Hankel operators on the Dirichlet space

Liankuo Zhao
School of Mathematics and Computer Science, Shanxi Normal University, No. 1, Gongyuan Street, Linfen 041004, PR China

Abstract
We study (small) Hankel operators on the Dirichlet space $\mathcal{D}$ with symbols in a class of function space, and show that such (small) Hankel operators are closely related to the corresponding Hankel operators on the Bergman space $L^2_a$ and the Hardy space $H^2$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk, $dA$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. The Dirichlet space $\mathcal{D}$ consists of analytic function $f$ on $\mathbb{D}$ such that

$$f(0) = 0 \quad \text{and} \quad D(f) = \int_{\mathbb{D}} |f'|^2 dA < \infty.$$

It is well known that $\mathcal{D}$ is a reproducing kernel Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z), \quad f, g \in \mathcal{D},$$

and reproducing kernel

$$R_\lambda(z) = \log \frac{1}{1 - \lambda z}, \quad \lambda, z \in \mathbb{D}.$$

The Sobolev space $\mathcal{S}$ is the completion of the space of functions $f$ in $C^1(\mathbb{D})$ under the norm

$$\|f\| = \left\{ \int_{\mathbb{D}} |f|^2 dA + \int_{\mathbb{D}} \left( \left| \frac{\partial f}{\partial z} \right|^2 + \left| \frac{\partial f}{\partial \bar{z}} \right|^2 \right) dA \right\}^{1/2} < \infty.$$
$S$ is a Hilbert space with inner product
\[ (f, g) = \int_{\mathbb{D}} f \, dA \int_{\mathbb{D}} g \, dA + \int_{\mathbb{D}} \left( \frac{\partial f}{\partial z} \frac{\partial g}{\partial \overline{z}} + \frac{\partial g}{\partial z} \frac{\partial f}{\partial \overline{z}} \right) \, dA, \quad f, g \in S, \]
and obviously, the Dirichlet space $D$ is the closed subspace of $S$ consisting of all holomorphic functions $f$ with $f(0) = 0$.

Denote
\[ L^{\infty,1} = \left\{ \phi \in C^1(\mathbb{D}) \mid \phi, \frac{\partial \phi}{\partial z}, \frac{\partial \phi}{\partial \overline{z}} \in L^{\infty}(\mathbb{D}) \right\}, \]
where $L^\infty(\mathbb{D})$ is the space of bounded Lebesgue functions on $\mathbb{D}$.

Let $P$ be the orthogonal projection from $S$ onto $D$, then for $\phi \in L^{\infty,1}$, define the Toeplitz operator $T_\phi : D \to D$,
\[ T_\phi f = P(\phi f), \]
(big) Hankel operator $H_\phi : D \to D^\perp$,
\[ H_\phi f = (1 - P)(\phi f) \]
and small Hankel operator $\Gamma_\phi : D \to D$, $\Gamma_\phi f = P(f(\phi f))$
for $f \in D$, where $J$ is the unitary $S \to S$ defined by $J h(z) = h(\overline{z})$ for $h \in S$, and $D^\perp$ is the orthogonal complement of $D$ in $S$.

The Fredholm properties [2] and algebraic properties [4] of Toeplitz operators on $D$ defined by symbols in $L^{\infty,1}$ have been studied. In this paper, we consider the compactness of Hankel operators, commutativity of small Hankel operators, commutativity of small Hankel operator and Toeplitz operator, etc. In fact, we convert such problem into the corresponding one in the Bergman space or the Hardy space, which has been studied in [1,5–8].

2. Main results

In this section, we present the main results and their proofs.

Let $L^2(\mathbb{D})$ be the Hilbert space of square integrable Lebesgue functions on $\mathbb{D}$. Throughout this paper, $(\cdot, \cdot), (\cdot, \cdot)_2$ denote the inner product in $S$ and in $L^2(\mathbb{D})$ respectively.

Let $L^2_0$ be the Bergman space on $\mathbb{D}$, which consists all holomorphic functions in $L^2(\mathbb{D})$, $P_a$ the Bergman projection from $L^2(\mathbb{D})$ onto $L^2_0$. Let
\[ K_\lambda(z) = \frac{1}{(1 - \lambda z)^2}, \quad \lambda, z \in \mathbb{D}, \]
be the reproducing kernel of $L^2_0$, $k_\lambda$ the normalization of $K_\lambda$.

For any $\phi \in L^\infty(\mathbb{D})$, $\hat{T}_\phi$ denotes the Toeplitz operator on $L^2_0$ defined by
\[ \hat{T}_\phi f = P_a(\phi f), \quad f \in L^2_0. \]

Let
\[ \hat{\phi}(z) = (\phi k_\lambda, k_\lambda)_2. \]
It is well known that $\hat{\phi}(z)$ is the Berezin transform of $\phi$ and plays an important role in the study of Toeplitz operators on $L^2_0$.

Denote $\hat{H}_\phi$ the corresponding Hankel operator on $L^2_0$, i.e.
\[ \hat{H}_\phi(f) = (1 - P_a)(\phi f), \quad f \in L^2_0. \]

The following theorem is one of main results in this paper.

**Theorem 2.1.** Let $\phi, \psi \in L^{\infty,1}$, then

1. $H^*_\phi H_\psi$ is compact on $D$ if and only if $\hat{H}^*_\psi \hat{H}_\phi$ is compact on $L^2_0$. In particular, $H_\psi$ is compact on $D$ if and only if $\hat{H}_\psi$ is compact on $L^2_0$.
2. $M_\psi H_\phi$ is compact if and only if $\hat{T}_\phi \hat{M}_\psi$ is compact. In particular, $M_\psi$ is compact if and only if $|\hat{\psi}(\lambda)| \to 0$ as $\lambda \to \partial \mathbb{D}$.

For the proof of Theorem 2.1, we give some discussion of the Toeplitz operators on the Dirichlet space.

**Lemma 2.2.** The identity operator $i$ from $D$ into $L^2_0$ is compact.
Proof. This result is well known, see [4, Lemma 12]. □

Denote \( U : D \to L^2_D \) to be the natural unitary that takes \( f \) into \( f' \). Then we have the following result.

**Proposition 2.3.** Let \( \psi \in \mathcal{L}^{\infty,1} \), then \( T_\psi = K_0 + U^* \tilde{T}_\psi U \) for some compact operator \( K_0 \) on \( D \).

**Proof.** For \( f, g \in D \), by direct computation, we have

\[
\langle T_\psi f, g \rangle = \langle \psi f, g \rangle = \left( \frac{\partial \psi}{\partial \bar{z}} f + \frac{\partial f}{\partial z} \frac{\partial \psi}{\partial z} + \frac{\partial g}{\partial z} \right)_{\bar{z}} = (U^* \tilde{T}_\psi if, g) + (U^* \tilde{T}_\psi Uf, g). \quad (1)
\]

Hence

\[
T_\psi = K_0 + U^* \tilde{T}_\psi U,
\]

where, by Lemma 2.2, \( K_0 = U^* \tilde{T}_\psi \psi \bar{z} i \) is compact on \( D \). □

**Remark 2.4.** In Eq. (1), we use the fact that for \( f \in D \), \( g \in L^2_D \), and \( \psi \in L^{\infty,1}(D) \),

\[
\langle \psi f, g \rangle_2 = (\tilde{T}_\psi if, g)_2,
\]

which will be used repeatedly in the following.

If \( T \) equals a finite sum of finite products of Toeplitz operators with symbols in \( \mathcal{L}^{\infty,1} \) on \( D \), let \( \tilde{T} \) be the corresponding operator on \( L^2_D \) with the same symbols, i.e., if \( T = \sum_{i,j} T_{\psi_{i1}} T_{\psi_{j2}} \cdots T_{\psi_{i|\lambda|}} \), then

\[
\tilde{T} = \sum_{i,j} \tilde{T}_{\psi_{i1}} \tilde{T}_{\psi_{j2}} \cdots \tilde{T}_{\psi_{i|\lambda|}}.
\]

Denote

\[
\tilde{T}(\lambda) = (\tilde{T}k_\lambda, k_\lambda)_2.
\]

Then we have the following result.

**Corollary 2.5.** If \( T \) equals a finite sum of finite products of Toeplitz operators, with symbols in \( \mathcal{L}^{\infty,1} \), on \( D \), then \( T \) is compact if and only if \( \tilde{T}(\lambda) \to 0 \) as \( \lambda \to \partial D \).

**Proof.** By Proposition 2.3, it is easy to check that

\[
T = K + U^* \tilde{T} U,
\]

for some compact operator \( K \) on \( D \). Hence \( T \) is compact if and only if \( \tilde{T} \) is compact. By [1, Theorem 2.2], \( \tilde{T} \) is compact if and only if

\[
\tilde{T}(\lambda) = (\tilde{T}k_\lambda, k_\lambda)_2 \to 0 \quad \text{as} \quad \lambda \to \partial D. \quad \square
\]

**Remark 2.6.** By Proposition 2.3 and Corollary 2.5, for \( \varphi, \psi, \mu \in \mathcal{L}^{\infty,1} \), \( \sigma_c(T_\psi) = \sigma_c(T_\mu) \), \( T_\psi T_\varphi - T_\mu \) is compact on \( D \) if and only if \( \tilde{T}_\psi \tilde{T}_\varphi - \tilde{T}_\mu \) is compact on \( L^2_D \). And for the corresponding results in the Bergman space, see [1,6].

**Proposition 2.7.** Let \( \varphi, \psi \in \mathcal{L}^{\infty,1} \), then

(i) \( M_{\varphi}^* M_{\psi} = K_1 + U^* \tilde{H}_\psi \tilde{H}_\varphi U \) for some compact operator \( K_1 \) on \( D \).

(ii) \( H^*_\psi H_\varphi = K_2 + U^* \tilde{H}_\psi \tilde{H}_\varphi U \) for some compact operator \( K_2 \) on \( D \).

**Proof.** For any \( f, g \in D \),

(i) \[
\langle M_{\varphi}^* M_{\psi} f, g \rangle = \langle \varphi f, \psi g \rangle = \int_D \varphi f \, dA \int_D \psi g \, dA + \left( \frac{\partial \varphi}{\partial \bar{z}} f + \frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial z} + \psi \frac{\partial g}{\partial z} \right)_{\bar{z}} + \left( \frac{\partial \varphi}{\partial \bar{z}} \frac{\partial \psi}{\partial z} g \right)_{\bar{z}} = \int_D \varphi f \, dA \int_D \psi g \, dA + \langle \tilde{T}_\psi \tilde{H}_\varphi f, g \rangle + \langle \tilde{T}_\psi \tilde{H}_\varphi Uf, g \rangle + \langle \tilde{T}_\psi \tilde{H}_\varphi \psi f, g \rangle + \langle \tilde{T}_\psi \tilde{H}_\varphi Uf, g \rangle.
\]

(ii) \[
\langle H^*_\psi H_\varphi f, g \rangle = \langle \varphi f, \psi g \rangle = \int_D \varphi f \, dA \int_D \psi g \, dA + \left( \frac{\partial \varphi}{\partial \bar{z}} f + \frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial z} + \psi \frac{\partial g}{\partial z} \right)_{\bar{z}} + \left( \frac{\partial \varphi}{\partial \bar{z}} \frac{\partial \psi}{\partial z} g \right)_{\bar{z}} = \int_D \varphi f \, dA \int_D \psi g \, dA + \langle \tilde{T}_\psi \tilde{H}_\varphi f, g \rangle + \langle \tilde{T}_\psi \tilde{H}_\varphi Uf, g \rangle + \langle \tilde{T}_\psi \tilde{H}_\varphi \psi f, g \rangle + \langle \tilde{T}_\psi \tilde{H}_\varphi Uf, g \rangle.
\]
Denote $L_ψ$ the linear functional on $D$ defined as $L_ψ f = \int_D ψ f \, dA$ $(f \in D)$, then

$$M_ψ^*M_ψ = K_1 + U^*\tilde{T}_ψU,$$

where

$$K_1 = L_ψ^*L_ψ + i\left(\frac{\partial}{\partial ψ} + \frac{\partial}{\partial ψ} \right)i + U^*\tilde{T}_ψU$$

is compact on $D$.

(ii) \( \langle H_ψ^*H_ψ f, g \rangle = \langle H_ψ f, H_ψ g \rangle = \langle (I - P)ψ f, ψ g \rangle \)

for some compact operator $P$. Hence $H_ψ^*H_ψ = M_ψ^*M_ψ - T_ψ^*T_ψ$.

By Proposition 2.3, $T_ψ^*T_ψ = K + U^*\tilde{T}_ψ\tilde{T}_ψU$ for some compact operator $K$ on $D$. Since $\tilde{H}_ψ^*\tilde{H}_ψ = \tilde{T}_ψ - \tilde{T}_ψ\tilde{T}_ψ$, with (i), we obtain

$$H_ψ^*H_ψ = K_2 + U^*\tilde{H}_ψ^*\tilde{H}_ψU$$

with $K_2$ compact on $D$. □

**Proof of Theorem 2.1.** It is easily followed from Proposition 2.7. □

For the characterization of compactness of Hankel operator on $L^2_a$, see [1]. Moreover for harmonic function $ψ, ψ \in L^∞_a$, by Zheng’s Theorem in [6], $H_ψ^*H_ψ$ is compact.

Next we study the algebraic properties of small Hankel operators on $D$.

Denote

$$\mathcal{H} = \{u \in L^∞_a \mid u \text{ is harmonic}\}.$$ 

In [4], the (semi-)commutativity of Toeplitz operators on $D$ is studied. In the following, we study the commutativity of Toeplitz operator and small Hankel operator, small Hankel operators on $D$ with symbols in $\mathcal{H}$. First, we fix some notation.

Denote $\Gamma = \partial \mathbb{D}$ be the unit circle. Let $L^2(\mathbb{T})$ be the square integrable Lebesgue functions on $\mathbb{T}$ with orthonormal basis $[s_n]_{n=-\infty}^{\infty}$ such that

$$s_n(\xi) = \xi^n, \quad \xi \in \mathbb{T},$$

and $L^2$ the Hardy space with orthonormal basis $[s_n]_{n=0}^{\infty}$.

Let $L^∞(\mathbb{T})$ be the space of bounded Lebesgue functions on $\mathbb{T}$ and $H^∞$, the functions in $L^∞(\mathbb{T})$ whose poisson integral are holomorphic on $\mathbb{D}$. It is well known that $L^∞(\mathbb{T})$ is one-to-one corresponding to bounded harmonic functions on $\mathbb{D}$ by the poisson integral [3].

For $ψ \in L^∞(\mathbb{T})$, the classical Toeplitz operator $\tilde{T}_ψ$ on $H^2$ is defined by

$$\tilde{T}_ψ(f) = \tilde{P}(ψ f),$$

and the classical Hankel operator $\tilde{H}_ψ$ on $H^2$ is defined by

$$\tilde{H}_ψ(f) = \tilde{P}(\tilde{f}(ψ f))$$

for $f \in H^2$, where $\tilde{P}$ is the Hardy projection from $L^2(\mathbb{T})$ onto $H^2$ and $\tilde{f}$ is the unitary $L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ defined by $\tilde{f}(g)(\xi) = g(\xi) \{g \in L^2(\mathbb{T})\}$.

Let $c_0 = \sqrt{2}$ and $e_n(z) = \frac{1}{c_n}z^n, z \in \mathbb{D}, n = 1, 2, \ldots$. Then $[e_n]_{n=1}^{\infty}$ forms an orthonormal basis of $\mathcal{D}$. Let $C$ be the diagonal matrix whose diagonal coefficients are given by $c_1, c_2, \ldots$, i.e.,

$$C = \begin{pmatrix}
  c_1 & 0 & 0 & \cdots \\
  0 & c_2 & 0 & \cdots \\
  0 & 0 & c_3 & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$ 

Let $u \in \mathcal{H}$ and $u(z) = \sum_{k<0} a_k z^{-k} + \sum_{k\geq 0} a_k z^k$. In [4, Proposition 4], the matrix of $T_u$ with respect to the orthonormal basis $e_n$ was given as $CM_uC^{-1}$, where
\[ M_u = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]  

(2)

Similarly, we have the following result.

**Proposition 2.8.** Let \( u \in \mathcal{H} \) and \( u(z) = \sum_{k<0} a_k \xi^{-k} + \sum_{k\geq 0} a_k \xi^k \). Then \( \Gamma_u \) has the matrix expression \( C N_u C^{-1} \) with respect to the orthonormal basis \( e_n \), where

\[ N_u = \begin{pmatrix} a_{-2} & a_{-3} & a_{-4} & a_{-5} & \cdots \\ a_{-3} & a_{-4} & a_{-5} & a_{-6} & \cdots \\ a_{-4} & a_{-5} & a_{-6} & a_{-7} & \cdots \\ a_{-5} & a_{-6} & a_{-7} & a_{-8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]  

(3)

**Proof.** By straightforward computation, for \( m, n \geq 1 \),

\[ \langle \Gamma_u e_n, e_m \rangle = \frac{\sqrt{m}}{\sqrt{n}} a_{-(n+m)}. \]

For \( u \in \mathcal{H} \) and \( u(z) = \sum_{k<0} a_k \xi^{-k} + \sum_{k\geq 0} a_k \xi^k \), let \( \tilde{u} \) denote the nontangential limit of \( u \) on the boundary \( \mathbb{T} \), then \( \tilde{u}(\xi) = \sum_{k=-\infty}^{\infty} a_k \xi^k \) for a.e. \( \xi \in \mathbb{T} \) and \( \tilde{u} \in L^\infty(\mathbb{T}) \). It is well known the Toeplitz operator \( \tilde{T}_u \) and the Hankel operator \( \tilde{H}_u \) on \( H^2 \) have the matrix expressions

\[ \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]

\[ \begin{pmatrix} a_{-1} & a_0 & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-1} & a_0 & a_{-2} & \cdots \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]

with respect to the orthonormal basis \( \{ s_n \}_{n=0}^{\infty} \) respectively. So straightforward computation show \( \tilde{T}_u \) has the matrix expression \( N_u \) as expressed in (3). Then we have the following result.

**Theorem 2.9.** Let \( u, v \in \mathcal{H} \) and \( \Gamma_u, \Gamma_v \) are nonzero on \( D \), then \( \Gamma_u \Gamma_v = \Gamma_v \Gamma_u \) on \( D \) if and only if there exists nonzero constant \( c \) such that \( u - cv = c_0 \tilde{z} + h \) for some constant \( c_0 \) and \( h \in H^\infty \).

**Proof.** By the proceeding observation, we have \( \Gamma_u \Gamma_v = \Gamma_v \Gamma_u \) on \( D \) if and only if \( \tilde{T}_u \tilde{T}_v = \tilde{T}_v \tilde{T}_u \) on \( H^2 \).

It is a well-known fact that for two bounded harmonic functions \( \phi, \psi \), \( \tilde{T}_\phi \tilde{T}_\psi = \tilde{T}_\psi \tilde{T}_\phi \) on \( H^2 \) if and only if there exists a constant \( c \) such that \( \phi - c \psi = \rho \) for \( \rho \in H^\infty \) with \( \rho(0) = 0 \). For the convenience, we outline the proof.

Let \( \phi(\xi) = \sum_{k<0} a_k \xi^{-k} + \sum_{k\geq 0} a_k \xi^k \), \( \psi(\xi) = \sum_{k<0} b_k \xi^{-k} + \sum_{k\geq 0} b_k \xi^k \). For \( m, n \geq 0 \), direct computation shows that

\[ \langle \tilde{T}_\phi \tilde{T}_\psi s_n, s_m \rangle = \sum_{k=0}^{\infty} a_{-m-k} b_{-n-k}, \quad \langle \tilde{T}_\phi \tilde{T}_\psi s_{n+1}, s_{m+1} \rangle = \sum_{k=1}^{\infty} a_{-m-k} b_{-n-k}, \]  

(4)

\[ \langle \tilde{T}_\psi \tilde{T}_\phi s_n, s_m \rangle = \sum_{k=0}^{\infty} b_{-m-k} a_{-n-k}, \quad \langle \tilde{T}_\psi \tilde{T}_\phi s_{n+1}, s_{m+1} \rangle = \sum_{k=1}^{\infty} b_{-m-k} a_{-n-k}. \]  

(5)

Compare Eqs. (4) and (5), by the fact \( \langle \tilde{T}_\phi \tilde{T}_\psi s_n, s_m \rangle = \langle \tilde{T}_\psi \tilde{T}_\phi s_n, s_m \rangle \), we have \( a_{-m-k} b_{-n-k} = b_{-m-k} a_{-n-k} \), which implies the conclusion.

So \( \Gamma_u \Gamma_v = \Gamma_v \Gamma_u \) on \( D \) if and only if there exists nonzero constant \( c \) such that \( \xi^2 \tilde{u}(\xi) - c \xi^2 \tilde{v}(\xi) = \xi f(\xi) \) for some \( f \in H^\infty \), i.e., \( \tilde{u}(\xi) - c \tilde{v}(\xi) = \xi f(\xi) \). By the poisson integral, we have

\[ u(z) - cv(z) = c_0 \tilde{z} + h(z) \]

with \( c_0 \) a constant and \( h \in H^\infty \), where \( c_0 \tilde{z} + h \) is the poisson integral of \( \xi f \).

In the following theorem we obtain the condition when the product of two small Hankel operators to be a small Hankel operator.

**Theorem 2.10.** Let \( u, v, \tau \in \mathcal{H} \). \( \Gamma_u \Gamma_v = \Gamma_\tau \) on \( D \) if and only if \( \tilde{T}_\xi \tilde{T}_\tau \tilde{T}_\xi = \tilde{T}_\xi \tilde{T}_\phi \) on \( H^2 \).
Since $T_u$ has matrix expression $CM_uC^{-1}$ and $\tilde{T}_u$ has matrix expression $M_u$, we have the following result.

**Theorem 2.11.** Let $u, v \in \mathcal{H}$ with $T_u$ not constant and $\Gamma_v$ nonzero, then $T_u \Gamma_v = \Gamma_v T_u$ on $\mathcal{D}$ if and only if $\tilde{T}_u \tilde{\Gamma}_v \tilde{z}_v = \tilde{\Gamma}_v \tilde{z}_v \tilde{T}_u$ on $H^2$.

Note that the conditions for Toeplitz operator and Hankel operator commuting and the product of Hankel operator being a Hankel operator on the Hardy space $H^2$ have been completely characterized in [7,8].

**Acknowledgment**

The author thanks the referee for numerous suggestions that helped make this paper more readable.

**References**