Comment. By iterating the inflation mapping σ_t on one of the non-occurring configurations (g) or (h), one can obtain a tiling x that is a fixed point for σ_t , but that does not belong to X_t . Such a tiling is not almost periodic since it has a vertex configuration (i.e., (g) or (h)) that occurs in just one place. It follows that the orbit closure is not minimal. This illustrates why we define rep-tiling spaces the way we do in this paper. The definition as the orbit closure of a fixed point for σ , which is common for 1-dimensional substitutions, does not always work in this case.

12. THE SPECTRUM OF THE TABLE

The purpose of this section is to give a simple direct proof of the following result, completing the proof of Theorem 8.2. We include this argument for the sake of completeness.

Proposition 12.1 [14]. *The table substitution system (X'_t, \mathbb{Z}^2, S) has a continuous component in its spectrum.*

In the 1-dimensional case, a similar result holds for *bijective substitutions* (cf. [7]). Our proof of Proposition 12.1 rests on the following lemma.

Lemma 12.2 [14]. *The table substitution system (X'_t, \mathbb{Z}^2, S) has $e^{2\pi i \mathbb{Z}[\frac{1}{2}]^2}$ as its eigenvalue group.*

Proof of Proposition 2.1. Let γ denote Haar measure on $Z = \mathbb{D}^2$. For $g \in L^2(Z, \gamma)$ we define $f \in L^2(X'_t, \mu)$ by $f(x) = (\varphi \circ g)(x)$. We denote the space of all such f by H_0 , and note that H_0 is T -invariant. Since φ is at least 4 to 1, it follows that $H_1 = L^2(X'_t, \mu) \ominus H_0$ is nontrivial. Now let w be an eigenvalue for (X'_t, \mathbb{Z}^2, S) , and let $g \in L^2(Z, \gamma)$ be the corresponding eigenfunction for (Z, \mathbb{Z}^2, A) . Then $f = \varphi \circ g$ is an eigenfunction for (X'_t, \mathbb{Z}^2, S) . By Lemma 12.2, all the eigenfunctions arise this way, so it follows that H_0 is the closure of the span of the eigenfunctions. Since $H_1 \neq \emptyset$, there exists a continuous component in the spectrum. \square

Now we proceed to the proof of Lemma 12.2. For this we use the following lemma, whose proof follows from expressing numbers in base 2.

Lemma 12.3. *Suppose $2^n w \rightarrow 0 \pmod{1}$. Then $w \in \mathbb{Z}[\frac{1}{2}]$.*

Proof of Lemma 12.2. Let E_n be the set of $x \in X'_t$ such that in $\varphi(x)$ the origin is the lower left corner of its k -locus for $k = 1, \dots, n$. In particular, $\varphi(x) = .3^n * * * \dots$. For $a \in \mathcal{A}$, let $C_a = \{x | x_0 = a\} \subseteq X'_t$, and let $K_{n,a} =$

$E_n \cap C_a$. Consider the locus $B^n = \{0, \dots, 2^n - 1\}^2 \subseteq \mathbb{Z}^2$. Then $P_n = \{K_{n,a,\mathbf{n}} = S^n K_{n,a} : a \in \mathcal{A}, \mathbf{n} \in B_n\}$ is a partition of X'_t with $\ell_n = \text{diam}(K_{n,a,\mathbf{n}})$ satisfying $\ell_n \rightarrow 0$.

Let f be an eigenfunction with eigenvalue \mathbf{w} for (X'_t, \mathbb{Z}^2, S) , and assume $|f| = 1$, so $f(x) = e^{2\pi i h(x)}$. We can approximate f in measure by a function f' constant on the sets of P_n . In particular, for some sequence $\epsilon_n \rightarrow 0$ there exist sets $K'_{n,a,0} \subseteq K_{n,a,0}$, with $\mu(K'_{n,a,0}) \geq (1 - \epsilon_n)\mu(K_{n,a,0})$ and for $x \in K'_{n,a,0}$, $f'(x) = e^{2\pi i h_{n,a}}$ where $|h(x) - h_{n,a}| < \epsilon_n$. Note that for $x \in K'_{n,a,\mathbf{n}} = S^n K'_{n,a,0}$, for any $\mathbf{n} \in B_n$, we have $f'(x) = e^{2\pi i(\mathbf{n}, \mathbf{w})} f'(T^{-\mathbf{n}}x) = e^{2\pi i(h_{n,a} + (\mathbf{n}, \mathbf{w}))}$. We let $e_n(x) = h(x) - h_{n,a}$.

For $n \in \mathbb{N}$, consider the four blocks (equivalent to) $\sigma^n(p), \sigma^n(q), \sigma^n(r)$ and $\sigma^n(s)$. Let us place such a block so the lower left corner is $\mathbf{0} \in \mathbb{Z}^2$. We denote that block with a at position $\mathbf{0}$ by $\beta_n(a)$. Note that $K_{n,a,0}$ is the *cylinder set* for this block: the set of all points x with $\beta_n(a)$ at the locus B_n . For $m \in \mathbb{N}$ and any $z \in \{1, 2, 3, 4\}^m$, let a^z denote the vertex in G_t^+ obtained by starting at a and following z . We have

$$\begin{aligned} \beta_{n+1}(a) = \sigma_t^{n+1}(a^{3^{n+1}}) &= \begin{vmatrix} \sigma_t^n(\sigma_t(a^{3^n})_1) & \sigma_t^n(\sigma_t(a^{3^n})_2) \\ \sigma_t^n(\sigma_t(a^{3^n})_3) & \sigma_t^n(\sigma_t(a^{3^n})_4) \end{vmatrix} \\ &= \begin{vmatrix} \sigma_t^n(a^{3^{n1}}) & \sigma_t^n(a^{3^{n2}}) \\ \sigma_t^n(a^{3^{n3}}) & \sigma_t^n(a^{3^{n4}}) \end{vmatrix} = \begin{vmatrix} \beta_n(a^{3^{n13^n}}) & \beta_n(a^{3^{n23^n}}) \\ \beta_n(a^{3^{n33^n}}) & \beta_n(a^{3^{n43^n}}) \end{vmatrix}, \end{aligned}$$

and by a simple induction, there exist $a_{n,1}, a_{n,2}, a_{n,3}, a_{n,4}$ such that

$$\beta_{n+1}(a) = \begin{vmatrix} \beta_n(a_{n,1}) & \beta_n(a_{n,2}) \\ \beta_n(a_{n,3}) & \beta_n(a_{n,4}) \end{vmatrix}.$$

In particular, $a_{n,k} = \sigma_t(a)_k$ for n odd, and $a_{n,k} = \sigma'_t(a)_k$ for n even, where σ'_t given by

$$p \mapsto \begin{matrix} p & p \\ q & r \end{matrix} \quad q \mapsto \begin{matrix} s & q \\ p & q \end{matrix} \quad r \mapsto \begin{matrix} r & s \\ r & p \end{matrix} \quad s \mapsto \begin{matrix} q & r \\ s & s \end{matrix}.$$

This shows that the $(n+1)$ -blocks $\beta_{n+1}(a)$ are each obtained by pasting together four n -blocks. This is what is known in ergodic theory as a *rank-4 construction*.

For n sufficiently large, there exists $x \in K'_{n,s,0}$ with $S^{(2^n, 0)}x \in K'_{n,s,0}$. For such an x we have

$$f(S^{(2^n, 0)}x) = e^{2\pi i h(S^{(2^n, 0)}x)} = e^{2\pi i(h_{n,s} + e_n(S^{(2^n, 0)}x))}.$$

We also have

$$f(S^{(2^n, 0)}x) = e^{2\pi i(\langle (2^n, 0), \mathbf{w} \rangle + h(x))} = e^{2\pi i(\langle (2^n, 0), \mathbf{w} \rangle + h_{n,s} + e_n(x))}.$$

Since $|e_n(S^{(2^n, 0)}x) - e_n(x)| \leq 2\epsilon_n$, it follows that $\langle (2^n, 0), \mathbf{w} \rangle \rightarrow 0 \pmod{1}$. Letting $\mathbf{w} = (w_1, w_2)$, we have $2^n w_1 \rightarrow 0 \pmod{1}$, and thus by Lemma 12.3, $w_1 \in \mathbb{Z}[\frac{1}{2}]$. Similarly, there exists $x \in K'_{n,r,0}$ such that $T^{(0, 2^n)}x \in K'_{n,r,0}$, and by the same argument $2^n w_2 \rightarrow 0 \pmod{1}$, so $w_2 \in \mathbb{Z}[\frac{1}{2}]$. \square

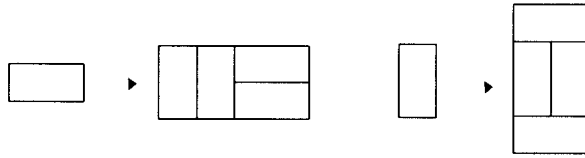


Figure 12. The modified table.

13. A DIFFERENT TABLE

If we replace the table rep-tile with the following modified rep-tile set we obtain a new set of tilings X_f , that we call the *modified table*. Using the same coding as for the table, we obtain the substitution σ_f

$$(13.1) \quad p \mapsto \begin{array}{cc} s & s \\ q & q \end{array} \quad q \mapsto \begin{array}{cc} q & q \\ p & r \end{array} \quad r \mapsto \begin{array}{cc} p & r \\ p & r \end{array} \quad s \mapsto \begin{array}{cc} p & r \\ s & s \end{array}.$$

One can check that in the forward substitution graph for this example, the path 31424 always leads to the vertex r . It follows that corresponding tiling dynamical system is an almost 1 to 1 extension of the \mathbb{R}^2 adding machine. Thus the *modified table tiling dynamical system* is almost topologically conjugate (and hence metrically isomorphic) to the chair. Since this implies that the modified table tiling dynamical system has pure discrete spectrum, it is not metrically isomorphic to the original table tiling dynamical system. Rather, it is metrically isomorphic to the chair.

We claim, however, that the modified table is not topologically conjugate to the chair. To see this, we note that in the tilings $x \in X_f$, the vertex types (c), (d), and (e) in Figure 11 that do not occur (all the other types do occur). Thus the number of $\frac{1}{4}$ -plane cases is $2 + 4 + 4 + 2 + 8 = 20$. One can also show that the number of $\frac{1}{2}$ -plane cases is 7. Hence for the the modified table, $|\varphi^{-1}(z)| \in \{1, 7, 20\} \neq \{1, 2, 5\}$.

14. COMMENTS

The *good blocks* that are used in the proof of Lemma 6.3 are what are called *magic words* in [5] or *synchronizing blocks* in [6]. In particular, the proof of Lemma 6.1 follows from a basic result in symbolic dynamics. Lemma 6.1 implies the factor map ψ is both *left and right closing* (cf. e.g. [6], [5]). Essentially the same argument – but very different looking – occurs in the work of Dekking [1] on 1-dimensional substitutions (and later generalized to tilings in [14]) where it is called *coincidence*.

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