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Stability of nonlinear elliptic systems with distributed parameters and variable boundary data☆

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Abstract

In this paper nonlinear partial differential equations of the elliptic type with the Dirichlet boundary data are investigated. Some sufficient conditions under which the solutions of considered equations depend continuously on parameters and boundary conditions are proved. The proofs of main results are based on variational methods. In the final part of the paper we give a short survey of the results and methods related to the question of stability of the boundary value problems.

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1. Introduction

In this work we investigate the elliptic systems of partial differential equations with variable distributed parameters and variable boundary conditions. The systems considered are of the form

$$\Delta z(x) = \varphi(x, z(x), \omega(x)), \tag{1.1}$$

$$z(x) = v(x) \text{ on } \partial\Omega, \tag{1.2}$$

where $x \in \Omega \subset \mathbb{R}^n$, $n \geqslant 2, \Omega$ is a bounded domain with Lipschitzian boundary $\partial\Omega$, $z(\cdot) \in H^1(\Omega, \mathbb{R}^N)$. We shall assume that the distributed parameter $\omega(\cdot)$ varies in the space $L^p(\Omega, \mathbb{R}^m)$ and the variable

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boundary data belong to the space of traces $H^{1/2}(\partial\Omega, \mathbb{R}^N)$, $N, p, m \geqslant 1$ (for details see Section 2). In the theory of boundary value problems and its applications we consider, first of all, the problem of the existence of a solution, next the question of stability, uniqueness, smoothness, etc. Following Hadamard, Courant and Hilbert, we say that a given problem is well-posed if this problem possesses at least one solution which continuously changes together with variable parameters of the system.

Courant and Hilbert in their monograph write: “A mathematical problem which is to correspond to physical reality should satisfy the following basic requirements: (1) The solution must exist. (2) The solution should be uniquely determined. (3) The solution should depend continuously on the data (requirement of stability)” and, next, they write: “The third requirement, particularly incisive, is necessary if the mathematical formulation is to describe observable natural phenomena. Data in nature cannot possibly be conceived as rigidly fixed: the mere process of measuring them involves small errors...” (cf. [5, Vol. II, Chapter III, Section 6.2]).

Further by stability of a boundary value problem we mean the continuous dependence of the solution of the problem on boundary data and parameters.

A problem is said to be ill-posed if it does not possess at least one of properties (1)–(3). However, the theory of ill-posed problems pays most attention to the third requirement. Hadamard gave a simple example of an ill-posed initial value problem for partial differential equations. Namely, consider the Laplace equation $z_{xx} + z_{yy} = 0$, $x \in (0, \pi)$, $y \in (-1, 1)$ with the initial conditions $z(x, 0) = \varphi_k(x) = (1/k^2)\sin(kx)$, $z_y(x, 0) = \psi_k(x) = (1/k)\sin(kx)$, $z(\cdot,\cdot) \in C^2$. By a direct inspection and Carleman's theorem, we can show that the function $z_k(x, y) = (1/k^2)\exp(ky)\sin(kx)$ is the unique solution of the above problem for $k = 1, 2, \ldots$. Passing with k to infinity, we see that φ_k and ψ_k tend to zero uniformly, but the sequence z_k does not converge to the function $z_0(x, y) = 0$ which is the unique solution of the Laplace equation with homogeneous initial data. Thus the above initial value problem is ill-posed.

Next, let us consider the boundary value problem

$$\Delta z(x, y) + 2z(x, y) = \alpha, \tag{1.3}$$

$$z(x, y) = 0 \text{ for } (x, y) \in \partial\tilde{\Omega}, \tag{1.4}$$

where $\tilde{\Omega} = (0, \pi) \times (0, \pi)$, $\alpha \in \mathbb{R}$.

It is easy to see that for $\alpha = 0$ any function of the form $z(x, y) = C \sin x \sin y$, $C \in \mathbb{R}$, is a solution of boundary problem (1.3)–(1.4). But for $\alpha \neq 0$ the above problem has no solutions in the space $H^2((0, \pi) \times (0, \pi), \mathbb{R})$. Indeed, multiplying (1.3) by $\sin x \sin y$ and using the Fubini theorem, we get

$$\int_0^\pi \left[\int_0^\pi z_{xx} \sin x \,\mathrm{d}x\right] \sin y \,\mathrm{d}y + \int_0^\pi \left[\int_0^\pi z_{yy} \sin y \,\mathrm{d}y\right] \sin x \,\mathrm{d}x$$
$$+ 2\int_0^\pi \int_0^\pi z \sin x \sin y \,\mathrm{d}x \,\mathrm{d}y = \alpha \int_0^\pi \sin x \,\mathrm{d}x \int_0^\pi \sin y \,\mathrm{d}y.$$

Integrating by parts we obtain $0 = 4\alpha$, a contradiction. We see that for $\alpha = 0$ the boundary value problem (1.3)–(1.4) possesses infinite many solutions, but if we change a little the parameter α that $\alpha \neq 0$, then a solution does not exist. Thus the boundary value problem (1.3)–(1.4) is ill-posed.

In this paper we prove some sufficient conditions under which the considered boundary value problems possess at least one solution which continuously depends on distributed parameters and the boundary data.

In Section 3 we consider system (1.2)–(1.3) with the variable boundary data and parameters. Using some variational methods we prove that this system is stable with respect to the norm topology in the space of distributed parameters $L^p(\Omega, \mathbb{R}^m)$, norm topology in the space of boundary data $H^{1/2}(\partial\Omega, \mathbb{R}^N)$ and norm topology in the space of solutions $H^1(\Omega, \mathbb{R}^N)$. Not going into details, we can formulate the main result of Section 3 as follows: if $v_k \to v_0$ in $H^{1/2}(\partial\Omega, \mathbb{R}^N)$, $\omega_k \to \omega_0$ in $L^p(\Omega, \mathbb{R}^m)$, then $z_k \to z_0$ in $H^1(\Omega, \mathbb{R}^N)$ where z_k is the solution of the boundary value problem (1.1)–(1.2) with fixed $\omega = \omega_k$ and $v = v_k$, $k = 0, 1, 2, \ldots$.

In the case when system (1.1) is linear with respect to ω, we can relax the topology in the space $L^p(\Omega, \mathbb{R}^m)$. In Section 4, we prove that $z_k \to z_0$ in $H^1(\Omega, \mathbb{R}^N)$ provided that $v_k \to v_0$ in $H^{1/2}(\partial\Omega, \mathbb{R}^N)$ and $\omega_k \to \omega_0$ weakly in $L^p(\Omega, \mathbb{R}^m)$.

In the final part of the paper we give some physical interpretation of the considered problem and a short survey of the results and methods related to the stability of the boundary value problems for the second-order partial and ordinary differential systems with the variable boundary conditions and parameters.

2. Formulation of the problem and basic assumptions

To begin with, we recall some definitions and notations.

By $H^1(\Omega, \mathbb{R}^N)$, $N \in \mathbb{N}$, $N > 0$ ($H^1(\Omega)$ for short), we shall denote the Sobolev space of functions $u = u(x)$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, such that $u(\cdot) \in L^2(\Omega, \mathbb{R}^N)$, whose (distributional) derivatives ∇u are elements of the space $L^2(\Omega, \mathbb{R}^{Nn})$. The norm in $H^1(\Omega, \mathbb{R}^N)$ is defined by formula

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2)\,\mathrm{d}x.$$

By $H^{1/2}(\Omega, \mathbb{R}^N)$ we denote the space of all functions $u(\cdot) \in L^2(\Omega, \mathbb{R}^N)$ for which

$$I_0(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}}\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

equipped with the norm

$$\|u\|^2_{H^{1/2}(\Omega)} = \|u\|^2_{L^2(\Omega)} + I_0(u)$$

(cf. [14, Definition 6.8.2]).

Covering $\partial\Omega$ by coordinate patches, we define the space $H^{1/2}(\partial\Omega, \mathbb{R}^N)$ as before via such charts (cf. [14, Section 6]) with an analogous norm.

$H^{1/2}(\partial\Omega, \mathbb{R}^N)$ is said to be the space of traces (boundary values) of functions from the space $H^1(\Omega, \mathbb{R}^N)$. Throughout the paper, we shall assume that Ω satisfies any condition which guarantees a compact embedding of $H^1(\Omega, \mathbb{R}^N)$ into $L^s(\Omega, \mathbb{R}^N)$ with $s \in (1, 2^*)$ where $2^* = 2n/(n-2)$ if $n \geqslant 3$ and $2^* = \infty$ if $n = 2$; for example, $\partial\Omega$ may be Lipschitzian, i.e., $\Omega \in C^{0,1}$ (see [14]).

By $R: H^1(\Omega, \mathbb{R}^N) \to H^{1/2}(\partial\Omega, \mathbb{R}^N)$ we shall denote the linear and continuous operator such that $Rz = z|_{\partial\Omega}$ for all $z \in C^\infty(\bar{\Omega})$. According to Theorem 6.8.13 (cf. [14]) there exists such operator and is uniquely determined. The value Rz is usually referred to as the trace of function z on the boundary $\partial\Omega$ and we often write z instead Rz. Further, we shall understand the boundary condition $z(x) = v(x)$ for $x \in \partial\Omega$, in the sense of the trace. We denote by $H_0^1(\Omega, \mathbb{R}^N)$ the subspace of $H^1(\Omega, \mathbb{R}^N)$ consisting of all functions z such that $z(x) = 0$ for $x \in \partial\Omega$ a.e., (in the sense of the trace). A norm in $H_0^1(\Omega, \mathbb{R}^N)$ can be defined by equality

$$\|z\|_{H_0^1}^2 = \int_\Omega |\nabla z(x)|^2 \, dx.$$

In our further considerations, an essential role is played by the inverse operator $T: H^{1/2}(\partial\Omega, \mathbb{R}^N) \to H^1(\Omega, \mathbb{R}^N)$. Theorem 6.9.2 (cf. [14]) implies that there exists a continuous linear mapping T defined on the space $H^{1/2}(\partial\Omega, \mathbb{R}^N)$ such that $Tv = z \in H^1(\Omega, \mathbb{R}^N)$ for any $v \in H^{1/2}(\partial\Omega, \mathbb{R}^N)$ and the trace of z is equal to v, i.e., $Rz = R(Tv) = v$.

Since the operator T is continuous, we have

$$\|Tv\|_{H^1} \leqslant c\|v\|_{H^{1/2}}, \tag{2.1}$$

where the constant $c > 0$ depends only on T and the description of $\partial\Omega$.

Further, in this paper we consider the case when the mapping $\varphi: \Omega \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$, which defined system (1.1), represents a potential vector field, i.e., there exists a scalar function $\phi: \Omega \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ such that

$$\varphi(x, z, \omega) = \nabla_z \phi(x, z, \omega) = \phi_z(x, z, \omega),$$

where $x \in \Omega$ a.e., $z \in \mathbb{R}^N$, $\omega \in \mathbb{R}^m$, $\nabla_z \phi = (\phi_{z^1}, \phi_{z^2}, \ldots, \phi_{z^N})$.

In this case the boundary value problem (1.1)–(1.2) may be written in the form

$$\Delta z(x) = \phi_z(x, z(x), \omega(x)), \tag{2.2}$$

$$z(x) = v(x) \quad \text{for } x \in \partial\Omega \text{ a.e.}, \tag{2.3}$$

where $\omega(\cdot) \in L^p(\Omega, \mathbb{R}^m)$ and $v(\cdot) \in H^{1/2}(\Omega, \mathbb{R}^N)$, $N, p, m \geqslant 1$. It is easy to see that system (2.2) represents the Euler–Lagrange equation for the following functional of action

$$F_{\omega,v}(z) = \int_\Omega [\tfrac{1}{2}|\nabla z(x)|^2 + \phi(x, z(x), \omega(x))] \, dx, \tag{2.4}$$

where $z(\cdot) \in H^1(\Omega, \mathbb{R}^N)$, $z(x) = v(x)$ for $x \in \partial\Omega$ a.e., $v(\cdot) \in H^{1/2}(\partial\Omega, \mathbb{R}^N)$, $\omega(\cdot) \in L^p(\Omega, \mathbb{R}^m)$, $p \geqslant 1$.

We shall impose the following conditions on the function ϕ:

(2.5) the functions ϕ and ϕ_z are measurable with respect to x for any $(z, \omega) \in \mathbb{R}^N \times \mathbb{R}^m$ and continuous with respect to (z, ω) for $x \in \Omega$ a.e.

(2.6) if $p \in [1, \infty)$, we assume that there exists a constant $c > 0$ such that

$$|\phi(x,z,\omega)| \leqslant c(1 + |z|^s + |\omega|^p),$$

$$|\phi_z(x,z,\omega)| \leqslant c(1 + |z|^{s-1} + |\omega|^{p(1-(1/s))}),$$

for $x \in \Omega$ a.e., $\omega \in \mathbb{R}^m$, $z \in \mathbb{R}^N$, where $s \in (1, 2^*)$;

if $p = \infty$, we assume that for any bounded set $W \subset \mathbb{R}^m$ there exists a constant $c > 0$ such that

$$|\phi(x,z,\omega)| \leqslant c(1 + |z|^s),$$

$$|\phi_z(x,z,\omega)| \leqslant c(1 + |z|^{s-1}),$$

for $x \in \Omega$ a.e., $\omega \in W$, $z \in \mathbb{R}^N$ and some $s \in (1, 2^*)$, where $2^* = 2n/(n-2)$ if $n \geqslant 3$ and $2^* = +\infty$ if $n = 2$.

(2.7) there exists a constant $b \in \mathbb{R}$ and some functions $\alpha(\cdot) \in L^2(\Omega, \mathbb{R}^N)$, $\beta(\cdot) \in L^1(\Omega, \mathbb{R})$, such that

$$\phi(x,z,\omega) \geqslant -b|z|^2 - \langle \alpha(x), z \rangle - \beta(x),$$

for $x \in \Omega$ a.e., $\omega \in \mathbb{R}^m$, $z \in \mathbb{R}^N$, where $\lambda - 2b > 0$ and λ is the principal eigenvalue of the Laplace operator $-\Delta z$ defined on the space $H_0^1(\Omega, \mathbb{R}^N)$.

Remark 2.1. The principal eigenvalue λ is given by the equality

$$\lambda = \inf \left\{ \frac{\int_\Omega |\nabla z(x)|^2 \, dx}{\int_\Omega |z(x)|^2 \, dx}; z \in H_0^1(\Omega, \mathbb{R}^N), \quad z \neq 0 \right\}$$

(cf. [6, Chapter 6.5.1]) and, in general, it is not easy to find λ if Ω is an arbitrary domain in $\mathbb{R}^n$. One can show that $\lambda \geqslant 1/d^2$ where d is the diameter of Ω (cf. [25, Appendix A]).

Under assumptions (2.5)–(2.6) the functional of action given by formula (2.4) is well-defined and Frechet differentiable. The derivative of $F_{\omega,v}(\cdot)$ acting on $h \in H_0^1(\Omega, \mathbb{R}^N)$ is defined by the formula

$$D_z F_{\omega,v}(z)h = \int_\Omega [\langle \nabla z(x), \nabla h(x) \rangle + \langle \phi_z(x, z(x), \omega(x)), h(x) \rangle] \, dx. \tag{2.8}$$

3. Continuous dependence of the solutions on parameters and boundary data: the case of the strong topology

Let $\{\omega_k\}$ and $\{v_k\}$ be some sequences of distributed parameters and boundary data, respectively. Denote by Z_k, $k = 0, 1, 2, \ldots$ the set of all minimizers of the functional $F_{\omega_k, v_k}(\cdot)$, i.e.,

$$Z_k = \{z \in H^1(\Omega, \mathbb{R}^N);\ F_{\omega_k,v_k}(z) = \min F_{\omega_k,v_k}(y), \quad y \in H^1(\Omega, \mathbb{R}^N) \quad \text{and}$$
$$y(x) = v_k(x) \quad \text{for } x \in \partial\Omega\}, \quad k = 0, 1, 2, \ldots .$$

Since the functional $F_{\omega,v}(\cdot)$ is differentiable, it follows that each minimizer $\bar{z} \in Z_k$ is a critical point of $F_{\omega_k,v_k}(\cdot)$, i.e., $D_z F_{\omega_k,v_k}(\bar{z})h = 0$ for any $h \in H_0^1(\Omega, \mathbb{R}^N)$ (cf. (2.8)) and consequently $\bar{z}$ is a weak solution of system (2.2)–(2.3). Inversely, if $\bar{z}$ is a weak solution of Eqs. (2.2)–(2.3), then $\bar{z} \in Z_k$ provided the functional $F_{\omega_k,v_k}(\cdot)$ is convex.

Of course, in general, the set Z_k is not singleton and hence the boundary value problem (2.2)–(2.3) has no unique solution.

We say that a set $\tilde{Z} \subset H^1(\Omega, \mathbb{R}^N)$ is an upper limit of the sets Z_k, $k = 1, 2, \ldots$ iff any point $\tilde{z} \in \tilde{Z}$ is a cluster point of some sequence $\{z_k\}$ (with respect to the norm topology of $H^1(\Omega, \mathbb{R}^N)$) where $z_k \in Z_k$ for $k = 1, 2, \ldots$. We will denote by $\operatorname{Lim\,sup} Z_k = \tilde{Z}$ the upper limit of the sets Z_k, $k = 1, 2, \ldots$.

Now, we can formulate and prove the main result of this paper.

Theorem 3.1. *Assume that*

(1) *the integrand ϕ satisfies conditions* (2.5)–(2.7),
(2) *the sequence of distributed parameters ω_k, $k = 1, 2, \ldots$ tends to ω_0 in $L^p(\Omega, \mathbb{R}^m)$*,
(3) *the sequence of boundary conditions v_k, $k = 1, 2, \ldots$ tends to v_0 in $H^{1/2}(\partial\Omega, \mathbb{R}^N)$*.

Then

(a) *for any ω_k and v_k the set Z_k, $k = 0, 1, 2, \ldots$ is a nonempty subset of $H^1(\Omega, \mathbb{R}^N)$*,
(b) *there exists a ball $B(0, \rho) \subset H^1(\Omega, \mathbb{R}^N)$ such that $Z_k \subset B(0, \rho)$ for $k = 0, 1, 2, \ldots$* ,
(c) *any sequence $\{z_k\}$, $z_k \in Z_k$ is relatively compact in the norm topology of $H^1(\Omega, \mathbb{R}^N)$, $\operatorname{Lim\,sup} Z_k = \tilde{Z}$ is a nonempty set and $\operatorname{Lim\,sup} Z_k \subset Z_0$, where Z_0 is the set of all minimizers of the functional $F_{\omega_0, v_0}(\cdot)$.*

If Z_k is a singleton, i.e., $Z_k = \{z_k\}$, $k = 0, 1, 2, \ldots$, then z_k tends to z_0 in the norm of $H^1(\Omega, \mathbb{R}^N)$.

In other words, if $Z(F_{\omega,v}) = Z_{\omega,v}$ denote the set of all minimizers of the functional $F_{\omega,v}(\cdot)$ defined by (2.4), then assertion (c) of Theorem 3.1 states that the set valued mapping $(\omega, v) \in L^p(\Omega, \mathbb{R}^m) \times H^{1/2}(\partial\Omega, \mathbb{R}^N) \to Z_{\omega,v} \subset H^1(\Omega, \mathbb{R}^N)$ is upper semicontinuous with respect to the norm topology of the spaces $L^p(\Omega, \mathbb{R}^m)$, $H^{1/2}(\partial\Omega, \mathbb{R}^N)$ and $H^1(\Omega, \mathbb{R}^N)$.

Proof. *Step* 1: In the first step we prove assertions (a) and (b) of our theorem.

Consider the functional

$$\bar{F}_{\omega_k, v_k}(y) = F_{\omega_k, v_k}(y + Tv_k)$$
$$= \int_\Omega [\tfrac{1}{2}|\nabla y(x) + \nabla(Tv_k)(x)|^2 + \phi(x, y(x) + Tv_k(x), \omega_k(x))]\, dx, \tag{3.2}$$

where $T : H^{1/2}(\partial\Omega, \mathbb{R}^N) \to H^1(\Omega, \mathbb{R}^N)$ is an inverse operator to R as introduced in Section 2 and $y(\cdot) \in H_0^1(\Omega, \mathbb{R}^N)$. Assumptions (2) and (3) of our theorem imply that $\|\omega_k\| \leqslant C_0$ and $\|v_k\| \leqslant C_0$ for some $C_0 > 0$ and $k = 0, 1, 2, \ldots$. By $\bar{Z}_k$ denote the set of all minimizers of the functional $\bar{F}_{\omega_k, v_k}(\cdot)$, i.e.,

$$\bar{Z}_k = \{\bar{y} \in H_0^1(\Omega, \mathbb{R}^N) \colon \bar{F}_{\omega_k, v_k}(\bar{y}) = \min \bar{F}_{\omega_k, v_k}(y), \quad y \in H_0^1(\Omega, \mathbb{R}^N)\}. \tag{3.3}$$

It is easily seen that $Z_k = \bar{Z}_k + Tv_k$. By (2.7) and (3.2), we have

$$\bar{F}_{\omega_k, v_k}(y) \geqslant \int_\Omega [\tfrac{1}{2}|\nabla y(x) + \nabla(Tv_k)(x)|^2 - b|y(x) + Tv_k(x)|^2$$
$$- \langle \alpha(x), y(x) + Tv_k(x) \rangle - \beta(x)]\, dx.$$

Applying the Poincaré inequality and inequality (2.1), we get

$$\bar{F}_{\omega_k,v_k}(y) \geqslant (\lambda - 2b)\|y\|^2 - C_1\|y\| - C_2 = p(y) \tag{3.4}$$

with $\lambda - 2b > 0$ (by (2.7)), where C_1, C_2 are some constants independent of ω_k and v_k. It is a well-known fact that under conditions (2.5) and (2.6) the integral functional $\bar{F}_{\omega_k,v_k}(\cdot)$ is weakly lower semicontinuous on $H_0^1(\Omega,\mathbb{R}^N)$. Since this functional is coercive (cf. (3.4)), we infer that the set $\bar{Z}_k$ is nonempty and weakly closed.

Putting $y=0$ in formula (3.2) and applying assumption (2.6), we get the following estimates:

$$\bar{F}_{\omega_k,v_k}(0) \leqslant \int_\Omega [\tfrac{1}{2}|\nabla(Tv_k)(x)|^2 + c(1+|Tv_k(x)|^s + |\omega_k(x)|^p)]\,\mathrm{d}x \leqslant D_1 \quad \text{if} \quad p<\infty,$$

$$\bar{F}_{\omega_k,v_k}(0) \leqslant \int_\Omega [\tfrac{1}{2}|\nabla(Tv_k)(x)|^2 + c(1+|Tv_k(x)|^s)]\,\mathrm{d}x \leqslant D_2 \quad \text{if} \quad p=\infty, \tag{3.5}$$

where the constants D_1 and D_2 are independent of ω_k and v_k. Directly from inequalities (3.4), (3.5) and formulas (3.3) it follows that

$$\bar{Z}_k \subset \{y\in H_0^1(\Omega,\mathbb{R}^N):\ p(y)\leqslant D\} \subset \bar{B}(0,\bar{\rho}) \quad \text{for some } \bar{\rho}>0, \tag{3.6}$$

where $\bar{B}(0,\bar{\rho})=\{y\in H_0^1(\Omega,\mathbb{R}^N);\, \|y\|\leqslant\bar{\rho}\}$. Since $Z_k=\bar{Z}_k+Tv_k$ and $\|Tv_k\|\leqslant C$, for all v_k and some $C>0$, we get the inclusion

$$Z_k \subset B(0,\rho) = \{z\in H^1(\Omega,\mathbb{R}^N);\, \|z\|\leqslant\rho\} \quad \text{for some } \rho>0.$$

We have thus proved assertions (a) and (b) of our theorem.

Step 2: Denote by $\bar{\mu}_k$ the minimal value of the functional $\bar{F}_{\omega_k,v_k}(\cdot)$, i.e.,

$$\bar{\mu}_k = \min_{y\in H_0^1} \bar{F}_{\omega_k,v_k}(y) = \bar{F}_{\omega_k,v_k}(\bar{y}), \quad k=0,1,2,\ldots,$$

where $\bar{y}\in\bar{Z}_k$. We shall observe that

$$\lim \bar{\mu}_k = \bar{\mu}_0 \tag{3.7}$$

provided that $\omega_k \to \omega_0$ and $v_k \to v_0$.

We begin by proving that the sequence $\bar{F}_{\omega_k,v_k}(y)$ tends to $\bar{F}_{\omega_0,v_0}(y)$ uniformly on any ball $\bar{B}(0,\rho)\subset H_0^1(\Omega,\mathbb{R}^N)$. By (3.2), we have

$$\begin{aligned}|\bar{F}_{\omega_k,v_k}(y) - \bar{F}_{\omega_0,v_0}(y)| \leqslant{}& \left|\int_\Omega \langle\nabla y(x), \nabla(Tv_k)(x) - \nabla(Tv_0)(x)\rangle\,\mathrm{d}x\right| \\ &+ \tfrac{1}{2}\left|\int_\Omega (|\nabla(Tv_k)(x)|^2 - |\nabla(Tv_0)(x)|^2)\,\mathrm{d}x\right| \\ &+ \left|\int_\Omega [\phi(x,y(x)+Tv_k(x),\omega_k(x)) \right. \\ &\left. -\phi(x,y(x)+Tv_0(x),\omega_0(x))]\,\mathrm{d}x\right|. \end{aligned} \tag{3.8}$$

Using the Hölder inequality, we can estimate the first integral I_k^1 in (3.8) as follows:

$$|I_k^1(y)| \leqslant \left(\int_\Omega |\nabla y(x)|^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla(Tv_k)(x)-\nabla(Tv_0)(x)|^2\,\mathrm{d}x\right)^{1/2}$$
$$\leqslant \rho\left(\int_\Omega |\nabla(Tv_k)(x)-\nabla(Tv_0)(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Hence $I_k^1(y)$ tends to zero uniformly on $\bar{B}(0,\rho)$, because $\nabla(Tv_k)\to\nabla(Tv_0)$ in $L^2(\Omega,\mathbb{R}^{Nn})$. The same reasoning applies to the second integral I_k^2 leads to the conclusion that I_k^2 tends to zero.

Now, suppose that the last integral $I_k^3(y)$ does not tend to zero uniformly on $\bar{B}(0,\rho)$. It means that there exists $\varepsilon_0>0$ and a sequence $\{y_k\}\subset\bar{B}(0,\rho)$ such that $|I_k^3(y_k)|>\varepsilon_0$. Passing, if necessary, to a subsequence, we can assume that y_k tends to some $\bar{y}$ weakly in $H_0^1(\Omega,\mathbb{R}^N)$. From the Sobolev embedding theorem (cf. [14]) we deduce that y_k+Tv_k tends to $\bar{y}+Tv_0$ in $L^s(\Omega,\mathbb{R}^N)$. By assumption, we know that ω_k tends to ω_0 in $L^p(\Omega,\mathbb{R}^m)$. Using the Krasnosielskii theorem (cf. [10,33]) and assumption (2.6) we infer that $I_k^3(y_k)\to 0$. Thus we have got a contradiction with the inequality $|I_k^3(y_k)|>\varepsilon_0$. It means that $I_k^3(y)$ tends to zero uniformly on $\bar{B}(0,\rho)$. Taking into account inequality (3.8), we see that $\bar{F}_{\omega_k,v_k}(y)$ converges to $\bar{F}_{\omega_0,v_0}(y)$ uniformly on $\bar{B}(0,\rho)$ provided that $\omega_k\to\omega_0$ in $L^p(\Omega,\mathbb{R}^m)$ and $v_k\to v_0$ in $H^{1/2}(\partial\Omega,\mathbb{R}^N)$.

From this, for any $\varepsilon>0$ and k sufficiently large, we have

$$\bar{\mu}_k=\min_{y\in H_0^1}\bar{F}_{\omega_k,v_k}(y)=\min_{y\in\bar{B}(0,\bar{\rho})}\bar{F}_{\omega_k,v_k}(y)\leqslant\min_{y\in\bar{B}(0,\bar{\rho})}\bar{F}_{\omega_0,v_0}(y)+\varepsilon$$
$$=\min_{y\in H_0^1}\bar{F}_{\omega_0,v_0}(y)+\varepsilon=\bar{\mu}_0+\varepsilon,$$

where $\bar{B}(0,\bar{\rho})$ is the ball given in (3.6).

Similarly, $\bar{\mu}_0\leqslant\bar{\mu}_k+\varepsilon$. We have thus proved equality (3.7).

Step 3: Finally, we shall prove assertion (c). Let $\{y_k\}$, $k=1,2,\dots$ be a sequence of minimizers, i.e., $y_k\in\bar{Z}_k$. Since $\bar{Z}_k\subset\bar{B}(0,\bar{\rho})$ for $k=0,1,2,\dots$, we infer that the sequence $\{y_k\}$ is weakly relatively compact. Passing, if necessary, to a subsequence, we can assume that y_k tends to some $\bar{y}\in\bar{B}(0,\bar{\rho})$ in the weak topology of $H_0^1(\Omega,\mathbb{R}^N)$. Let us prove that $\bar{y}\in\bar{Z}_0$, i.e., $\bar{y}$ is a minimizer of the functional $\bar{F}_{\omega_0,v_0}(\cdot)$. Indeed, suppose that $\bar{y}$ does not belong to $\bar{Z}_0$. The set $\bar{Z}_0$ is nonempty therefore there exists some $y_0\in\bar{Z}_0$. Obviously, $\bar{F}_{\omega_0,v_0}(\bar{y})-\bar{F}_{\omega_0,v_0}(y_0)=\alpha>0$ and we have

$$\bar{\mu}_k-\bar{\mu}_0=\bar{F}_{\omega_k,v_k}(y_k)-\bar{F}_{\omega_0,v_0}(y_0)=\bar{F}_{\omega_k,v_k}(y_k)-\bar{F}_{\omega_0,v_0}(\bar{y})+\alpha.$$

Since $y_k\to\bar{y}$ in $L^s(\Omega,\mathbb{R}^N)$, $\omega_k\to\omega_0$ in $L^p(\Omega,\mathbb{R}^m)$ and $v_k\to v_0$ in $H^{1/2}(\partial\Omega,\mathbb{R}^N)$, the growth condition (2.6) and the Krasnosielskii theorem imply that $\bar{F}_{\omega_k,v_k}(y_k)-\bar{F}_{\omega_0,v_0}(\bar{y})\to 0$. Thus we have got a contradiction with (3.7). It means that $\bar{y}\in\bar{Z}_0$. Note that we have just proved that $\operatorname{Lim\,sup}\bar{Z}_k$ is a nonempty set with respect to the weak topology of $H_0^1(\Omega,\mathbb{R}^N)$ and $\operatorname{Lim\,sup}\bar{Z}_k\subset\bar{Z}_0$.

To complete the proof, we shall show that the sequence y_k converges to $\bar{y}$ in the norm of $H_0^1(\Omega, \mathbb{R}^N)$. By (2.8), we have

$$0 = \left\langle \frac{\partial}{\partial y} \bar{F}_{\omega_k, v_k}(y_k) - \frac{\partial}{\partial y} \bar{F}_{\omega_0, v_0}(\bar{y}), y_k - \bar{y} \right\rangle$$
$$= \int_\Omega |\nabla y_k(x) - \nabla \bar{y}(x)|^2 \, dx + \int_\Omega \langle \nabla(Tv_k)(x) - \nabla(Tv_0)(x), \nabla y_k(x) - \nabla \bar{y}(x) \rangle \, dx$$
$$+ \int_\Omega \langle \phi_z(x, y_k(x) + Tv_k(x), \omega_k(x)) - \phi_z(x, \bar{y}(x) + Tv_0(x), \omega_0(x)), y_k(x) - \bar{y}(x) \rangle \, dx. \qquad (3.9)$$

By assumption (3) of our theorem, $\nabla(Tv_k)$ tends to $\nabla(Tv_0)$ in $L^2(\Omega, \mathbb{R}^{Nn})$. Meanwhile, $\nabla y_k \to \nabla y_0$ weakly in $L^2(\Omega, \mathbb{R}^{Nn})$. Hence the second integral in equality (3.9) tends to zero. Using the Hölder inequality and the growth condition (2.6), we can estimate the last integral I_k in (3.9) as follows:

$$I_k \leqslant \left(\int_\Omega |\phi_z(x, y_k(x) + Tv_k(x), \omega_k(x)) - \phi_z(x, \bar{y}(x) + Tv_0(x), \omega_0(x))|^{s/(s-1)} \, dx \right)^{(s-1)/s}$$
$$\times \left(\int_\Omega |y_k(x) - \bar{y}(x)|^s \, dx \right)^{1/s}$$
$$\leqslant C_4 \left(\int_\Omega (1 + |y_k(x) + Tv_k(x)|^s + |\bar{y}(x) + Tv_0(x)|^s + |\omega_k(x)|^p + |\omega_0(x)|^p) \, dx \right)^{(s-1)/s}$$
$$\times \|y_k - \bar{y}\|_{L^s} \quad \text{if } p < \infty,$$

and in a similar manner

$$I_k \leqslant C_5 \left(\int_\Omega (1 + |y_k(x) + Tv_k(x)|^s + |\bar{y}(x) + Tv_0(x)|^s) \, dx \right)^{(s-1)/s} \|y_k - \bar{y}\|_{L^s} \text{ if } p = \infty,$$

where C_4 and C_5 are some positive constants. We have just assumed that y_k converges to $\bar{y}$ weakly in $H_0^1(\Omega, \mathbb{R}^N)$. It implies that y_k tends to $\bar{y}$ in $L^s(\Omega, \mathbb{R}^N)$ which together with assumptions (2) and (3) lead to the conclusion that the last integral in formula (3.9) converges to zero. Consequently, the first integral $\|y_k - \bar{y}\|_{H_0^1}^2 = \int_\Omega |\nabla y_k(x) - \nabla \bar{y}(x)|^2 \, dx$ tends to zero. Thus, we have shown that the weak convergence of the minimizers $y_k \in \bar{Z}_k$ to $\bar{y} \in \bar{Z}_0$ implies the strong convergence in $H_0^1(\Omega, \mathbb{R}^N)$.

Since $Z_k = \bar{Z}_k + Tv_k$ and $Tv_k \to Tv_0$ strongly in $H^1(\Omega, \mathbb{R}^N)$, we obtain assertion (c), which completes the proof. □

Let us return to the boundary value problem (2.2)–(2.3). Denote by S_k, $k = 0, 1, 2, \ldots$ the set of the solutions of the problem which correspond to the parameter ω_k and to the boundary data v_k. It is a well-known fact (see for instance [17,18]) that for the convex functional of action the set of minimizers Z_k coincides with the set S_k. Hence Theorem 3.1 implies the following:

Corollary 3.1. *If*

(1) *the integrand* ϕ *and the sequences* $\{\omega_k\}$ *and* $\{v_k\}$ *satisfy assumptions* (1–3) *of Theorem* 3.1,
(2) *the functional of action* (2.4) *is convex*,

then the sequence S_k, $k=0,1,2,\dots$ *satisfies assertions* (a–c) *of Theorem* 3.1 *with* $Z_k=S_k$, $k=0,1,2,\dots$.

If the functional of action is strictly convex, then problem (2.2)–(2.3) *possesses uniquely determined solution* z_k, $k=0,1,2,\dots$ *and* $\lim z_k = z_0$ *in* $H^1(\Omega,\mathbb{R}^N)$.

4. Continuous dependence of the solutions on parameters and boundary data: the case of the weak topology in the space of distributed parameters

In this section, we shall assume that the integrand ϕ is linear with respect to the distributed parameter ω, i.e.,

$$\phi(x,z,\omega) = \phi^1(x,z) + \langle \phi^2(x,z),\omega\rangle, \tag{4.1}$$

where $\phi^1 : \Omega\times\mathbb{R}^N \to \mathbb{R}$, $\phi^2 : \Omega\times\mathbb{R}^N \to \mathbb{R}^m$ and $\omega(\cdot)\in L^\infty(\Omega,\mathbb{R}^m)$. In this case, the boundary value problem (2.2)–(2.3) takes the form

$$\Delta z(x) = \phi^1_z(x,z(x)) + \langle \phi^2_z(x,z(x)),\omega(x)\rangle \tag{4.2}$$

$$z(x) = v(x) \text{ on } \partial\Omega \tag{4.3}$$

and the functional of action is given by the equality

$$F^1_{\omega,v}(z) = \int_\Omega [\tfrac{1}{2}|\nabla z(x)|^2 + \phi^1(x,z(x)) + \langle\phi^2(x,z(x)),\omega(x)\rangle]\,\mathrm{d}x, \tag{4.4}$$

where $z(\cdot)\in H^1(\Omega,\mathbb{R}^N)$, $z(x)=v(x)$ on $\partial\Omega$, $\omega(\cdot)\in L^\infty(\Omega,\mathbb{R}^m), v(\cdot)\in H^{1/2}(\partial\Omega,\mathbb{R}^N)$.

Let $\{\omega_k\}$ and $\{v_k\}$ be sequences of the distributed parameters and the boundary conditions, respectively. Denote by Z^1_k a set of all minimizers of the functional of action (4.4) with $\omega=\omega_k$ and $v=v_k$. We shall prove:

Theorem 4.1. *Suppose that*

(1) *the integrand* ϕ *is of the form* (4.1) *and satisfies conditions* (2.5)–(2.7),
(2) *the sequence of distributed parameters* ω_k, $k=1,2,\dots$ *tends to* ω_0 *in the weak* $*$ *topology of the space* $L^\infty(\Omega,\mathbb{R}^m)$,
(3) *the sequence* v_k, $k=1,2,\dots$ *tends to* v_0 *in* $H^{1/2}(\partial\Omega,\mathbb{R}^N)$.

Then the sequence Z^1_k, $k=0,1,2,\dots$ *satisfies assertions* (a–c) *of Theorem* 3.1 *with* Z_k *replaced by* Z^1_k, $k=0,1,2,\dots$.

Proof. As in the proof of Theorem 3.1, we put

$$\begin{aligned}\bar{F}^1_{\omega_k,v_k}(y) &= F^1_{\omega_k,v_k}(y+Tv_k)\\ &= \int_\Omega [\tfrac{1}{2}|\nabla y(x)+\nabla(Tv_k)(x)|^2+\phi^1(x,y(x)+Tv_k(x))\\ &\quad +\langle\phi^2(x,y(x)+Tv_k(x)),\omega_k(x)\rangle]\,\mathrm{d}x, \end{aligned}\tag{4.5}$$

where $\omega_k\in L^\infty(\Omega,\mathbb{R}^m)$, $v_k\in H^{1/2}(\partial\Omega,\mathbb{R}^N)$, $k=0,1,2,\ldots$ and $y(\cdot)\in H^1_0(\Omega,\mathbb{R}^N)$.

Since the integrand ϕ defined by (4.1) satisfies conditions (2.5)–(2.7), Theorem 3.1 implies immediately that the sequence Z^1_k, $k=0,1,2,\ldots$ fulfills assertions (a) and (b).

Let us notice that the weak $*$ convergence of ω_k to ω_0 in $L^\infty(\Omega,\mathbb{R}^m)$ and the strong convergence of v_k to v_0 in $H^{1/2}(\partial\Omega,\mathbb{R}^N)$ imply that the sequence $\bar{F}^1_{\omega_k,v_k}(\cdot)$ converges to $\bar{F}^1_{\omega_0,v_0}(\cdot)$ uniformly on any ball $\bar{B}(0,\rho)\subset H^1_0(\Omega,\mathbb{R}^N)$. By (4.5), we have

$$\begin{aligned}|\bar{F}^1_{\omega_k,v_k}(y)-\bar{F}^1_{\omega_0,v_0}(y)| \leqslant{}& \left|\int_\Omega\langle\nabla y(x),\nabla(Tv_k)(x)-\nabla(Tv_0)(x)\rangle\,\mathrm{d}x\right|\\ &+\tfrac{1}{2}\left|\int_\Omega[|\nabla(Tv_k)(x)|^2-|\nabla(Tv_0)(x)|^2]\,\mathrm{d}x\right|\\ &+\left|\int_\Omega[\phi^1(x,y(x)+Tv_k(x))-\phi^1(x,y(x)+Tv_0(x))]\,\mathrm{d}x\right|\\ &+\left|\int_\Omega[\langle\phi^2(x,y(x)+Tv_k(x)),\omega_k(x)\rangle\right.\\ &\left.-\langle\phi^2(x,y(x)+Tv_0(x)),\omega_0(x)\rangle]\,\mathrm{d}x\right|.\end{aligned}\tag{4.6}$$

The same reasoning as in the proof of Theorem 3.1, Step 2, gives that the first three integrals in (4.6) converge to zero uniformly on the ball $\bar{B}(0,\rho)$.

Suppose that the last integral $I^4_k(y)$ does not tend uniformly on $\bar{B}(0,\rho)$ to zero. It means that there exists $\varepsilon_0>0$ and a sequence $\{y_k\}\subset\bar{B}(0,\rho)$ such that $|I^4_k(y_k)|>\varepsilon_0$. Passing, if necessary, to a subsequence, we can assume that y_k tends to some $\bar{y}$ weakly in $H^1_0(\Omega,\mathbb{R}^N)$ and we have

$$I^4_k(y_k)=\left|\int_\Omega[\langle\phi^2(x,y_k(x)+Tv_k(x)),\omega_k(x)\rangle-\langle\phi^2(x,\bar{y}(x)+Tv_0(x)),\omega_0(x)\rangle]\,\mathrm{d}x\right|.$$

By the Sobolev embedding theorem y_k tends to $\bar{y}$ in $L^s(\Omega,\mathbb{R}^N)$. From assumption (3) Tv_k tends to Tv_k in $H^1(\Omega,\mathbb{R}^N)$ and consequently Tv_k tends to Tv_k in $L^s(\Omega,\mathbb{R}^N)$. Furthermore, due to the growth conditions (2.6) and the Krasnosielskii theorem both sequences $\phi^2(x,y_k(x)+Tv_k(x))$ and $\phi^2(x,y_k(x)+Tv_0(x))$ tend to $\phi^2(x,\bar{y}(x)+Tv_0(x))$ in $L^1(\Omega,\mathbb{R}^m)$ and by assumption (2), ω_k tends to ω_0 in the weak $*$ topology of $L^\infty(\Omega,\mathbb{R}^m)$. Therefore, the last integral in (4.6) tends to zero, which contradicts the inequality $|I^4_k(y_k)|>\varepsilon_0$. It means that $I^4_k(y)$ tends to zero uniformly on the ball $\bar{B}(0,\rho)$ and, in consequence, the sequence $\bar{F}^1_{\omega_k,v_k}(y)$ tends to $\bar{F}^1_{\omega_0,v_0}(y)$ uniformly on any ball from $H^1_0(\Omega,\mathbb{R}^N)$ provided that $\omega_k\to\omega_0$ in the weak $*$ topology of $L^\infty(\Omega,\mathbb{R}^m)$ and $v_k\to v_0$ in

$H^{1/2}(\partial\Omega, \mathbb{R}^N)$. Using this fact, one can prove, quite similarly as in the proof of Theorem 3.1 (see Step 3), assertion (c). □

Theorem 4.1 implies the following:

Corollary 4.1. *If the integrand* $\phi(x,z,\omega) = \phi^1(x,z) + \langle \phi^2(x,z), \omega\rangle$ *and the sequences* $\{\omega_k\}$ *and* $\{v_k\}$ *satisfy the assumptions of Theorem* 4.1 *and the functional of action defined by* (4.4) *is convex, then the sequence* S_k^1 *of solutions of problem* (4.2)–(4.3) *satisfies conditions* (a–c) *of Theorem* 4.1 *with* $Z_k^1 = S_k^1$. *If the functional of action is strictly convex, then the set* S_k^1 *is a singleton, i.e.,* $S_k^1 = \{z_k\}$ *and* $z_k \to z_0$ *in* $H^1(\Omega, \mathbb{R}^N)$ *provided that* $\omega_k \to \omega_0$ *weakly* $*$ *in* $L^\infty(\Omega, \mathbb{R}^m)$ *and* $v_k \to v_0$ *in* $H^{1/2}(\partial\Omega, \mathbb{R}^N)$.

Now, we give an example of applications of Theorems 3.1 and 4.1.

Example 4.1. Suppose that Ω is a three-dimensional block heated from outside by radiant heater which generates some temperature $v = v(x)$ on the surface $\partial\Omega$. Next, assume that there is a compressor inside the block Ω which generate the pressure $\omega = \omega(x)$ at the point $x \in \Omega$. In this case, the temperature inside the block Ω is given by the function $z = z(x)$ which satisfies the elliptic equation of the form (1.1) with the boundary condition (1.2) (cf. [5,27]). Suppose that system (1.1)–(1.2) is of the form

$$\Delta z(x) = \alpha|\omega(x)|^2|z(x)|^2 z(x) - \tfrac{1}{4} z(x) + \omega(x),$$

$$z(x) = v(x) \text{ on } \partial\Omega \tag{4.7}$$

with the boundary data

$$z(x) = v(x) \text{ on } \partial\Omega.$$

We assume that $x \in \Omega = \{x \in \mathbb{R}^3;\ |x| \leqslant \frac{1}{2}\}$, $\alpha \geqslant 0$, $z(\cdot) \in H^1(\Omega, \mathbb{R})$, $\omega(\cdot) \in L^\infty(\Omega, \mathbb{R})$, $v(\cdot) \in H^{1/2}(\partial\Omega, \mathbb{R})$.

It is easy to see that the functional of action for system (4.7) takes the form

$$F_{\omega,v}(z) = \int_\Omega \left[\tfrac{1}{2}|\nabla z(x)|^2 + \tfrac{\alpha}{4}|\omega(x)|^2|z(x)|^4 - \tfrac{1}{8}|z(x)|^2 + \omega(x)z(x)\right] \mathrm{d}x$$

and is strictly convex for any distributed parameter $\omega(\cdot) \in L^\infty(\Omega, \mathbb{R})$. Thus, for any $\omega = \omega_k$ and $v = v_k$, there exists exactly one solution $z_k(\cdot) \in H^1(\Omega, \mathbb{R})$ of system (4.7). By Theorem 3.1 it follows that $z_k \to z_0$ in $H^1(\Omega, \mathbb{R})$ provided that $\omega_k \to \omega_0$ in $L^\infty(\Omega, \mathbb{R})$ and $v_k \to v_0$ in $H^{1/2}(\partial\Omega, \mathbb{R})$. Next, let us put $\omega_0 \equiv 1$, $v_0 \equiv 4$ and $\alpha = 0$. It is easy to verify that in this case $z_0(x) \equiv 4$ and Theorem 4.1 implies that $\|z_k - z_0\|^2 = \int_\Omega [|z_k(x) - 4|^2 + |\nabla z_k(x)|^2]\,\mathrm{d}x$ tends to zero provided that ω_k tends to $\omega_0 = 1$ weakly $*$ in $L^\infty(\Omega, \mathbb{R})$, i.e., $\int_\Omega \langle a(x), \omega_k(x) - 1\rangle\,\mathrm{d}x \to 0$, for any $a(\cdot) \in L^1(\Omega, \mathbb{R})$ and $v_k \to 4$ in $H^{1/2}(\partial\Omega, \mathbb{R})$. It means that the function $z_0(x) = 4$ is a "good" approximation of a solution of the boundary value problem

$$\Delta z(x) = -\tfrac{1}{4} z(x) + \omega_k(x),$$

$$z(x) = v_k(x) \text{ for } x \in \partial\Omega$$

with ω_k and v_k sufficiently close to $\omega_0 = 1$ and $v_0 = 4$, respectively, in appropriate topologies.

5. Historical and bibliographical notes

The question of the existence of a solution for the initial value problem and its continuous dependence on the parameters and boundary data was investigated many years ago. Under simple and natural assumptions one can prove that the Cauchy problem for ODE of the form $\dot{y}(t) = \varphi(t, y(t), \omega_k(t))$, $y(t_0) = a_k$ possesses a unique solution $y = y_k(t)$, for any ω_k and a_k, and $y_k(\cdot)$ converges to $y_0(\cdot)$ provided that ω_k tends to ω_0 and a_k tends to a_0 in appropriate spaces and topologies (cf. for e.g., [4,7,21,22]).

In the case of the Cauchy problem for PDE of the first order the similar results have been proved in papers [21,26,32].

The question of the existence of a solution for the boundary value problem of the Dirichlet type, periodic, homocyclic type etc. was investigated in many papers and monographs. A wide survey of results and research methods can be found in monographs [6,17,18,23,25,33] and the references given there.

The literature on stability issues for the boundary value problems described by the ODE and PDE of the elliptic type is not very extensive. The stability of solutions of second-order ODE's with two-point boundary conditions was considered in the 1970s in papers [11,12,16,24,28] (see also the references therein).

All this works are dealt with the scalar equations and based on some direct methods related to the implicit function theorem.

The question of the continuous dependence of solutions of the linear elliptic equations with the variable Dirichlet boundary data and parameters was investigated, first time, in paper [20]. In this work sufficient conditions for stability of the linear PDE defined in the classical spaces are given. Similar results for scalar linear PDE with the Dirichlet boundary conditions defined in the Sobolev spaces $H^1(\Omega, \mathbb{R})$ and $H^{1/2}(\partial\Omega, \mathbb{R})$ are proved in paper [13]. The first results related to the stability of N-dimensional nonlinear boundary value problems with variable parameters appeared in the 1990s. In papers [1–3,8,9,19,29,31] ordinary differential equations with two-point boundary conditions and variable functional parameters were investigated, and the stability conditions with respect to the strong and weak topology have been proved.

Similar results for PDE with distributed parameters are given in papers [15,30].

In this paper we have investigated the stability problem for N-dimensional elliptic systems defined in the Sobolev space $H^1(\Omega, \mathbb{R}^N)$ with distributed parameters $\omega(\cdot)$ from the space $L^p(\Omega, \mathbb{R}^m)$ and variable boundary data $v(\cdot)$ from the space of traces $H^{1/2}(\partial\Omega, \mathbb{R}^N)$.

All the above works related to the N-dimensional nonlinear elliptic systems are based on the variational methods.

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