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p-adic analogues of the law of large numbers and the central limit theorem

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1. INTRODUCTION

To solve the problem of the statistical interpretation of p-adic valued wave functions in non-Archimedean quantum physics a p-adic valued theory of probability was proposed (for non-Archimedean physics see books [3, 17]). Here we have p-adic coefficients, which must be considered as probabilities from the physical point of view (density matrix), but they belong to the field of p-adic numbers Q_p. Thus, these coefficients cannot be probabilities within Kolmogorov's axiomatic theory of probability. This problem forces us to re-analyze the foundations of modern probability theory. We propose a new, more general probability theory, a special case of which is the ordinary probability theory.

The first step in this direction was the communication [4], where a concrete non-Archimedean probability distribution was proposed, the theory of a non-Archimedean white noise. This theory was correct from a mathematical point of view. But I could not give the answer to the following question. What can we say about a *probability* in such an unusual situation?

Foundations of a non-Archimedean probability theory were proposed in [5, 6]. It is well known that A.N. Kolmogorov [10] constructed an axiomatic system for the modern theory of probability, using the frequency theory of probability of R. von Mises [12] (see the remarks in Kolmogovor's book [10]).

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We also began the construction of the new theory of probability with a frequency definition of probability [5], [3]. Then we considered the theorems of this frequency theory as axioms in the new theory [6], [3].

What is our main idea?

We study a statistical stabilization of relative frequencies $\{\nu_N\}$, not only in the standard real topology on the field of rational numbers Q, but also in an arbitrary topology on Q (relative frequencies $\{\nu_N\}$ always belong to Q). We present a general frequency theory of probability containing von Mises' theory as the particular case coming from the real topology of the statistical stabilization of relative frequencies; see [5], [3]. Then the properties of these frequency probabilities are considered as a basis for a measure-theoretical approach. In particular, p-adic probability is defined as a bounded Q_p-valued measure P_p with the normalization condition, $P_p(\Omega) = 1$. The main difference with ordinary probability is that a p-adic probability measure P_p may take every x in Q_p as a value and the ordinary probability measure P_∞ has its range only in a part of R, the segment $[0, 1]$. This is a consequence of the fact [5], [3] that every p-adic number can be a limit of relative frequencies with respect to a p-adic metric.

The role of the Bernoulli limit theorem for sums of independent equally distributed random variables is well known. In this paper I present a p-adic analogue of the Bernoulli theorem. As in the ordinary case, we consider sums

$$S_n(\omega) = \xi_1(\omega) + \cdots + \xi_n(\omega) \tag{1}$$

of independent random variables ($\xi_n = 0$ or 1 with equal probabilities $\frac{1}{2}$) but the limit distribution of the normalized sums $T_n(\omega) = S_n(\omega)/n$ differs from the ordinary case. Moreover it depends on the way in which n approaches infinity. A limit distribution of the sequence $T_{n_k}(\omega)$ depends on the sequence $\{n_k\}$.

The p-adic Bernoulli theorem has a natural interpretation as an asymptotic of the p-adic probability that p divides $S_{n_k}(\omega)$ or not. From the p-adic point of view these probabilities are equal. Then we study the nonsymmetric case, $\xi_n(\omega) = 0, 1$ with probabilities q and $q' = 1 - q$, $q \in Q_p$. In particular, here we have a limit theorem for negative probabilities; for example, $q = -1$, $q' = 2$. A connection with negative probabilities which arise in quantum physics [15] can be made (Wigner's function, Dirac's relativistic quantization, Einstein–Podolsky–Rosen paradox); see [9] for applications of p-adic probabilities to the problem of Bell's inequality violations.

At the end, we try to find a p-adic analogue of the central limit theorem. Here we have found the form of the limiting distribution. In some sense, it must be considered as a p-adic Gaussian distribution, but it differs from the p-adic Gaussian distribution which has been introduced in [4] using the Laplace transform. At the moment, we cannot prove in this case the weak convergence of distributions (only the convergence for analytic functions).

2. BERNOULLI THEOREM

Let $\Omega_B = \{0, 1\}^N$ be the standard Bernoulli probability space, i.e. the space of sequences $\omega = (\omega_j)_{j=1}^{\infty}$, $\omega_j = 0, 1$. Let $I = \bigcup I_n$, where I_n is the set of all vectors of length n with coordinates $0, 1$. Let $i \in I_n$ and $B_i = \{\omega \in \Omega\colon \omega_1 = i_1, \dots, \omega_n = i_n\}$; this is a cylindrical subset. Denote by the symbol F the algebra generated by all cylindrical subsets. Then the standard Bernoulli measure μ is defined by $\mu(B_i) = 1/2^n$, extended as an additive set-function on the algebra F. It can be extended to the standard σ-additive Bernoulli probability P_∞ on the σ-algebra F_∞ generated by F. $(\Omega_B, F_\infty, P_\infty)$ is a probability space in the sense of Kolmogorov's axiomatics.

Remark 1. The symbol Q_∞ is often used instead of R in the theory of numbers. It is convenient for us to use a label ∞ for real objects and, in particular, for real valued probabilities.

As usual, set $\xi_n(\omega) = \omega_n$. These are independent random variables. $P_\infty(\xi_n = 0) = \mu(\xi_n = 0) = \frac{1}{2}$ and $P_\infty(\xi_n = 1) = \mu(\xi_n = 1) = \frac{1}{2}$.

Let η be a random variable on the probability space $(\Omega_B, F_\infty, P_\infty)$. Denote its probability distribution by the symbol $P_{\infty,\eta}$.

According to the Bernoulli theorem:

(2) $$T_n(\omega) \to \xi_{\lim}(\omega)$$

where $\xi_{\lim}(\omega) = \frac{1}{2}$ a.s. In particular, we have the limit theorem for distributions of random variables $\{T_n(\omega)\}$:

(3) $$P_{\infty,T_n} \to P_{\infty,\xi_{\lim}},$$

where the limit distribution is $\delta_{1/2}$ (everywhere δ_a denotes a δ-measure concentrated at the point a) and (3) means weak convergence of distributions

(4) $$Mf(T_n(\omega)) = \int f(x) P_{\infty,T_n}(\mathrm{d}x) \to \int f(x) P_{\infty,\xi_{\lim}}(\mathrm{d}x)$$

for all bounded continuous functions f, with limit equal to $f(\frac{1}{2})$.

Further,

$$Mf(T_n(\omega)) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \sum_{i_1 + \cdots + i_n = k} \mu(B_i).$$

Hence

(5) $$Mf(T_n(\omega)) = \frac{1}{2^n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k,$$

where the C_n^k are the binomial coefficients.

Remark 2. These computations were based only on the cylindrical measure μ. Therefore we can rewrite (4):

(6) $$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k \to f(\tfrac{1}{2}).$$

But we are interested in another extension of μ. As μ assumes its values in Q, it is also possible to consider it as a p-adic valued measure.

Everywhere in what follows Q_p denotes the field of p-adic numbers, $|\cdot|_p$ is the

p-adic valuation. We denote by $U_r(a)$ a ball of radius r with center at a, $U_r(a) = \{x \in Q_p : |x-a|_p \le r\}$ and set $U_r = U_r(0)$. As usual we denote by Z_p the ring of p-adic integers, $Z_p = U_1$.

Here we present a brief review of the theory of integration with respect to Q_p-valued measures [11, 13, 14, 16]. Let Ω be a compact ultrametric space. Thus, we have the strong triangle inequality

$$\rho(x,y) \le \max[\rho(x,z), \rho(z,y)], \quad x,y,z \in \Omega.$$

The balls $W_r(a) = \{x \in \Omega : \rho(x,a) \le r\}$ are at the same time open and closed, i.e. 'clopen'. The p-adic metric given by $\rho_p(x,y) = |x-y|_p$ is an ultrametric and every ball $U_r(a)$ in Q_p is a compact ultrametric space.

Denote by $F(\Omega)$ the algebra of all clopen subsets of Ω. A measure ν on $F(\Omega)$ is an additive set function, $\nu : F(\Omega) \to Q_p$. A measure ν is said to be bounded if

$$\operatorname{var}(\nu) = \sup\{|\nu(A)|_p : A \in F(\Omega)\} < \infty.$$

Denote the space of continuous functions $f : \Omega \to Q_p$ by the symbol $C(\Omega, Q_p)$. It is a Q_p-linear Banach space with respect to the uniform norm $\|f\| = \max_{x \in \Omega} |f(x)|_p$. There is a one-to-one correspondence between bounded measures on $F(\Omega)$ and bounded Q_p-linear functionals on the space $C(\Omega, Q_p)$ and $\|\nu\| = \operatorname{var}(\nu)$. The integral of a continuous function $f(x)$ with respect to a bounded measure ν is defined as a limit of Riemann sums:

$$\int_\Omega f(x)\nu(\mathrm{d}x) = \lim_\alpha \sum_{i=1}^n f(a_i)\nu(A_i),$$

where $\alpha = (A_i)$ is a cover of Ω, $A_i \in F(\Omega)$, $A_i \cap A_j = \emptyset$, $a_i \in A_i$.

A bounded normalized measure $\nu : F(\Omega) \to Q_p$, $\nu(\Omega) = 1$, is said to be *a p-adic valued probability*, where Ω is a probability space, $F(\Omega)$ is an algebra of events, $(\Omega, F(\Omega), \nu)$ is a p-adic probability model (this is a particular case of p-adic probability axiomatics [3], [6], [5]). A function $\xi : \Omega \to Q_p$, $\xi \in C(\Omega, Q_p)$, is said to be a (continuous) random variable. The consideration of continuous random variables is sufficient for our purpose (see [3] about general theory of Q_p-valued random variables). A mean value of ξ is defined as an integral with respect to a probability measure

$$M\xi(\omega) = \int_\Omega \xi(\omega)\nu(\mathrm{d}\omega).$$

The definition of independent random variables is standard (see [3]) and

$$M\xi(\omega)\eta(\omega) = M\xi(\omega)M\eta(\omega) \tag{7}$$

for independent random variables.

Let ξ be a random variable and $\xi(\Omega) \subset U_r(a)$, then as usual we can define a probability distribution of ξ on the ball $U_r(a)$:

$$\nu_\xi(D) = \nu(\xi^{-1}(D)), \quad D \in F(U_r(a)).$$

It is a probability on $U_r(a)$. Thus $(U_r(a), F(U_r(a)), \nu_\xi)$ is a probability model.

Now we come back to the Bernoulli measure. The Bernoulli space Ω_B is isomorphic to the ring of 2-adic integers Z_2, a cylindrical set B_i is a ball $U_{2^{-n}}(a)$, where a is an arbitrary point of Q_2 with the property $a_0 = i_1, \ldots, a_{n-1} = i_n$. The algebra F coincides with the algebra of clopen sets $F(Z_2)$ and the Bernoulli measure μ on F can be considered as a Q_p-valued measure. This measure is bounded if $p \neq 2$ (see, for example, [13, 14, 16]) and a bounded Q_p-linear functional on $C(Z_2, Q_p)$ corresponds to μ.

Thus for $p = 3, 5, \ldots, 127, \ldots$, we have p-adic probability models $(Z_2, F(Z_2), P_p)$, where $P_p = \mu$ are p-adic valued Bernoulli probabilities.

It is interesting that the probabilities P_∞ and P_p coincide for all events which depend on a finite number of experiments, (B_i). Thus, we cannot distinguish these two distributions on the basis of experiment. But these probabilities have different asymptotic properties.

The Bernoulli random variables $\xi_n(\omega)$ have rational values and are locally constant functions on Z_2. That is why we can consider these functions as p-adic valued random variables. Moreover as in the ordinary case, we have $P_p(\xi_n(\omega) = 0) = \mu(\xi_n(\omega) = 0) = \frac{1}{2}$ and $P_p(\xi_n(\omega) = 1) = \mu(\xi_n(\omega) = 1) = \frac{1}{2}$. Now it is possible to study the asymptotic behaviour (2) of sums (1). We shall study the asymptotics of subsequences $S_{n_k}(\omega)$. This depends very much on $\{n_k\}$. We shall get different limit random variables $\xi_{\lim}$ for different sequences $\{n_k\}$. Weak convergence of distributions $P_{p, T_{n_k}}$ is defined as in the ordinary theory of probability, (4).

The method of characteristic functions is one of the most powerful methods in the ordinary theory of limit theorems. At the moment there are no theorems on the correspondence between convergence of characteristic functions and probability distributions in the p-adic case. We cannot apply this technique directly. But we have such a theorem for the convergence of generalized functions; see [3], [7]. That is why at first we consider the limit procedure (2) in the sense of the theory of generalized functions and find asymptotic laws, which will be probability measures in 'good' cases. Then we shall prove weak convergence directly. The generalized function point of view permits us to study a larger class of limiting theorems than the language of a measure theory. In particular, we can find a limit theorem for the 'generalized' probability P_2, which is not a bounded measure (see [16], [1–8] on an integration theory with respect to unbounded non-Archimedean valued measures). But of course, we must consider another functional space instead of the space $C(Z_2, Q_p)$ to define a limit procedure for generalized functions.

3. CHARACTERISTIC FUNCTIONS METHOD

We use a theory of analytic generalized functions [3], [7], [9]. It is more convenient to work over the field of complex p-adic numbers C_p, instead of Q_p. Here C_p is a completion of an algebraic closure Q_p^a of Q_p (see, for example, [11], [16]), $|\cdot|_p$ is the extension of the p-adic valuation on C_p, $|C_p| = \{r \in R_+ : r = |z|_p, z \in C_p\}$. Set $\mathcal{U}_r = \{z \in C_p : |z|_p \leq r\}$. These are balls in C_p with center zero.

The function $f : \mathcal{U}_r \to C_p$, $r \in |C_p|$, is said to be analytic if the series

$$f(x) = \sum_n f_n x^n, \quad f_n \in C_p,$$

converges (uniformly) on $\mathcal{U}_r$. The topology in the space $\mathcal{A}(\mathcal{U}_r) \equiv \mathcal{A}_r$ of functions analytic on the ball $\mathcal{U}_r$ is defined by the non-Archimedean norm

$$\|f\|_r = \max_{z \in \mathcal{U}_r} |f(z)|_p = \max_n |f_n|_p r^n.$$

$\mathcal{A}_r$ is a non-Archimedean Banach space.

A function $f : C_p \to C_p$ is *entire* if its Taylor series converges on the ball $\mathcal{U}_r$ for every r. The topology in the space of entire functions $\mathcal{A}(C_p) \equiv \mathcal{A}$ is defined by the system on non-Archimedean norms $\{\|\cdot\|_r\}_{r \in |C_p|}$. A sequence of entire functions $\{f_n\}$ converges in $\mathcal{A}$ if it converges uniformly on each ball $\mathcal{U}_r$. $\mathcal{A}$ is a non-Archimedean Fréchet space. Such a type of topology is known as the topology of the projective limit of Banach spaces

$$\mathcal{A} = \lim \text{proj}_{r \to \infty} \mathcal{A}(\mathcal{U}_r).$$

A function $f : C_p \to C_p$ is said to be analytic at zero if there exists an r such that $f \in \mathcal{A}(\mathcal{U}_r)$. The space of functions analytic at zero, $\mathcal{A}_0(C_p) \equiv \mathcal{A}_0$ is provided with a topology of an inductive limit:

$$\mathcal{A}_0 = \lim \text{ind}_{r \to 0} \mathcal{A}(\mathcal{U}_r).$$

We choose the spaces $\mathcal{A}$ and $\mathcal{A}_0$ as the spaces of test functions and the spaces of C_p-linear continuous functionals $\mathcal{A}'$ and $\mathcal{A}_0'$ as the spaces of distributions (generalized functions).

The following simple result will be very important.

Proposition 1. *Let ν be a bounded measure on a ball U_r of Q_p. Then the functional*

$$f \to \nu(f) = \int_{U_r} f(x)\nu(\mathrm{d}x), \quad f \in \mathcal{A},$$

is continuous on the space $\mathcal{A}$.

That is why every bounded measure defines a distribution $\nu \in \mathcal{A}'$. Different distributions correspond to different measures.

Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of distributions in the space $\mathcal{A}'$. If this sequence converges to a distribution γ in the weak topology of $\mathcal{A}'$, then we speak about $\mathcal{A}$-weak convergence. This means that $(\gamma_n, f) \to (\gamma, f)$ for every entire analytic function f.

Definition 1. The (two-sided) Laplace transform of the distribution $g \in \mathcal{A}_0'$ is the function $L(g)(y) = (g(x), \exp\{yx\})$.

Theorem 1 [7], [3]. *The Laplace transformation $L : \mathcal{A}_0' \to \mathcal{A}$ is an isomorphism.*

Thus we have a non-Archimedean Laplace calculus

$$\mathcal{A}_0' \xrightarrow{L} \mathcal{A}, \qquad \mathcal{A}_0 \xleftarrow{L'} \mathcal{A}'.$$

The Laplace transformation has all the standard properties of the usual Laplace transformation.
If a distribution $\gamma \in \mathcal{A}'$ is the image of a bounded measure on U_r then

$$L'(\gamma)(y) = \int_{U_r} e^{xy}\gamma(\mathrm{d}x)$$

for sufficiently small $y \in C_p$.

If $p \neq 2$ then the probability distributions $P_{p,T_{n_k}}$ of the random variables $T_{n_k}(\omega)$ are bounded measures on the corresponding balls and define distribution, belonging to the space $\mathcal{A}'$. Thus

$$\phi_{n_k}(z) = M\exp\{zT_{n_k}\}(\omega) = \int_{U_r} \exp\{zx\}P_{p,T_{n_k}}(\mathrm{d}x) = L'(P_{p,T_{n_k}})(z).$$

But as usual we have for independent random variables (see (7)):

$$\phi_{n_k}(z) = \prod_{j=1}^{n_k} Me^{z(\xi_j(\omega)/n_k)} = [(1+e^{z/n_k})/2]^{n_k}. \tag{8}$$

If $p = 2$ then the Bernoulli measure μ is not bounded. It can be realized as a distribution $P_2 \in \mathcal{A}'(C_2)$, see [13], [2]. It is also possible to introduce distributions $P_{2,T_{n_k}} \in \mathcal{A}'(C_2)$ in this case. Formula (8) is valid. To prove this formula, we need to use the property that the Laplace transform of a convolution of distributions is a product of Laplace transforms.
Set

$$\phi(z;a) = [(1+e^{z/a})/2]^a$$

for $a \in Z_p$, $a \neq 0$. This function is defined for sufficiently small z. As $L' : \mathcal{A}' \to \mathcal{A}_0$ is an isomorphism, there exists a distribution $\kappa_a \in \mathcal{A}'$ such that $L'(\kappa_a)(z) = \phi(z;a)$.

Set $r(p) = 1/p$ if $p \neq 2$ and $r(2) = \frac{1}{4}$.

Lemma 1. *Let a number a, $a \neq 0$, belong to Z_p and let $\{n_k\}_{k=1}^{\infty}$ be a sequence of natural numbers such that $\lim_{k\to\infty} n_k = a$ in Q_p. Then the functional sequence $\{\phi_{n_k}(z)\}$ converges to $\phi(z;a)$ in $\mathcal{A}_0(C_p)$.*

Proof. We can write $n_k = m_k + a$, where $m_k \to 0$. Then $|n_k|_p = |a|_p$ for large k. Further

$$\phi_{n_k}(z) = [(1+e^{z/n_k})/2]^a[(1+e^{z/n_k})/2]^{m_k} = (1+\Delta_{n_k})^a(1+\Delta_{n_k})^{m_k}.$$

Then we have

$$\begin{aligned}
&|(1+\Delta_{n_k})^a - [1+(e^{z/a}-1)/2]^a|_p \\
&\quad = |\exp\{a\ln(1+\Delta_{n_k})\} - \exp\{a\ln(1+(e^{z/a}-1)/2)\}|_p \\
&\quad = |a|_p|\ln(1+\Delta_{n_k}) - \ln(1+(e^{z/a}-1)/2)|_p \\
&\quad = |a/2|_p|e^{z/n_k} - e^{z/a}|_p = |m_k|_p|z|_p/|2a|_p \to 0
\end{aligned}$$

uniformly on the ball $\mathcal{U}_r$, $r = r(p)|a|_p$. Here we have used the isometry proper-

ties of the exponential function e^z and the logarithmic function $\ln(1+z)$ on $\mathcal{U}_r$, see, for example [11], [16]. Note also that these functions are well defined for $z \in \mathcal{U}_r$.
We can prove in the same way that

$$|(1+\Delta_{n_k})^{m_k} - 1|_p \to 0$$

uniformly on the same ball. □

The following theorem is a simple consequence of the previous lemma.

Theorem 2 (Bernoulli theorem for generalized functions). *Let a sequence of natural numbers $\{n_k\}$ approach a p-adic integer a, $a \neq 0$. Then the sequence of distributions $\{P_{p,T_{n_k}}\}$ converges $\mathcal{A}$-weakly to the distribution κ_a.*

Example 1. Choose $a = 1$. Then $\kappa_1 = (\delta_0 + \delta_1)/2$. Hence the limit distribution for sums $T_{n_k}(\omega)$ coincides with the original distribution of the random variables $\xi_i(\omega)$. For example, we can choose $n_k = 1 + p^k$, $k = 0, 1, \ldots$ and (in the sense of $\mathcal{A}$-weak convergence):

$$\frac{\xi_1(\omega) + \cdots + \xi_{1+p^k}(\omega)}{1+p^k} \to \xi_{\lim}(\omega), \tag{9}$$

where $\xi_{\lim}(\omega)$ is a random variable with the distribution κ_1. When we calculated the mean value (5) with respect to a Bernoulli cylindrical measure μ, we used only rational numbers. Hence the same answer is also valid for the p-adic case. That is why the asymptotic formula (9) means

$$\frac{1}{2^{p^k+1}} \sum_{j=0}^{1+p^k} f\left(\frac{j}{1+p^k}\right) C_{1+p^k}^j \to (f(0) + f(1))/2 \tag{10}$$

for every entire analytic function $f(x)$.
Let $p = 3$ then $n_k = 4^{3^k} \to 1$ and

$$\frac{\xi_1(\omega) + \cdots + \xi_{4^{3^k}}(\omega)}{4^{3^k}} \to \xi_{\lim}(\omega). \tag{11}$$

This asymptotic formula means

$$(1/2^{4^{3^k}}) \sum_{j=0}^{4^{3^k}} f(j/4^{3^k}) C_{4^{3^k}}^j \to (f(0) + f(1))/2. \tag{12}$$

Let $p = 2$ then, for example, $n_k = (2m+1)^{2^k} \to 1$ for $m = 1, 2, \ldots$ and

$$\frac{\xi_1(\omega) + \cdots + \xi_{(2m+1)^{2^k}}(\omega)}{(2m+1)^{2^k}} \to \xi_{\lim}(\omega). \tag{13}$$

Example 2. Choose $a = 2$ then a limit distribution is $\kappa_2 = \frac{1}{4}\delta_0 + \frac{1}{2}\delta_{1/2} + \frac{1}{4}\delta_1$. Moreover if $a = m$ is a natural number then the limiting distribution is

$$\kappa_m = \frac{1}{2^m} \sum_{j=0}^{m} C_m^j \delta_{j/m}.$$

Thus, if $n_k = m + p^k$ then

$$(14) \qquad \frac{1}{2^{m+p^k}} \sum_{j=0}^{m+p^k} C_{m+p^k}^j f(j/(m+p^k)) \to \frac{1}{2^m} \sum_{j=0}^{m} C_m^j f(j/m)$$

for every entire analytic function f.

4. BERNOULLI THEOREM FOR PROBABILITY DISTRIBUTIONS

Now we wish to extend Theorem 2 to the weak convergence of probability measures. Hence, we first of all restrict our considerations to the case $p \neq 2$. To show the main ideas of the proof, we restrict our considerations to the case where n_k *approaches 1 in* Q_p. A general proof will be presented in the nonsymmetric case.

Now the random variables $T_{n_k}(\omega)$ take their values in the ring of p-adic integers Z_p (for sufficiently large k). Hence all probability distributions $P_{p,T_{n_k}}$ are defined on Z_p. We shall use Mahler's integration theory [11], [16] on Z_p. Set $C(x,k) = C_x^k = (x(x-1)\cdots(x-k+1))/k!$. Every function $f \in C(Z_p, Q_p)$ is expanded into a series (Mahler expansion)

$$(15) \qquad f(x) = \sum_{k=0}^{\infty} a_k C(x,k).$$

This series converges uniformly on Z_p because $a_k \to 0$, $k \to \infty$, and $\|f\| = \max\{|a_k|_p\}$. If ν is a bounded measure on Z_p then

$$\int_{Z_p} f(x)\nu(\mathrm{d}x) = \sum a_k \int_{Z_p} C(x,n)\nu(\mathrm{d}x).$$

In this section $\xi_{\lim}$ is a random variable with the symmetric distribution κ_1.

Further, $S_n(\omega)$ is the same as in (1). Set $\lambda_{mn} = MC(S_n(\omega), m)$.

Lemma 2. *If $m \leq n$ then $\lambda_{mn} = C_n^m/2^m$ and if $m > n$ then $\lambda_{nm} = 0$.*

Proof. It is sufficient to use the formula [11], [16]:

$$e^{tx} = \sum_{m=0}^{\infty} (e^t - 1)^m C(x,m).$$

Thus

$$M\exp\{tS_n(\omega)\} = \sum_{m=0}^{\infty} (e^t - 1)^m \lambda_{mn}.$$

And we have also

$$M\exp\{tS_n(\omega)\} = [(1+e^t)/2]^n = \sum_{m=0}^{n} C_n^m (e^t - 1)^m / 2^m. \qquad \square$$

Lemma 3. *If $m \neq 0, 1$ then $\lambda_{mn_k} \to 0$, $n_k \to 1$.*

Proof. We have

$$\lambda_{mn_k} = n_k(n_k - 1)\cdots(n_k - m + 1)/m!2^m.$$

If $m \neq 0, 1$ then the factor $n_k - 1$ is present in λ_{mn_k} and that is why $\lambda_{mn_k} \to 0$, $k \to \infty$. □

Lemma 4. *Let $f(x)$ be a continuous function and suppose the coefficients a_0 and a_1 in the Mahler expansion are equal to zero. Then*

$$Mf(S_{n_k}(\omega)) \to 0,\ k \to \infty.$$

Proof. We can write $f(x) = \sum_{m=2}^{N} a_m C(x,m) + \sum_{m=N+1}^{\infty} a_m C(x,m) = g_N(x) + u_N(x)$. We choose N so large that

$$|u_N(S_{n_k}(\omega))|_p = \left| \int_{\Omega_B} u_N(S_{n_k}(\omega)) P_p(\mathrm{d}\omega) \right|_p \leq \|P_p\| \max_{x\in Z_p} |u_N(x)|_p < \epsilon.$$

But $Mg_N(S_{n_k}(\omega)) \to 0$ by Lemma 3.

Note that the coefficient $\lambda_{1n_k} = n_k/2 \to \frac{1}{2}$, $n_k \to 1$, and $\lambda_{0n_k} = 1$. □

Theorem 3. *If $n_k \to 1$ in Q_p then*

$$S_{n_k}(\omega) \to \xi_{\lim}(\omega),\ k \to \infty, \tag{16}$$

in the sense of weak convergence of probability distributions.

Proof. By the previous lemmas it is sufficient to observe that $a_0 + a_1/2 = (f(0) + f(1))/2$. □

Theorem 4 (Bernoulli theorem). *If $n_k \to 1$ in Q_p then*

$$T_{n_k}(\omega) \to \xi_{\lim}(\omega) \tag{17}$$

in the sense of weak convergence of probability distributions.

Proof. Using the estimate $|S_n(\omega)|_p \leq 1$, we get

$$|S_{n_k}(\omega) - T_{n_k}(\omega)|_p \leq |n_k - 1|_p$$

for large k.

Every function $f \in C(Z_p, Q_p)$ is uniformly continuous on Z_p. Hence

$$\sup_{\omega\in\Omega} |f(S_{n_k}(\omega)) - f(T_{n_k}(\omega))|_p \to 0,\ k \to \infty.$$

To finish the proof, we use the mean value estimate

$$|Mf(S_{n_k}(\omega)) - Mf(T_{n_k}(\omega))|_p \leq \|P_p\| \max_{\omega\in\Omega} |f(S_{n_k}(\omega)) - f(T_{n_k}(\omega))|_p. \quad □$$

As a consequence of this theorem we get (10), (12), (13) for every continuous function $f : Z_p \to Q_p$.

It is an *open problem* to prove weak convergence if $a \neq 1$.

Let $A \in F(Z_p)$ be an arbitrary clopen subset of Z_p. The characteristic function χ_A of this set, $\chi_A(x) = 1$, $x \in A$, and $\chi_A(x) = 0$, $x \notin A$, is a continuous function. Hence

$$P_p(\{\omega \in \Omega_B : T_{n_k}(\omega) \in A\}) = M\chi_A(T_{n_k}(\omega)) \to (\chi_A(0) + \chi_A(1))/2.$$

Now choose $A = U_\epsilon$, $\epsilon < 1$, then we have

$$P_p\left(\left|\frac{\xi_1(\omega) + \cdots + \xi_{n_k}(\omega)}{n_k}\right|_p \le \epsilon\right) \to \frac{1}{2}, \ k \to \infty, \tag{18}$$

and

$$P_p\left(\left|\frac{\xi_1(\omega) + \cdots + \xi_{n_k}(\omega)}{n_k} - 1\right|_p \le \epsilon\right) \to \frac{1}{2}, \ k \to \infty. \tag{19}$$

If $c \neq 0, 1$ then for sufficiently small ϵ

$$P_p\left(\left|\frac{\xi_1(\omega) + \cdots + \xi_{n_k}(\omega)}{n_k} - c\right|_p \le \epsilon\right) \to 0, \ k \to \infty. \tag{20}$$

We can consider (18), (19), (20) as a p-adic analogue of the law of large numbers.

Set $S_r = \{x \in Z_p : |x|_p = r\}$, $r = 1/p^l$, $l = 0, 1, 2, \ldots$ These are spheres in Z_p with center in zero. These sets are also clopen and thus belong to the algebra $F(Z_p)$. We have

$$P_p\left(\left|\frac{\xi_1(\omega) + \cdots + \xi_{n_k}(\omega)}{n_k}\right|_p = 1\right) \to \frac{1}{2}, \ k \to \infty, \tag{21}$$

and

$$P_p\left(\left|\frac{\xi_1(\omega) + \cdots + \xi_{n_k}(\omega)}{n_k}\right|_p = \frac{1}{p^l}\right) \to 0, \ k \to \infty, \tag{22}$$

for $l = 1, 2, \ldots$

5. COMBINATORIAL CONSEQUENCES

This section is more interesting for specialists in the theory of numbers and non-Archimedean analysis; specialists in the theory of probability might wish to continue with the next section.

It is well known that the usual Bernoulli theorem has a simple combinatorial interpretation (we can compute probabilities directly and get the asymptotics of these expressions). What about the p-adic case? We restrict our considerations to the case $n_k \to 1$. Here

$$A_{l,n} = \{\omega \in \Omega_B : |T_{n_k}(\omega)|_p = 1/p^l\} = \{\omega \in \Omega_B : |S_{n_k}(\omega)|_p = 1/p^l\}.$$

What is the meaning of the events $A_{l,n}$? The event $A_{l,n}$ means that exactly p^l divides the sum of units in the first n trials (already p^{l+1} does not divide this sum). The meaning of (22) is that the p-adic probability of the events A_{l,n_k}, $l \neq 0$, approaches 0, $k \to \infty$, i.e. the p-primary part of $P_p(A_{l,n_k})$ goes to infinity. On the other hand, we can compute this probability directly using combinatorial computations.

Let $m(l,n)$ be the number of $i \in I_n$ such that p^l is the power of p dividing the sum $|i| = i_1 + \cdots + i_n$. There is no problem in writing an expression for $m(l,n)$ as a sum of C_n^j. We have $P_p(A_{l,n}) = m(l,n)/2^n$. But p does not divide 2^n and that is why (22) is equivalent to the property that the p-primary part of $m(l, n_k)$ goes to infinity. Then $P_p(A_{0,n}) = m(0,n)/2^n$ and (21) means that $m(0,n_k)/2^{n_k} \to \frac{1}{2}$ in Q_p.

For example, let $p = 3$ and $n_k = 1 + 3^k$. Then $2^{n_k} \to -2$ in Q_3. Thus $m(0,n_k) \to -1$, i.e. $m(0,n_k) + 1 \to 0$. That is why the p-primary part of $m(0,n_k) + 1$ goes to infinity.

Note that there is no problem in finding (22) directly, using the combinatorial expression for the probability. For example, $m(1,n)$ is equal to the sum of C_n^{jp}, where p does not divide j. But

$$|m(1, 1+p^k)|_p \le \max |C^{jp}_{1+p^k}|_p$$
$$= \left| \frac{p^k}{pj} \cdot \frac{p^k - p}{p} \cdot \frac{p^k - 2p}{2p} \cdots \frac{p^k - p(j-1)}{p(j-1)} \right|_p = \frac{1}{p^{k-1}}.$$

Here we get more exact asymptotics of (22): the probability $P(A_{1,1+p^k})$ decreases by a factor p after every step.

There is no problem to generalize this result to every sequence $n_k \to 1$ in Q_p. We have for large $k : n_k = 1 + j_k p^{m_k}$, where $m_k \to \infty$, $k \to \infty$, and j_k are such natural numbers that p does not divide j_k. Then

$$|C^{jp}_{n_k}|_p = \left| \frac{(p^{m_k} j_k)(p^{m_k} j_k - p) \cdots (p^{m_k} j_k - (j-1)p)}{p(2p) \cdots (j-1)p(jp)} \right|_p = \frac{1}{p^{m_k - 1}}.$$

Here the probability $P(A_{1,n_k})$ decreases by a factor $p^{m_{k+1} - m_k}$ after every step.
Now consider the event A_{l,n_k}, $l \ne 0$. Here

$$|C^{jp^l}_{n_k}|_p = \left| \frac{(p^{m_k} j_k) \cdots (p^{m_k} j_k - jp^l + p)}{p \cdots (jp^l - p)(jp^l)} \right|_p = \frac{1}{p^{k-l}}.$$

But the situation is not so easy for the event A_{1,n_k}. At the moment, it is not clear how to show directly that $P_p(A_{1,n_k}) \to \frac{1}{2}$ in Q_p.
We must study the expression:

$$\gamma_{n_k} = 2^{-n_k} \sum_j C^j_{n_k},$$

where p does not divide j. The simplest case is $n_k = 1 + p^k$. In that case the normalization factor 2^{1+p^k} approaches $2\omega_p(2)$, where $\omega(x)$ is a Teichmuller character of $x \in Q_p$. The Teichmuller character $\omega_p(2)$ is the $(p-1)$th root of 1 in Q_p. In particular, $\omega_3(2) = -1$ and we must show that

$$\text{(23)} \qquad \sum_{j:(3,j)=1} C^j_{1+3^k} \to -1$$

in Q_3.

Thus, as we have seen, the p-adic Bernoulli theorem has generated a lot of surprising new p-adic asymptotics for simple combinatorial expressions.

What is the intuitive meaning of (21) and (22)? Consider a large number of trials, $n_k = 1 + j_k p^{m_k}$. We are interested in the random variables $S_{n_k}(\omega)$. From the p-adic point of view the number of ω such that p divides $S_{n_k}(\omega)$ is approximately equal to the number of ω such that p does not divide $S_{n_k}(\omega)$. This is a kind of p-adic symmetry.

6. NONSYMMETRIC BERNOULLI DISTRIBUTIONS

We shall study the case of the random variables $\xi_n(\omega) = 0, 1$ with probabilities q and $q' = 1 - q$, where q is a p-adic number. It is useful to consider a general p-adic probability space $(\Omega, F(\Omega), P)$, where Ω is a compact ultrametric space, $F(\Omega)$ is the algebra of clopen subsets, P is the bounded normalized Q_p-valued measure on $F(\Omega)$.

The first question is about the existence of a probability space for a sequence of independent equally distributed random variables. Are there restrictions on the probability q? The answer is 'yes', the probability q must belong to Z_p. The proof is very easy. Let us consider the clopen sets

$$C_n = \{\omega \in \Omega : \xi_1(\omega) = 0, \ldots, \xi_n(\omega) = 0\}.$$

Then $P(C_n) = q^n$ and $|P(C_n)|_p = |q|_p^n \to \infty$ if $|q|_p > 1$. The same proof can be applied to the situation when $\{\eta_n(\omega)\}_{n=1}^{\infty}$ is an infinite sequence of independent equally distributed RV and $\eta_n(\omega) = x_1, \ldots, x_m \in Q_p$ with probabilities $q_1, \ldots, q_m \in Q_p$, $\sum_{j=1}^{m} q_j = 1$. Here all these probabilities must belong to Z_p.

This fact is unexpected from the point of view of the p-adic frequency theory of probability [5, 3].

The next question is about the existence of such p-adic probability spaces. The simplest construction is based on product measures (generalization of the Bernoulli probability), see [3].

We consider again the space of sequences Ω_B and define a measure of the cylindrical set B_i, $i \in I_n$, by the equality $\mu_q(B_i) = q^{n-|i|}(1-q)^{|i|}$, where $|i| = i_1 + \cdots + i_n$. The map $\mu_q : F(\Omega_B) \to Q_p$ is a bounded normalized measure in p-adic probability.

We begin by studying $\mathcal{A}$-weak convergence of distributions. Here we consider the more general case that a normalization sequence is different from $\{n_k\}$. Let $\{c_k\}_{k=1}^{\infty}$ be a sequence of p-adic numbers and $\lim_{k\to\infty} c_k = c$ and $c \neq 0$. Consider normalized sums

$$T_{n_k,c}(\omega) = S_{n_k}(\omega)/c_k.$$

We have characteristic functions

$$\phi_{n_k}(z) = M \exp\{zT_{n_k,c}(\omega)\} = (1 + q'(e^{z/c_k} - 1))^{n_k}.$$

Set $\phi(z, q, a, c) = (1 + q'(e^{z/c} - 1))^a$. This function belongs to $\mathcal{A}_0$ for small $z \in C_p$. There exists a distribution $\kappa_{q,a,c} \in \mathcal{A}'$ with Laplace transform $\phi(z, q, a, c)$.

Everywhere below $\{n_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ are sequences of natural and p-adic numbers respectively with $\lim_{k\to\infty} n_k = a$, $\lim_{k\to\infty} c_k = c$ and $c \neq 0$.

Lemma 5. *The sequence* $\{\phi_{n_k}(z)\}$ *converges to the function* $\phi(z,q,a,c)$ *in the space* $\mathcal{A}_0$.

The proof is the same as that of Lemma 1. As a consequence of this lemma and Theorem 1, we get

Theorem 5. *The sequence of distributions* $\{P_{T_{n_k,c}}\}$ *converges* $\mathcal{A}$*-weakly to the distribution* $\kappa_{q,a,c}$.

Further we shall study the weak convergence of probability distributions. We begin with the distributions of the sums $S_{n_k}(\omega)$. Here the limiting characteristic function is $\phi(z,q,a) = (1+q'(e^z-1))^a$. We are interested in the boundedness of the corresponding distribution $\kappa_{q,a} \equiv \kappa_{q,a,1}$. Using the expansion of $\phi(z,q,a)$, we get

$$\lambda_m^q = \int_{Z_p} C(x,m)\kappa_{q,a}(\mathrm{d}x) = (1-q)^m C(a,m).$$

But $|C(a,m)|_p \le 1$ for $a \in Z_p$ and that is why the distribution $\kappa_{q,a}$ is a bounded measure on Z_p. We compute

$$\lambda_{mn_k}^q = MC(S_{n_k}(\omega),m) = (1-q)^m C_{n_k}^m.$$

Thus $\lambda_{mn_k}^q \to \lambda_m^q$, $n_k \to a$. In the same way as in the previous sections, we get

Theorem 6. *The sequence of probability distributions* $\{P_{S_{n_k}}\}$ *converges weakly to* $\kappa_{q,a}$.

Now we consider the case of the normalized random variables $T_{n_k,c}(\omega)$. As $|T_{n_k,c}(\omega)|_p \le 1/|c|_p$, we can study weak convergence in the space $C(U_{1/|c|_p}, Q_p)$. Define the functional $\beta_{q,a,c}$ on this space by

$$(\beta_{q,a,c}, g) = \int_{Z_p} g(x/c)\kappa_{q,a}(\mathrm{d}x).$$

This functional is bounded. Thus it is a bounded measure on $U_{1/|c|_p}$.

Proposition 2. *The distribution* $\kappa_{q,a,c}$ *is a bounded measure on* $U_{1/|c|_p}$.

Proof. Compute the Laplace transform of $\beta_{q,a,c}$:

$$L'(\beta_{q,a,c})(z) = \int_{Z_p} \exp\{zx/c\}\kappa_{q,a}(\mathrm{d}x) = \phi(z,q,a,c).$$

Thus, $\beta_{q,a,c}$ coincides with $\kappa_{q,a,c}$. □

Theorem 7. *The sequence of probability distributions* $\{P_{T_{n_k,c}}\}$ *converges weakly on the space* $C(U_{1/|c|_p}, Q_p)$ *to* $\kappa_{q,a,c}$.

Proof. Let $f \in C(U_{1/|c|_p}, Q_p)$. Then

$$Mf(S_{n_k}(\omega)/c) = \int_{Z_p} f(x/c) P_{S_{n_k}}(\mathrm{d}x)$$

$$\to \int_{Z_p} f(x/c)\kappa_{q,a}(\mathrm{d}x) = \int_{U_{1/|c|_p}} f(x)\kappa_{q,a,c}(\mathrm{d}x).$$

Further,

$$|Mf(S_{n_k}(\omega)/c) - Mf(T_{n_k,c}(\omega))|_p$$

$$\leq (\operatorname{var} P) \max_{\omega \in Z_2} |f(S_{n_k}(\omega)/c) - f(T_{n_k,c}(\omega))|_p.$$

To finish the proof, we need only to use the fact that the function $f(x)$ is uniformly continuous on the ball $U_{1/|c|_p}$. □

7. THE CENTRAL LIMIT THEOREM

Here we restrict our considerations to the case of a symmetric random variable $\xi_n(\omega) = 0, 1$ with probabilities $\frac{1}{2}$. We study a p-adic asymptotic of normalized sums $G_n(\omega) = (S_n(\omega) - MS_n(\omega))/\sqrt{DS_n(\omega)}$. Here $MS_n = n/2$, $D\xi_n = M\xi^2 - (M\xi)^2 = \frac{1}{4}$ and $DS_n = n/4$. Hence

$$G_n(\omega) = \frac{S_n(\omega) - n/2}{\sqrt{n}/2} = \sum_{j=1}^{n} \frac{2\xi_n}{\sqrt{n}} - \sqrt{n}.$$

At the moment, we can only find a form of the limiting distribution (is it an analogue of the Gaussian distribution?) and prove the $\mathcal{A}$-weak convergence. Compute the characteristic function of the random variables $G_n(\omega)$:

$$\psi_n(z) = Me^{zG_n(\omega)} = (ch\{z/\sqrt{n}\})^n,$$

where $chz = (e^z + e^{-z})/2$ is the hyperbolic cos. Set $\psi(z, a) = (ch\{z/\sqrt{a}\})^a$, $a \in Z_p$, $a \neq 0$. This function belongs to the space $\mathcal{A}_0$ and there exists a distribution $\gamma_a \in \mathcal{A}'$ with Laplace transform $L'(\gamma_a)(z) = \psi(z, a)$. In the same way as in the previous sections, we prove

Theorem 8 (Central limit theorem). *If the sequence of natural numbers $\{n_k\}$ approaches a, $a \neq 0$, then the sequence of distributions $\{P_{G_{n_k}}\}$ approaches the distribution γ_a.*

Remark 3. A non-Archimedean analogue of the Gaussian distribution was introduced in the author's papers [7, 8]. It was defined as the distribution (generalized function) with the Laplace transform $e^{z^2/2b}$. It differs from the limiting distribution γ_a. It was proved by M. Endo and the author [2] that p-adic Gaussian distributions are unbounded. There are no bounded measures corresponding to these distributions (but our proof works only in the case $p \neq 2$).

Theorem 9. *The distribution γ_1 is a bounded measure on Z_p.*

Proof. As

$$chz = 1 + \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (e^z - 1)^n,$$

then $\lambda_{m,\infty} = \int_{Z_p} C(x,m)\gamma_a(\mathrm{d}x) = (-1)^m/2, m = 2,3,\ldots$ and $\lambda_{1,\infty} = 0$, $\lambda_{0,\infty} = 1$. Thus, there exists a bounded measure corresponding to the distribution γ_1.

Hence the limiting distribution γ has more ordinary properties than the Gaussian distributions of [7, 8, 3] (we hope that the analogue of Theorem 9 is valid also for $a \neq 1$). □

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