Stochastic Processes and their Applications 3 (1975) 1–18

LIMIT DISTRIBUTIONS FOR THE MAXIMA OF STATIONARY GAUSSIAN PROCESSES

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Received 18 January 1974

Let $\{X_n\}$ be a stationary Gaussian sequence with $\mathrm{E}\{X_0\} = 0$, $\mathrm{E}\{X_0^2\} = 1$ and $\mathrm{E}\{X_0 X_n\} = r_n$. Let $c_n = (2 \ln n)^{1/2}$, $b_n = c_n - \frac{1}{2} c_n^{-1} \ln(4\pi \ln n)$, and set $M_n = \max_{0 \leqslant k \leqslant n} X_k$. A classical result for independent normal random variables is that

$$\mathrm{P}[c_n(M_n - b_n) \leqslant x] \to \exp[-e^{-x}] \text{ as } n \to \infty \quad \text{for all } x. \tag{1}$$

Berman has shown that (1) applies as well to dependent sequences provided $r_n \ln n = \mathrm{o}(1)$. Suppose now that $\{r_n\}$ is a convex correlation sequence satisfying $r_n = \mathrm{o}(1)$, $(r_n \ln n)^{-1}$ is monotone for large n and $\mathrm{o}(1)$. Then

$$\mathrm{P}[r_n^{-1/2}(M_n - (1-r_n)^{1/2}\, b_n) \leqslant x] \to \Phi(x) \quad \text{for all } x, \tag{2}$$

where Φ is the normal distribution function. While the normal can thus be viewed as a second natural limit distribution for $\{M_n\}$, there are others. In particular, the limit distribution is given below when r_n is (sufficiently close to) $\gamma/\ln n$. We further exhibit a collection of limit distributions which can arise when r_n decays to zero in a nonsmooth manner. Continuous parameter Gaussian processes are also considered. A modified version of (1) has been given by Pickands for some continuous processes which possess sufficient asymptotic independence properties. Under a weaker form of asymptotic independence, we obtain a version of (2).

AMS Subj. Class.: Primary 60G10, 60G15; Secondary 60F99

limit distributions	stationary Gaussian sequences
maxima	stationary Gaussian processes

* Research supported in part by NSF Grant No. GP-28576 while at Northwestern University.
** Research supported in part by NSF Grant No. GP-33431X.

1. Introduction

$\{X_n\}$ denotes a stationary Gaussian sequence with $\mathrm{E}\{X_0\} = 0$, $\mathrm{E}\{X_0^2\} = 1$ and $\mathrm{E}\{X_0 X_n\} = r_n$. We set $c_n = (2 \ln n)^{1/2}$, $b_n = c_n - \frac{1}{2} c_n^{-1} \ln(4\pi \ln n)$ and take $M_n = \max_{0 \leqslant k \leqslant n} X_k$. Fisher and Tippett [2] first established the result that if $r_n = 0$, $n \neq 0$,

$$\mathrm{P}[c_n(M_n - b_n) \leqslant x] \to \exp[-\mathrm{e}^{-x}] \text{ as } n \to \infty \quad \text{for all } x. \tag{1.1}$$

Their work was later subsumed in the general theory of maxima of independent random variables as given by Gnedenko [3]. Since that time there has been some interest in the limit distribution question as it applied to dependent normal variables. Thus Watson [10], in treating M-dependent stationary sequences, found that (1.1) obtained if r_n was zero for $|n| \geqslant M$. Berman [1] subsequently proved that $r_n \ln n = \mathrm{o}(1)$ is a sufficient condition for (1.1). It may be seen below that Berman's result pushes matters about as far as is possible in this direction.

In Section 2 we are concerned with maxima when $r_n = \mathrm{o}(1)$ but $r_n \ln n \neq \mathrm{o}(1)$. In this setting no simple characterization of the set of possible limit distributions seems possible – in contrast to what occurs in the study of the maxima of independent random variables. However, a fairly complete picture of the situation will emerge from the theorems and discussion given below.

We first consider the boundary case $r_n = \gamma/\ln n$, $|n| \geqslant M$. In Theorem 2.3 the limit distribution is shown to be a convolution of the extreme-value distribution of (1.1) with a normal distribution whose parameters depend on γ. Next it is assumed that r_n is convex for $n \geqslant 0$, $r_n = \mathrm{o}(1)$ and that $(r_n \ln n)^{-1}$ is monotone for large n and $\mathrm{o}(1)$. Then (Theorem 2.4) the limit distribution for M_n is normal. We note here that Theorem 2.4 remains true when the convexity condition on r_n is replaced by a variety of weaker conditions (see the remark about this which follows the proof of Theorem 2.4).

Our proofs rely heavily on Berman's Lemma [1] (see also [5]). It says, implicitly, that some perturbation of the correlation sequence leaves the limit distribution for M_n unchanged. In this sense, condition (2.4) of Theorem 2.3 can be viewed as describing an appropriate neighborhood of the correlation sequence $\gamma/\ln n$, $|n| \geqslant M$. In Theorem 2.4 on the other hand, it is not worthwhile carrying through the same neighborhood argument. A use of Lemma 2.2 which is more appropriate to that context surfaces in the remark following Theorem 2.4.

At the end of Section 2 we emphasize the crucial role of some sort of smoothness in the decrease of r_n to zero once $r_n \ln n \neq o(1)$. We consider correlation sequences of the form $\{r_n p_n\}$, where $\{p_n\}$ is a periodic correlation sequence. Assuming that the maxima under $\{r_n\}$ can be handled, e.g. as in Theorems 2.3 and 2.4, one can track down the limit distribution under $\{r_n p_n\}$. It is hoped that the discussion given there will shed some light on the problem of characterizing limit distributions in the present context.

In Section 3, $\{X(t)\}$ denotes a continuous parameter stationary Gaussian process with mean value zero and correlation function $r(t)$. It is assumed that r satisfies

$$r(t) = 1 - c|t|^\alpha + o(|t|^\alpha), \quad 0 < \alpha \leq 2, c > 0, \tag{1.2}$$

for t in a neigborhood of zero. Accordingly, X may be taken to have continuous paths, and one can define $M_T = \max_{0 \leq t \leq T} X(t)$. Pickands [7] has shown that the extreme-value distribution of (1.1) is a limit distribution for M_T if $r(t) \ln t = o(1)$ (The exact statement is given at (3.2)). We show that if $r(t)$ is convex and $o(1)$ (now $0 < \alpha \leq 1$) and $(r(t) \ln t)^{-1}$ is monotone for large t and $o(1)$, then the limit distribution for M_T is normal (Theorem 3.1).

2. Limit distributions for M_n

Throughout this section, M_n will denote $\max_{0 \leq k \leq n} X_k$, where $\{X_n\}$ is stationary Gaussian, $E\{X_0\} = 0$, $E\{X_0^2\} = 1$ and $E\{X_0 X_n\} = r_n$. H will be the extreme-value distribution function of (1.1), while Φ will be the normal distribution function with $\Phi' = \varphi$. By $M_n(\rho)$ we mean the maximum of $n + 1$ standard normal variables with constant correlation ρ between any two. With this notation, (1.1) becomes

$$P[c_n(M_n(0) - b_n) \leq x] \to H(x) \text{ as } n \to \infty \quad \text{for all } x. \tag{2.1}$$

Observe further, and this is essential to us, that if U is standard normal and independent of $M_n(0)$, $M_n(\rho)$ may be represented as $(1-\rho)^{1/2} M_n(0) + \rho^{1/2} U$.

Repeated use is to be made of the (comparison) Lemmas of Slepian [9] and Berman [1]. For the present purposes, suppose $M_n' = \max_{0 \leq k \leq n} Y_k$, where $\{Y_n\}$ is stationary Gaussian, $E\{Y_0\} = 0$, $E\{Y_0^2\} = 1$ and $E\{Y_0 Y_n\} = \rho_n$.

Lemma 2.1 (Slepian). *If $\rho_k \leqslant r_k$ for all k, then*

$$\mathrm{P}[M'_n \leqslant c] \leqslant \mathrm{P}[M_n \leqslant c] \quad \text{for all } c. \tag{2.2}$$

Lemma 2.2 (Berman). *Let $\omega_k = \max\{r_k, \rho_k\}$. Then*

$$|\mathrm{P}[M_n \leqslant c] - \mathrm{P}[M'_n \leqslant c]| \leqslant n \sum_{k=1}^{n} |r_k - \rho_k| (1-\omega_k^2)^{-1/2} \exp[-c^2/(1+\omega_k)] \quad \text{for all } c \tag{2.3}$$

In demonstrating that $r_n \ln n = o(1)$ is sufficient for (1.1), Berman was able to use (2.3) for a direct comparison of M_n with $M_n(0)$. A straightforward application of (2.3) is also possible when $r_n \ln n = O(1)$. Indeed, Theorem 2.3 contains Berman's result and its proof bears a strong resemblance to his. On the other hand, Theorem 2.4 requires a more devious use of this comparison – the derivation of (2.3) requires estimates which become rather crude once $r_n \ln n \neq o(1)$.

Theorem 2.3. *Suppose $r_n \ln n = O(1)$ and*

$$\#\{1 \leqslant k \leqslant n \mid |r_k \ln k - \gamma| > \epsilon\} = o(n) \quad \text{for all } \epsilon > 0. \tag{2.4}$$

Then

$$\mathrm{P}[c_n(M_n - b_n) \leqslant x] \to \int_{-\infty}^{\infty} \exp[-\exp[-(x+\gamma-(2\gamma)^{1/2}y)]]\,\varphi(y)\,\mathrm{d}y$$

$$= \int_{-\infty}^{\infty} H(x-z)\,(2\gamma)^{-1/2}\varphi((z+\gamma)/2\gamma)\mathrm{d}z. \tag{2.5}$$

Proof. First note that $\gamma \geqslant 0$ follows from (2.4) and the fact that the sum of all entries in the covariance matrix of $X_0, X_1, \ldots, X_n$ must be nonnegative. Set $\rho_n = \gamma/\ln n$. The theorem will follow if we establish that $\mathrm{P}[M_n(\rho_n) \leqslant b_n + x/c_n]$ has the required limit in n and that

$$|\mathrm{P}[M_n \leqslant b_n + x/c_n] - \mathrm{P}[M_n(\rho_n) \leqslant b_n + x/c_n]| \leqslant$$

$$\leqslant n \sum_{k=1}^{n} |r_k - \rho_n| (1-\omega_k^2)^{-1/2} \exp[-(b_n + x/c_n)^2/(1+\omega_k)] = o(1), \tag{2.6}$$

where $\omega_k = \max\{r_k, \rho_n\}$. We first demonstrate (2.6). Let $\bar{r}(k) = \sup_{i \geqslant k} r_i$, note that $\bar{r}(1) < 1$, and take $0 < \theta < (1-\bar{r}(1))/(1+\bar{r}(1))$. Let $m = [n^\theta]$, and observe that for large n the right-hand side of (2.6) is no larger than

$$2n^{1+\theta}(1-r^{-2}(1))^{-1/2} \exp\left[-\frac{(b_n+x/c_n)^2}{1+\bar{r}(1)}\right] +$$

$$+\, n(1-r^{-2}(m))^{-1/2} \exp\left[-\frac{(b_n+x/c_n)^2}{1+\bar{r}(m)}\right] \sum_{k=m}^{n} |r_k - \rho_n|. \quad (2.7)$$

The first term of (2.7) goes to zero since $b_n^2 \sim 2 \ln n$.

Now $\bar{r}(m) \ln n = O(1)$, so

$$\frac{n^2}{\ln n} \exp\left[-\frac{(b_n+x/c_n)^2}{1+\bar{r}(m)}\right] = \exp\left[\frac{\bar{r}(m)}{1+\bar{r}(m)}(2\ln n - \ln\ln n) + o(1)\right]$$

$$= O(1).$$

Hence the second term of (2.7) will also be o(1) if

$$n^{-1} \ln n \sum_{k=m}^{n} |r_k - \rho_n| = o(1). \quad (2.8)$$

For $k \geqslant m$,

$$|r_k - \gamma/\ln k| = O(1/\ln n),$$

and we have for any ϵ,

$$\sum_{k=m}^{n} |r_k - \rho_n| \leqslant \sum_{k=m}^{n} \left|r_k - \frac{\gamma}{\ln k}\right| + \sum_{k=m}^{n} \left|\frac{\gamma}{\ln k} - \frac{\gamma}{\ln n}\right|$$

$$\leqslant \#\{m \leqslant k \leqslant n \mid |r_k \ln k - \gamma| > \epsilon\}\; O\left(\frac{1}{\ln n}\right)$$

$$+ \frac{\epsilon(n-m)}{\ln m} + \gamma \sum_{k=m}^{n} \left|\frac{1}{\ln k} - \frac{1}{\ln n}\right|.$$

(2.8) now follows if

$$\sum_{k=m}^{n} \left(\frac{1}{\ln k} - \frac{1}{\ln n}\right) = o\left(\frac{n}{\ln n}\right). \quad (2.9)$$

Using an integration by parts we find that

$$\sum_{k=m}^{n} \frac{1}{\ln k} \leqslant \frac{1}{\ln m} + \int_{m}^{n} \frac{1}{\ln x}\,\mathrm{d}x$$

$$\leqslant \frac{1}{\ln m} + \frac{n}{\ln n} - \frac{m}{\ln m} + \frac{n-m}{(\ln m)^2}.$$

Hence

$$\frac{\ln n}{n} \sum_{k=m}^{n} \left(\frac{1}{\ln k} - \frac{1}{\ln n}\right) \leqslant$$

$$\leqslant \left(\frac{1}{\ln m} + \frac{n}{\ln n} - \frac{m}{\ln m} + \frac{n-m}{(\ln m)^2}\right)\frac{\ln n}{n} - \frac{n-m}{n} = \mathrm{o}(1),$$

and (2.9) is established as in (2.6).

Now consider $\mathrm{P}[M_n(\rho_n) \leqslant b_n + x/c_n]$. Write this as

$$\mathrm{P}[(1-\rho_n)^{1/2} M_n(0) + \rho_n^{1/2}\, U \leqslant b_n + x/c_n] =$$

$$= \int_{-\infty}^{\infty} \mathrm{P}[M_n(0) \leqslant (b_n + x/c_n - \rho_n^{1/2} u)\,(1-\rho_n)^{-1/2}]\,\varphi(u)\,\mathrm{d}u$$

$$= \int_{-\infty}^{\infty} \mathrm{P}[M_n(0) \leqslant b_n + (x(u) + \mathrm{o}(1))/c_n]\,\varphi(u)\,\mathrm{d}u, \qquad (2.10)$$

where $x(u) = x - (2\gamma)^{1/2} u + \gamma$. Using (2.1) in (2.10), dominated convergence gives us

$$\int_{-\infty}^{\infty} \mathrm{P}[M_n(0) \leqslant b_n + (x(u) + \mathrm{o}(1))/c_n]\,\varphi(u)\,\mathrm{d}u \to$$

$$\int_{-\infty}^{\infty} \exp[-\exp[-x(u)]]\,\varphi(u)\,\mathrm{d}u,$$

which completes the proof of the theorem. □

Before beginning on Theorem 2.4 let us make the following observation. Assume in Theorem 2.3 that

$$r_n = \gamma/\ln n \quad \text{for } |n| \geqslant M.$$

Then according to (2.6), M_n has the same asymptotic distribution as $M_n(r_n)$. In Theorem 2.4 the same conclusion is ultimately made, viz. M_n and

$M_n(r_n)$ have the same asymptotic distribution. However, we are unable to demonstrate this fact directly using (2.3). Our proof of Theorem 2.4 involves trimming back to a maximum obtained from an appropriate set of $n \exp[-(\ln n)^{1/2}]$ variables. The trimming invokes the convexity of r_n and while this device ensures a lengthy proof, it appears that something akin to this is an essential step in obtaining the result.

Theorem 2.4. *Suppose that r_n is convex for $n \geqslant 0$, $r_n = o(1)$, and that $(r_n \ln n)^{-1}$ is monotone for large n and* o(1). *Then*

$$\mathbf{P}[r_n^{-1/2}(M_n-(1-r_n)^{1/2} b_n) \leqslant x] \to \Phi(x) \text{ as } n \to \infty \quad \text{for all } x. \tag{2.11}$$

Proof. The proof proceeds by showing that the lim inf of the left-hand side of (2.11) is at least $\Phi(x-\epsilon)$ for all $\epsilon > 0$, while the lim sup is at most $\Phi(x+\epsilon)$. The first statement is almost immediate. r_n is decreasing, so (2.2) gives us a first comparison of

$$\mathbf{P}[M_n \leqslant r_n^{1/2}x + (1-r_n)^{1/2} b_n] \geqslant$$

$$\geqslant \mathbf{P}[M_n(r_n) \leqslant r_n^{1/2}x + (1-r_n)^{1/2} b_n]. \tag{2.12}$$

But the right-hand side of (2.12) satisfies

$$\mathbf{P}[M_n(r_n) \leqslant (1-r_n)^{1/2} b_n + r_n^{1/2} x] =$$

$$= \mathbf{P}[(1-r_n)^{1/2} M_n(0) + r_n^{1/2} U \leqslant (1-r_n)^{1/2} b_n + r_n^{1/2} x]$$

$$= \int_{-\infty}^{\infty} \mathbf{P}[M_n(0) \leqslant b_n + r_n^{1/2}(x-u)/(1-r_n)^{1/2}]\, \varphi(u)\, du$$

$$\geqslant \mathbf{P}[M_n(0) \leqslant b_n + \epsilon\, r_n^{1/2}/(1-r_n)^{1/2}]\; \Phi(x-\epsilon),$$

and the last expression $\to \Phi(x-\epsilon)$ as $n \to \infty$ in view of (2.1) and the fact that $r_n^{1/2} c_n \to \infty$ as $n \to \infty$.

To do the other inequality, first note that the convexity of r_n ensures that there is a stationary Gaussian sequence $\{Y_k\} = \{Y_k(n)\}$ with the correlations

$$\rho_k = \rho_k(n) = (r_k - r_n)/(1-r_n) \quad \text{for } k = 1, 2, \ldots, n.$$

(For example, one may take $\rho_k(n) = 0$ for $k > n$ and apply Polya's criterion.)

Further, if U is supposed independent of $\{Y_k(n)\}$, X_k may be represented as $(1-r_n)^{1/2} Y_k(n) + r_n^{1/2} U$ for $k = 0, 1, ..., n$. Thus if M'_n denotes the maximum of $Y_0(n), ..., Y_n(n)$, we may write

$$M_n = (1-r_n)^{1/2} M'_n + r_n^{1/2} U.$$

Now

$$\begin{aligned} P[M_n \leqslant (1-r_n)^{1/2} b_n + r_n^{1/2} x] &= \\ &= P[(1-r_n)^{1/2} M'_n + r_n^{1/2} U \leqslant (1-r_n)^{1/2} b_n + r_n^{1/2} x] \\ &= \int_{-\infty}^{\infty} P[M'_n \leqslant b_n + (r_n^{1/2}(x-u))/(1-r_n)^{1/2}]\, \varphi(u)\, du \\ &\leqslant \Phi(x+\epsilon) + P[M'_n \leqslant b_n - \epsilon\, r_n^{1/2}/(1-r_n)^{1/2}], \end{aligned}$$

so the theorem will follow if

$$P[M'_n \leqslant b_n - \epsilon\, r_n^{1/2}/(1-r_n)^{1/2}] = o(1)$$

or

$$P[M'_n \leqslant b_n - \epsilon r_n^{1/2}] = o(1). \tag{2.13}$$

Let $t(n) = [n \exp[-(\ln n)^{1/2}]]$. The convexity of $\{\rho_k(n)\}$ again ensures that there is a stationary Gaussian sequence $\{Z_k\} = \{Z_k(n)\}$ with the correlations

$$\sigma_k = \sigma_k(n) = \rho_k \vee \rho_{t(n)}, \quad k = 1, 2, ..., n.$$

Let M''_n be the maximum of $Z_0(n), ..., Z_n(n)$ and use (2.2) to produce

$$\begin{aligned} P[M'_n \leqslant b_n - \epsilon r_n^{1/2}] &\leqslant P[M''_n \leqslant b_n - \epsilon r_n^{1/2}] \\ &\leqslant P[M_n(\rho_{t(n)}(n)) \leqslant b_n - \epsilon r_n^{1/2}] \\ &\quad + |P[M_n(\rho_{t(n)}) \leqslant b_n - \epsilon r_n^{1/2}] - P[M''_n \leqslant b_n - \epsilon r_n^{1/2}]|. \end{aligned} \tag{2.14}$$

We shall complete matters by showing that both terms on the right-hand side of (2.14) tend to zero, the first by direct means and the second by appealing to (2.3).

To show that

$$P[M_n(\rho_{t(n)}) \leqslant b_n - \epsilon r_n^{1/2}] = o(1),$$

we need an estimate of the size of $\rho_{t(n)}$. For $k < n$,

$$r_k - r_n = \sum_{j=k}^{n-1} (r_j - r_{j+1}) \leqslant \sum_{j=k}^{n-1} \frac{r_k}{r_j} (r_j - r_{j+1}) = r_k \sum_{j=k}^{n-1} 1 - \frac{r_{j+1}}{r_j}$$

$$\leqslant r_k \sum_{j=k}^{n-1} -\ln \frac{r_{j+1}}{r_j} = r_k \ln \frac{r_k}{r_n}.$$

But if k is sufficiently large that $r_n \ln n$ is monotone for $n \geqslant k$,

$$\rho_k(n) = \frac{r_k - r_n}{1-r_n} \leqslant \frac{r_k}{1-r_n} \ln \frac{r_k}{r_n} \leqslant \frac{r_k}{1-r_n} \ln \frac{\ln n}{\ln k}. \tag{2.15}$$

Now

$$\mathbf{P}[M_n(\rho_{t(n)}) \leqslant b_n - \epsilon r_n^{1/2}] = \mathbf{P}[(1-\rho_{t(n)})^{1/2} M_n(0) + \rho_{t(n)}^{1/2} U \leqslant b_n - \epsilon r_n^{1/2}]$$

$$= \int_{-\infty}^{\infty} \mathbf{P}\left[M_n(0) \leqslant b_n + \frac{b_n \rho_{t(n)}}{(1-\rho_{t(n)})^{1/2}(1+(1-\rho_{t(n)})^{1/2})} - \frac{\epsilon r_n^{1/2} + \rho_{t(n)}^{1/2} u}{(1-\rho_{t(n)})^{1/2}}\right] \varphi(u)\, du$$

$$\leqslant \Phi(-\epsilon r_n^{1/2} / 2\rho_{t(n)}^{1/2})$$

$$+ \mathbf{P}\left[M_n(0) \leqslant b_n + \frac{b_n \rho_{t(n)}}{(1-\rho_{t(n)})^{1/2}(1+(1-\rho_{t(n)})^{1/2})} - \frac{\epsilon r_n^{1/2}}{2(1-\rho_{t(n)})^{1/2}}\right]. \tag{2.16}$$

For the first term on the right-hand side of (2.16), note that (2.15) gives

$$\frac{r_n}{\rho_{t(n)}} \geqslant \frac{r_n (1-r_n)}{r_{t(n)} \ln(\ln n / \ln t(n))} \sim \frac{r_n (1-r_n)(\ln n)^{1/2}}{r_{t(n)}} \to \infty$$

since $1 < r_{t(n)}/r_n \leqslant \ln n / \ln t(n) \to 1$ as $n \to \infty$. To see that the second term on the right-hand side of (2.16) is o(1), note that

$$\frac{b_n \rho_{t(n)}}{r_n^{1/2}} \leqslant \frac{b_n r_{t(n)}}{(1-r_n) r_n^{1/2}} \ln \frac{\ln n}{\ln t(n)}$$

$$\sim \frac{b_n}{(\ln n)^{1/2}} \frac{r_{t(n)}}{(1-r_n) r_n^{1/2}} \to 0,$$

and invoke (2.1).

It remains to be seen that the last term in (2.14) is o(1). Recall that the correlation sequences associated with $M_n(\rho_{t(n)})$ and M_n'' agree for all $k \geqslant t(n)$.

Then according to (2.3),

$$|\mathbf{P}[M_n(\rho_{t(n)}) \leqslant b_n - \epsilon r_n^{1/2}] - \mathbf{P}[M_n'' \leqslant b_n - \epsilon r_n^{1/2}]| \leqslant$$
$$\leqslant n \sum_{k=1}^{t(n)} \rho_k (1-\rho_k^2)^{-1/2} \exp[-(b_n - \epsilon r_n^{1/2})^2/(1+\rho_k)]. \quad (2.17)$$

Select an increasing sequence of integers $t_i(n)$ as follows:

$$t_0(n) = [n^\theta], \qquad 0 < \theta < (1-\rho_1)/(1+\rho_1),$$

$$t_i(n) = [\exp[(1-r_n^{1/2}) \ln n]], \quad i = 1, 2, \ldots, q(n),$$

$$t_{q(n)+1}(n) = t(n),$$

where $q(n)$ satisfies

$$r_n^{q(n)/2} > (\ln n)^{-1/2} \geqslant r_n^{(q(n)+1)/2}.$$

Observe that $q(n) < \ln \ln n / |\ln n| < \ln \ln n$ for large n. Now the right-hand side of (2.17) is bounded for large n by

$$n^{1+\theta} \rho_1 \exp\left[-\frac{(b_n - \epsilon r_n^{1/2})^2}{1+\rho_1}\right]$$
$$+ n \sum_{i=0}^{q(n)} t_{i+1}(n) \rho_{t_i(n)} \exp\left[\frac{-(b_n - \epsilon r_n^{1/2})^2}{1+\rho_{t_i(n)}}\right]. \quad (2.18)$$

The first term in (2.18) is negligible as at (2.7). The term in (2.18) indexed by $i < q(n)$ is bounded by

$$\exp\left[(2-r_n^{(i+1)/2}) \ln n - \frac{2 \ln n}{1+\rho_{t_i(n)}} + o((\ln n)^{1/2}\right] =$$
$$= \exp\left[\ln n\left(\frac{2\rho_{t_{i(n)}}}{1+\rho_{t_{i(n)}}} - r_n^{(i+1)/2}\right) + o((\ln n)^{1/2})\right], \quad (2.19)$$

while the term indexed by $q(n)$ is at most

$$\exp\left[2 \ln n - (\ln n)^{1/2} - \frac{2 \ln n}{1+\rho_{t_{q(n)}}} + o((\ln n)^{1/2})\right] =$$
$$= \exp\left[(\ln n)^{1/2}\left(\frac{2\, \rho_{t_{q(n)}} (\ln n)^{1/2}}{1+\rho_{t_{q(n)}}} - 1\right) + o((\ln n)^{1/2})\right]. \square \quad (2.20)$$

We shall find that (2.19) is bounded by

$$\exp[-\tfrac{1}{2}(\ln n)^{1/2} + o((\ln n)^{1/2}] \quad \text{for } i = 0, 1, ..., q(n)-1,$$

and that (2.20) admits the same bound. This will complete the theorem as (2.18) is then no larger than

$$(q(n)+1)\exp[-\tfrac{1}{2}(\ln n)^{1/2} + o((\ln n)^{1/2})] + o(1) <$$

$$< \ln\ln n \exp[-\tfrac{1}{2}(\ln n)^{1/2} + o((\ln n)^{1/2})] + o(1).$$

To bound (2.19) and (2.20), first use (2.15) to obtain

$$\rho_{t_0(n)} \leqslant 2\, r_{t_0(n)}\,(-\ln\theta),$$

$$\rho_{t_i(n)} \leqslant 2\, r_{t_i(n)} \ln(1-r_n^{1/2})^{-1} \leqslant 2\, r_{t_i(n)} r_n^{1/2} \quad \text{for } i = 1, 2, ..., q(n).$$

With $i = 0$, (2.19) is bounded by

$$\exp[\ln n(4\, r_{t_0(n)}\,(-\ln\theta) - r_n^{1/2}) + o((\ln n)^{1/2})] =$$

$$= \exp[(\ln n) r_n^{1/2}(4 r_{t_0(n)} r_n^{-1/2}\,(-\ln\theta) - 1) + o((\ln n)^{1/2})]$$

$$< \exp[-\tfrac{1}{2} r_n^{1/2} \ln n + o((\ln n)^{1/2})] < \exp[-\tfrac{1}{2}(\ln n)^{1/2} + o((\ln n)^{1/2})],$$

where the last inequality follows from $r_n^{1/2} > (\ln n)^{-1/2}$. For $0 < i < q(n)$ we again have $r_n^{(i+1)/2} > (\ln n)^{-1/2}$ and so (2.19) is no more than

$$\exp[r_n^{(i+1)/2} \ln n(4 r_{t_i(n)}\, r_n^{1/2} - 1) + o((\ln n)^{1/2})] <$$

$$< \exp[-\tfrac{1}{2}(\ln n)^{1/2} + o((\ln n)^{1/2})].$$

Lastly, (2.20) is at most

$$\exp[(\ln n)^{1/2}(4 r_{t_{q(n)}}\, r_n^{q(n)/2}\,(\ln n)^{1/2} - 1) + o((\ln n)^{1/2})] <$$

$$< \exp[(\ln n)^{1/2}(4 r_{t_{q(n)}}\, r_n^{-1/2} - 1) + o((\ln n)^{1/2})] <$$

$$< \exp[-\tfrac{1}{2}(\ln n)^{1/2} + o((\ln n)^{1/2})]$$

using the fact that $r_n^{(q(n)+1)/2}(\ln n)^{1/2} \leqslant 1$. The proof is now complete since (2.14) has been shown to be o(1) via (2.16), (2.17) and (2.18). □

Remark. The convexity condition on r_n in Theorem 2.4 can be weakened. It is first of all clear that r_n convex for $n \geqslant M$ would suffice – such a correlation sequence differs from some convex one for finitely many

n and one then appeals to Berman's Lemma. As a more substantial weakening, we can prove (rather arduously) that r_n convex may be replaced by r_n monotone. The first part of the proof follows exactly as at (2.12). The other part of the proof uses the blocking technique described preceding [4, (2.1)]. Whereas [4] uses a one-stage blocking, we are forced here to do a $(q(n)+2)$-stage blocking into the block sizes $t_i(n)$ as given above (2.18). The proof is long and not particularly instructive, so we have chosen not to give it. With this fact at hand it is subsequently easy to prove that r_n monotone may be replaced by r_n positive and monotone for large n.

We would like at this point to indicate how nonsmooth behavior in $\{r_n\}$ can lead one away from the limit distributions of Theorems 2.3 and 2.4. The device we invoke is the following: Given $\{r_n\}$ and a periodic correlation sequence $\{p_n\}$, $\{r_n p_n\}$ is a new correlation sequence which can have rather large oscillations. If $r_n \ln n = o(1)$, the limit distribution for M_n is unaffected by this change, but this is no longer true when $r_n \ln n \neq o(1)$. Here then is a first simple example which already suffices to distinguish this context from the independent variables setting. Take

$$p_k = 0 \quad \text{for } k = 1, 2, \dots, d-1, \qquad p_d = 1,$$

and suppose that $\{r_n\}$ satisfies the assumptions of Theorem 2.4. If $\tilde{M}_n$ denotes the maximum under the correlation sequence $\{r_n p_n\}$, $\tilde{M}_n$ is itself the maximum of d independent maxima of sequences with the correlation $\{r_{kd}\}$. Applying Theorem 2.4 to each of these d sequences and checking the behavior of the centering and scaling constants leads one to

$$\mathrm{P}[r_n^{-1/2}(\tilde{M}_n - (1-r_n)^{1/2} b_n) \leqslant x] \to \Phi^d(x) \quad \text{for all } x. \tag{2.21}$$

Now distribution types which are closed with respect to taking powers are, equivalently, those which arise as limit distributions for the maxima of independent variables. In Theorems 2.3 and 2.4 we do not have such closure and new laws arise.

Here is the information we have about other limit laws that arise in this way. Suppose $\{p_n\}$ is arbitrary and $d = \min\{n > 0: p_n = 1\}$. Let $U_0, \dots, U_{d-1}$ be a stationary Gaussian sequence with $\mathrm{E}\{U_0\} = 0$, $\mathrm{E}\{U_0^2\} = 1$ and $\mathrm{E}\{U_0 U_n\} = p_n$, and let $\{\tilde{M}_n\}$ denote the maxima associated with the correlation sequence $\{r_n p_n\}$. If $\{r_n\}$ satisfies the assumptions of Theorem 2.3, we can show that

$$\mathrm{P}[c_n(\tilde{M}_n - b_n) \leqslant x] \to \mathrm{E} \prod_{i=0}^{d-1} H(x + \ln d + \gamma - (2\gamma)^{1/2} U_i)). \tag{2.22}$$

If $\{r_n\}$ satisfies the assumptions of Theorem 2.4 we expect that

$$\mathrm{P}[r_n^{-1/2}(\widetilde{M}_n - (1-r_n)^{1/2} b_n) \leqslant x] \to G(x), \tag{2.23}$$

where G is the distribution function of $\max_{0 \leqslant k \leqslant d-1} U_k$. We have not proved (2.23) in this generality. In case $\{p_k\}$ is nonnegative however, this can be demonstrated by two simple comparisons. A suitable lower bound for the probability in (2.23) is obtained by comparing $\widetilde{M}_n$ with $\max_{0 \leqslant k \leqslant n}((1-r_n)^{1/2} W_k + r_n^{1/2} U_k)$ where the W_k's are independent, independent of $\{U_k\}$ and standard normal. An upper bound can be found by representing $\widetilde{M}_n$ by $\max_{0 \leqslant k \leqslant n}((1-r_n)^{1/2} Y_k(n) + r_n^{1/2} U_k)$ where $\{Y_k(n)\}$ is the sequence used in Theorem 2.4.

3. Limit distributions for M_T

In this section $\{X(t)\}$ is to be a continuous parameter, stationary Gaussian process with

$$\mathrm{E}\{X(0)\} = 0, \qquad \mathrm{E}\{X^2(0)\} = 1, \qquad \mathrm{E}\{X(0)X(t)\} = r(t).$$

We suppose throughout that

$$r(t) = 1 - c|t|^\alpha + \mathrm{o}(|t|^\alpha), \tag{3.1}$$

for t a neighborhood of zero where α and c are constants satisfying $0 < \alpha \leqslant 2$ and $c > 0$. As a consequence of (3.1) we may take X to have continuous paths and M_T to be defined by $\max_{0 \leqslant t \leqslant T} X(t)$.

We first state a result of Pickands [7] which shows that the extreme-value distribution H is a limit distribution for M_T under suitable asymptotic conditions on r. For this, consider a separable, nonstationary Gaussian process $Y(t)$, $t \geqslant 0$, with $\mathrm{E}\{Y(t)\} = -|t|^\alpha$ and

$$\mathrm{Cov}(Y(s), Y(t)) = |s|^\alpha + |t|^\alpha - |s-t|^\alpha.$$

Set

$$H_\alpha = \lim_{T \to \infty} T^{-1} \int_0^\infty e^u \, \mathrm{P}[\max_{0 \leqslant t \leqslant T} Y(t) > u] \, \mathrm{d}u,$$

$$c_T = (2 \ln T)^{1/2},$$

$$\beta_T = c_T + \{(\alpha^{-1} - \tfrac{1}{2}) \ln \ln T + \ln((2\pi)^{-1/2} c^{1/\alpha} H_\alpha 2^{(2-\alpha)/2\alpha})\} c_T^{-1},$$

in which it should be noted that H_α is positive and finite [7]. With (3.1) in force, Pickands has shown that $r(t) \ln t = \mathrm{o}(1)$ implies

$$\mathrm{P}[c_T(M_T-\beta_T)\leqslant x] \to \exp[-e^{-x}] \text{ as } T\to\infty \quad \text{for all } x. \tag{3.2}$$

We are now in a position to state:

Theorem 3.1. *Suppose that* (3.1) *holds with* $0<\alpha\leqslant 1$, $r(t)$ *is convex for* $t\geqslant 0$ *and* o(1), *and that* $(r(t)\ln t)^{-1}$ *is monotone for large* t *and* o(1). *Then*

$$\mathrm{P}[r^{-1/2}(T)\,(M_T-(1-r(T))^{1/2}\,\beta_T)\leqslant x]\to \Phi(x) \text{ as } T\to\infty \quad \text{for all } x. \tag{3.3}$$

Proof. We proceed, as in Theorem 2.4, to show first that the lim inf of the left-hand side of (3.3) is at least $\Phi(x-\epsilon)$ and then that its lim sup is at most $\Phi(x+\epsilon)$.

Let $\rho(t)$ be a correlation function satisfying

$$\rho(t) = 1 - 2c\,|t|^\alpha + o(|t|^\alpha) \quad \text{as } t\to 0.$$

Then there exists a $\tau>0$ such that for all sufficiently large T,

$$\rho(t)\,(1-r(T))+r(T)\leqslant r(t) \quad \text{for all } 0\leqslant t\leqslant \tau. \tag{3.4}$$

Consider a process $\{Y_1(t),\, t\geqslant 0\}$, composed of standard normal variables, which, on each interval $[k\tau, (k+1)\tau)$, has the correlation function ρ, has continuous paths and is independent of $Y_1(t)$, $t\in[j\tau,(j+1)\tau)$ if $j\neq k$. If T is large and U is a standard normal variable independent of $\{Y_1(t),\, t\geqslant 0\}$, then (3.4) ensures that

$$\mathrm{E}\{X(s)X(t)\}\geqslant \mathrm{E}\{[(1-r(T))^{1/2}\,Y_1(s)+r^{1/2}(T)\,U]$$
$$[(1-r(T))^{1/2}\,Y_1(t)+r^{1/2}(T)\,U]\}$$

for all s and t. Applying (2.2) we find that

$$\mathrm{P}[M_T\leqslant (1-r(T))^{1/2}\,\beta_T + r^{1/2}(T)\,x]\geqslant$$

$$\geqslant \mathrm{P}[(1-r(T))^{1/2}\sup_{0\leqslant t\leqslant T} Y_1(t)+r^{1/2}(T)\,U\leqslant (1-r(T))^{1/2}\,\beta_T + r^{1/2}(T)\,x]$$

$$\geqslant \int_{-\infty}^{\infty} \mathrm{P}^{[T/\tau]+1}\left[\sup_{0\leqslant t\leqslant \tau} Y_1(t)\leqslant \beta_T+\left(\frac{r(T)}{(1-r(T))}\right)^{1/2}(x-u)\right]\varphi(u)\,du$$

$$\geqslant \Phi(x-\epsilon)\,\mathrm{P}^{[T/\tau]+1}\,[\sup_{0\leqslant t\leqslant\tau} Y_1(t)\leqslant \beta_T+\epsilon\, r^{1/2}(T)]. \tag{3.5}$$

Thus the lim inf of the left-hand side of (3.3) is at least $\Phi(x-\epsilon)$ provided

$$\mathbf{P}^{[T/\tau]+1}[\sup_{0\leq t\leq \tau} Y_1(t) \leq \beta_T + \epsilon\, r^{1/2}(T)] \to 1 \quad \text{as } T\to\infty. \tag{3.6}$$

Consider $A = \inf_{0\leq t\leq\tau}(1-\rho(t))|t|^{-\alpha}$. We claim that $A > 0$ since otherwise there is a $t_0 \neq 0$ for which $\rho(t_0) = 1$. But then $r(t_0) = 1$ because of (3.4), and this contradicts $r(t) = o(1)$. Since $A > 0$, [6, Lemma 2.9] is applicable (see also [8, Lemma 1.2]) and

$$\frac{\mathbf{P}[\sup_{0\leq t<\tau} Y_1(t) > \beta_T + \epsilon\, r^{1/2}(T)]}{\tau\, c_T^{2/\alpha-1}\,\varphi(\beta_T + \epsilon\, r^{1/2}(T))} \to (2c)^{1/\alpha} H_\alpha \quad \text{as } T\to\infty. \tag{3.7}$$

Now the log of (3.6) is

$$\begin{aligned}([T/\tau]+1)\ln \mathbf{P}[\sup_{0\leq t<\tau} Y_1(t) \leq \beta_T + \epsilon r^{1/2}(T)] &\sim \\ \sim -(T/\tau)\,\mathbf{P}[\sup_{0\leq t\leq\tau} Y_1(t) > \beta_T + \epsilon\, r^{1/2}(T)]& \\ \sim -T\, c_T^{2/\alpha-1}(2c)^{1/\alpha} H_\alpha\, \varphi(\beta_T + \epsilon\, r^{1/2}(T)),& \end{aligned} \tag{3.8}$$

and this tends to zero as $T\to\infty$, completing the first part of the proof.

To do the other part, first note that the convexity of r ensures that there is a separable stationary Gaussian process $\{Y_T\} = \{Y_T(t)\}$ with the correlation function

$$\rho_T = \rho_T(t) = (r(t)-r(T))/(1-r(T)) \quad \text{for } t\leq T.$$

Let

$$M'_T = \max_{0\leq t\leq T} Y_T(t),$$

and represent M_T by

$$(1-r(T))^{1/2} M'_T + r^{1/2}(T)\, U$$

for U independent of $\{Y_T\}$. As in (2.13), the proof will be complete when it is shown that

$$\mathbf{P}[M'_T \leq \beta_T - \epsilon\, r^{1/2}(T)] \to 0 \quad \text{as } T\to\infty. \tag{3.9}$$

Let $Q(T) = T\exp[-(\ln T)^{1/2}]$. The convexity of ρ_T again ensures that there is a separable stationary Gaussian process $\{Z_T\} = \{Z_T(t)\}$ with the correlation function

$$\sigma_T = \sigma_T(t) = \rho_T(t) \vee \rho_T(Q(T)).$$

Let

$$M''_T = \max_{0\leq t\leq T} Z_T(t),$$

and conclude from (2.2) that

$$\mathrm{P}[M'_T \leqslant \beta_T - \epsilon\, r^{1/2}(T)] \leqslant \mathrm{P}[M''_T \leqslant \beta_T - \epsilon\, r^{1/2}(T)]. \tag{3.10}$$

The problem is now made discrete as follows. Take $I_1, ..., I_{[T]}$ to be $[T]$ consecutive unit intervals with an interval of length δ removed from the right-hand end of each one. Let G_T be the set of integer multiples of $(2 \ln T)^{-3/\alpha} = c_T^{-6/\alpha}$ and let $S(T) = G_T \cap (\bigcup I_i)$. According to [5], M''_T and $\max_{t\in S(T)} Z_T(t)$ have the same limit distribution, so it will be enough to show that

$$\mathrm{P}[\max_{t\in S(T)} Z_T(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)] \to 0 \quad \text{as } T \to \infty. \tag{3.11}$$

Factoring out a common Gaussian variable in the usual way, we may write, in parallel with (2.16),

$$\begin{aligned}
&\mathrm{P}[\max_{t\in S(T)} Z_T(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)] = \\
&= \mathrm{P}[(1-\sigma_T(T))^{1/2} \max_{t\in S(T)} Z'_T(t) + \sigma_T^{1/2}(T)\, U \leqslant \beta_T - \epsilon\, r^{1/2}(T)] \\
&\leqslant \Phi\left(-\tfrac{1}{2}\epsilon\left(\frac{r(T)}{\sigma_T(T)}\right)^{1/2}\right) \\
&\quad + \mathrm{P}\left[\max_{t\in S(T)} Z'_T(t) \leqslant \beta_T + \frac{\beta_T\, \sigma_T(T)}{(1-\sigma_T(T))^{1/2}(1+(1-\sigma_T(T))^{1/2}} \right. \\
&\qquad \left. - \frac{\epsilon\, r^{1/2}(T)}{2(1-\sigma_T(T))^{1/2}}\right],
\end{aligned} \tag{3.12}$$

where Z'_T has the correlation function

$$\sigma'_T(t) = (\sigma_T(t) - \sigma_T(T))/(1 - \sigma_T(T)) \quad \text{for } t \leqslant T.$$

Because $r(t) \ln t$ is monotone for large t, we find as in the discrete case that

$$r(t) - r(T) \leqslant r(t) \ln (\ln T/\ln t)$$

whenever $t \leqslant T$. Hence

$$\rho_T(t) \leqslant \frac{r(t)}{1-r(T)} \ln \frac{\ln T}{\ln t},$$

$$\sigma_T(T) = \rho_T(T \exp[-(\ln T)^{1/2}]) \leqslant \frac{r(T \exp[-(\ln t)^{1/2}]}{1-r(T)} \frac{2}{(\ln T)^{1/2}}.$$

The first term on the right-hand side of (3.12) is now easily seen to be o(1), while the second term is handled as above (2.17) to reduce the problem to that of showing

$$P[\max_{t\in S(T)} Z'_T(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)] = o(1). \tag{3.13}$$

(3.13) is derived as follows. Let $\{Y_2(t), t \geqslant 0\}$ be a process of standard normal variables which on each interval $[k, k+1)$ has the correlation function σ'_T and is independent of $Y_2(t)$, $t \in [j, j+1)$ if $j \neq k$. We have

$$\begin{aligned} P[\max_{t\in S(T)} Z'_T(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)] &\leqslant \\ \leqslant P^{[T]}[\max_{t\in S(T)\cap[0,1)} Y_2(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)] \\ + |P[\max_{t\in S(T)} Y_2(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)] \\ - P[\max_{t\in S(T)} Z'_T(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)]|. \end{aligned} \tag{3.14}$$

Since $\sigma'_T(t) \leqslant r(t)$ on $[0, 1]$,

$$\begin{aligned} P^{[T]}[\max_{t\in S(T)\cap[0,1)} Y_2(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)] \\ \leqslant P^{[T]}[\max_{t\in S(T)\cap[0,1]} X(t) \leqslant \beta_T - \epsilon\, r^{1/2}(T)], \end{aligned}$$

and this is o(1) by [6, Lemma 2.5] (see also, [8, Lemma 1.3]). Finally, the last term in (3.14) is estimated according to (2.3) and the resulting sum is handled exactly as (2.17). These details are omitted. □

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