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On the existence of directed rings and algebras with negative squares

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Abstract

We show that there exist many directed rings and algebras with negative squares.

Keywords: Directed ring; Directed algebra; Valuation; Negative square

1. Introduction

In [1], Artin and Schreier observed that a totally ordered field cannot have negative squares, and Johnson in [11] and Fuchs in [7] extended this result to totally ordered domains with unit element. In [12], Schwartz showed that an archimedean lattice-ordered field that has $1 > 0$ and that is algebraic over its maximal totally ordered subfield cannot have negative squares, and in [6], DeMarr and Steger showed that in a partially ordered finite-dimensional real linear algebra no square can be the negative of a strong unit. In [4], Birkhoff and Pierce asked whether the complex numbers have a compatible lattice-order. Below we use valuations to construct directed algebras that have negative squares. By the results of Johnson and Fuchs, the order on such an algebra cannot be extended to a compatible total order.

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For the general theory of partially ordered algebraic systems, we refer the reader to [2,3,8,10].

Throughout this note, all fields are commutative; all rings are associative and have identity; $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the totally ordered ring of integers, the totally ordered field of rational numbers, the totally ordered field of real numbers, and the filed of complex numbers respectively; x denotes an indeterminate.

2. Directed rings with negative squares

Let F be a ring with unit element 1, let G be a nontrivial, totally ordered, additive, abelian group, and let $G_{-\infty} = G \cup \{-\infty\}$, where $(-\infty) + (-\infty) = -\infty = (-\infty) + g = g + (-\infty)$ for all $g \in G$. A function $v : F \to G_{-\infty}$ is a *negative valuation* if v is onto and the following conditions hold for all $a, b \in F$:

(1) $v(a) = -\infty$ if and only if $a = 0$;
(2) $v(ab) = v(a) + v(b)$;
(3) $v(a + b) \leqslant \max\{v(a), v(b)\}$.

As noted in the proof of Corollary 2.3 below, the degree function on the quotient field of $\mathbb{R}[x]$ is a negative valuation into $\mathbb{Z}$. For later use, note that by (2), $v(1) = 0 = v(-1)$ and hence $v(-a) = v(a)$ for all $a \in F$.

Proposition 2.1. *Let F be a totally ordered ring with unit element, let G be a nontrivial totally ordered abelian group, and let $v : F \to G_{-\infty}$ be a negative valuation. Then the following statements are equivalent*:

(1) *for all* $0 \leqslant a, b \in F$, $v(a + b) = \max\{v(a), v(b)\}$;
(2) *for all* $0 \leqslant a, b \in F$, *if* $v(a) > v(b)$, *then* $a > b$.

Proof. Assume that (1) holds, that $0 \leqslant a, b \in F$, that $v(a) > v(b)$, and that $a \leqslant b$. Then $0 \leqslant b - a$ and $v(b) = v((b - a) + a) = \max\{v(b - a), v(a)\} \geqslant v(a)$, a contradiction. So $a > b$.

Conversely suppose that (2) holds, that $0 \leqslant a, b \in F$, and that $v(a + b) \neq \max\{v(a), v(b)\}$. Then since v is a negative valuation, $v(a + b) < \max\{v(a), v(b)\}$. If $\max\{v(a), v(b)\} = v(a)$, then $v(a) > v(a + b)$, and hence by (2), $a > a + b$ so that $0 > b$, a contradiction. A similar contradiction occurs if $\max\{v(a), v(b)\} = v(b)$ and thus $v(a + b) = \max\{v(a), v(b)\}$. $\square$

Theorem 2.2. *Let F be a totally ordered field, and let $F(i)$ be an extension of F by an element i, where $i^2 = -1$. If there exist a totally ordered nontrivial abelian group G and a negative valuation $v : F \to G_{-\infty}$ that satisfies the equivalent conditions of Proposition* 2.1, *then there exists a partial order on $F(i)$ with respect to which $F(i)$ is a directed field.*

Proof. Let

$$P = \{a + bi \mid a \geqslant 0,\ b \geqslant 0,\ \text{and if } b \neq 0,\ \text{then } v(a) > v(b)\}.$$

By [8, pp. 13 and 105], P will be the positive cone of a directed order on $F(i)$ if

(1) $P \cap (-P) = \{0\}$,
(2) $P + P \subseteq P$,
(3) $PP \subseteq P$, and
(4) $F(i) = P - P$.

That condition (1) holds is clear. For (2) and (3), let $a + bi, c + di \in P$.

Suppose first that $a = 0$. If $b \neq 0$, then $-\infty > v(b)$, a contradiction. So $b = 0$ and hence $(a + bi) + (c + di) = c + di \in P$ and $(a + bi)(c + di) = 0 \in P$. A similar situation occurs when $c = 0$ and hence we may assume that $a > 0$ and $c > 0$. Note that if $b = 0$, $v(a) > -\infty = v(b)$ and if $b \neq 0$, $v(a) > v(b)$ because $a + bi \in P$, and that similarly $v(c) > v(d)$.

For (2), we certainly have $a + c > 0$ and $b + d \geqslant 0$. So if $b + d = 0$, $(a + bi) + (c + di) = (a + c) + (b + d)i \in P$. If $b + d \neq 0$, then since $v(a) > v(b)$ and $v(c) > v(d)$, Proposition 2.1 implies that

$$v(a + c) = \max\{v(a), v(c)\} > \max\{v(b), v(d)\} = v(b + d).$$

So in this case as well, $(a + bi) + (c + di) \in P$, and therefore, (2) holds.

For (3), we have $ad + bc \geqslant 0$ and $v(ac) = v(a) + v(c) > v(b) + v(d) = v(bd)$. So by Proposition 2.1, $ac - bd > 0$, and thus if $ad + bc = 0$, $(a + bi)(c + di) = (ac - bd) + (ad + bc)i \in P$. If $ad + bc \neq 0$, then by Proposition 2.1, $v(ad + bc) = \max\{v(ad), v(bc)\}$ and $v(ac) = v((ac - bd) + bd) = \max\{v(ac - bd), v(bd)\}$. So since $v(ac) > v(bd)$, $v(a) + v(c) = v(ac) = v(ac - bd)$. Then by Proposition 2.1, if $v(ad + bc) = v(ad)$,

$$v(ad + bc) = v(a) + v(d) < v(a) + v(c) = v(ac - bd),$$

and similarly if $v(ad + bc) = v(bc)$,

$$v(ad + bc) = v(b) + v(c) < v(a) + v(c) = v(ac - bd).$$

It follows that $(a + bi)(c + di) = (ac - bd) + (ad + bc)i \in P$ and hence that (3) holds.

For (4), we first show that $i \in P - P$. Since v is onto and G is nontrivial, $v(f) > 0$ for some $f \in F$, and thus since $v(f) = v(-f)$ and F is totally ordered, we may assume $f > 0$. Furthermore, $v(f) > 0 = v(1)$ and since F is a totally ordered field, $1 > 0$. So $f + i \in P$ and thus $i = (f + i) - f \in P - P$. Now suppose that $a + bi \in F(i)$. Then $a, b \in P \cup (-P)$ and we have shown that $i = \alpha + \beta$ for $\alpha \in P$ and $\beta \in -P$. So by (3), $b\alpha \in P$ and $b\beta \in -P$ or $b\alpha \in -P$ and $b\beta \in P$, and by (2), writing $a + bi$ as $(a + b\alpha) + b\beta$ or $(a + b\beta) + b\alpha$ shows that $a + bi \in P - P$. So (4) holds and thus $F(i)$ is a directed field with respect to the order induced by P. $\square$

Corollary 2.3. *Let F be a totally ordered field and let Q denote the quotient field of the polynomial ring $F[x]$. If i is a solution of $x^2+1=0$, then there exists a partial order on $Q(i)$ with respect to which $Q(i)$ is a directed field.*

Proof. By [8, p. 107], $F[x]$ is a totally ordered ring with respect to the order: $a_0+a_1x+\cdots+a_nx^n>0$ if and only if $a_n>0$, and by [8, p. 110], the quotient field Q of $F[x]$ is a totally ordered field with respect to the order: $\frac{f(x)}{g(x)}>0$ if and only if $f(x)g(x)>0$. It is straightforward to show that the function $v: Q\to\mathbb{Z}_{-\infty}$ defined by letting

$$v\left(\frac{f(x)}{g(x)}\right)=\begin{cases}\deg(f(x))-\deg(g(x)) & \text{if } f(x)\neq 0,\\ -\infty & \text{if } f(x)=0,\end{cases}$$

is a well-defined negative valuation on Q that satisfies condition (1) of Proposition 2.1 ([5, p. 7] does this for the degree function on $F[x]$). So by Theorem 2.2, there is a partial order on $Q(i)$ with respect to which it is a directed field. □

Since we have not shown that the order defined in the proof of Theorem 2.2 is a lattice-order, we have not answered the question of Birkhoff and Pierce mentioned in Section 1. In view of the work above, a more general question would be the following. Note that since a lattice-ordered field is directed, a negative answer to this question would yield a negative answer to the question of Birkhoff and Pierce.

Question 2.4. Do the complex numbers have a compatible directed order?

3. Directed algebras

Let T be a totally ordered ring and recall (see [6]) that a *directed T-algebra* is an algebra D over T with a partial order that makes it a directed ring with the following compatibility property: if $0\leqslant\tau\in T$ and $0\leqslant d\in D$, then $0\leqslant\tau d$.

In [4], Birkhoff and Pierce showed that $\mathbb{Q}(i)$ admits no partial order with respect to which it is a lattice-ordered field and that $\mathbb{C}$ admits no partial order with respect to which it is a lattice-ordered algebra over $\mathbb{R}$. In [12], Schwartz proved that the field of algebraic numbers admits no partial order with respect to which it is a lattice-ordered field. And in [6], DeMarr and Steger proved that if A is a finite-dimensional nontrivial algebra over $\mathbb{R}$ whose center contains a square root of -1, then A admits no partial order with respect to which it is a directed algebra over $\mathbb{R}$. We first note that the proof of DeMarr and Steger may be easily generalized to prove the following result.

Proposition 3.1. *Let T be a totally ordered archimedean ring such that the map $x\mapsto x+x$ is onto, and let A be a finite-dimensional nontrivial T-algebra whose identity is a strong order unit and whose center contains a square root of -1. Then A admits no partial order with respect to which it is a directed algebra over T.*

On the other hand, since the proofs in Section 2 remain unchanged if the rings are viewed as algebras over a totally ordered ring T, we have the following result for directed algebras.

Proposition 3.2. *Let F be a totally ordered field, and let $F(i)$ be an extension of F by an element i, where $i^2 = -1$. If there exist a nontrivial totally ordered abelian group G and a negative valuation $v : F \to G_{-\infty}$ that satisfies the equivalent conditions of Proposition* 2.1, *then there exists a partial order on $F(i)$ with respect to which $F(i)$ is a directed F-algebra. In particular, if Q is the quotient field of the polynomial ring $F[x]$, then there exists a partial order on $Q(i)$ with respect to which $Q(i)$ is a directed Q-algebra.*

We conclude this note with three examples. All are examples of directed algebras that are not lattice-ordered, the first over $\mathbb{R}$ and the second and third over $\mathbb{Z}$. The third shows that a directed $\mathbb{Z}$-algebra need not satisfy the condition: if $na \geqslant 0$, then $a \geqslant 0$.

Example 3.3. Fuchs' tight Riesz orders [9] give many examples of directed $\mathbb{R}$-algebras that are not lattice-ordered. For instance,

$$P = \left\{a_0 + a_1x + \cdots + a_nx^n \in \mathbb{R}[x] \mid 0 < a_i \text{ for all } i\right\} \cup \{0\}$$

is the positive cone of a partial order on $\mathbb{R}[x]$ with respect to which $\mathbb{R}[x]$ is a directed $\mathbb{R}$-algebra that is not lattice-ordered.

Example 3.4. It is easy to check that with respect the coordinatewise operations, $\mathbb{Z} \times \mathbb{Z}$ is a $\mathbb{Z}$-algebra and that $P = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid |m| \leqslant n\}$ is the positive cone of a partial order $\succcurlyeq$ on $\mathbb{Z} \times \mathbb{Z}$ with respect to which $\mathbb{Z} \times \mathbb{Z}$ is a directed $\mathbb{Z}$-algebra. To see that this order is not a lattice-order, observe that $(1, 1) \succcurlyeq (1, 0)$ and $(1, 1) \succcurlyeq (0, 0)$, $(0, 1) \succcurlyeq (1, 0)$, and $(0, 1) \succcurlyeq (0, 0)$. But if $(m, n) \succcurlyeq (1, 0)$ and $(m, n) \succcurlyeq (0, 0)$, then $|m - 1| \leqslant n$ and $|m| \leqslant n$. So if as well $(1, 1) \succcurlyeq (m, n)$, then $|1 - m| \leqslant 1 - n$ and hence $2|m - 1| \leqslant (1 - n) + n = 1$, i.e., $m = 1$. And if as well $(0, 1) \succcurlyeq (m, n)$, then $|-m| \leqslant 1 - n$ and hence $2|m| \leqslant (1 - n) + n = 1$, i.e., $m = 0$. It follows that $\succcurlyeq$ is neither a lattice-order nor a tight Riesz order.

Example 3.5. It is easy to check that $P = \{n \in \mathbb{Z} \mid 1 < n\} \cup \{0\}$ is the positive cone of a partial order $\succeq$ on $\mathbb{Z}$ with respect to which $\mathbb{Z}$ is a directed $\mathbb{Z}$-algebra. Note that $3 \succeq 1$ and $3 \succeq 0$ and if $3 \succeq n$ and $3 \neq n$, then $1 \geqslant n$ so that $n \not\succeq 0$. But $4 \succeq 1$, $4 \succeq 0$, and $4 \not\succeq 3$. It follows that $\succeq$ is neither a lattice-order nor a tight Riesz order. Note that $2 \cdot 1 \succeq 0$ but $1 \not\succeq 0$.

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References

[1] E. Artin, O. Schreier, Algebraische Konstruction reeler Körper, Abh. Math. Sem. Univ. Hamburg 5 (1926) 85–99.
[2] A. Bigard, K. Keimel, S. Wolfenstein, Groupes et Anneuax Réticulés, Lecture Notes in Math., vol. 608, Springer-Verlag, Berlin, ISBN 3-540-08436-3, 1977.
[3] G. Birkhoff, Lattice Theory, third ed., Amer. Math. Soc. Colloquium Pub., vol. 25, Amer. Math. Soc., Providence, RI, 1973.
[4] G. Birkhoff, R.S. Pierce, Lattice-ordered rings, An. Acad. Brasil. Cienc. 28 (1956) 41–69.
[5] N. Bourbaki, Éléments de Mathématique, Algèbre, Chapitre 4, Polynomes et Fractions Rationelles, Actaualités Scientifiques et Industrielles, vol. 1102, Hermann, Paris, 1959.
[6] R. DeMarr, A. Steger, On elements with negative squares, Proc. Amer. Math. Soc. 31 (1972) 57–60.
[7] L. Fuchs, Note on ordered groups and rings, Fund. Math. 46 (1958) 167–174.
[8] L. Fuchs, Partially Ordered Algebraic Systems, Pergamon Press, Oxford, 1963.
[9] L. Fuchs, Riesz Vector Spaces and Riesz Algebras, Queen's Papers in Pure and Appl. Math., vol. 1, Queen's Univ., Kingston, ON, 1966.
[10] A.M.W. Glass, Partially Ordered Groups, World Scientific, Singapore, ISBN 981-02-3493-7, 1999.
[11] R.E. Johnson, On ordered domains of integrity, Proc. Amer. Math. Soc. 7 (1956) 414–416.
[12] N. Schwartz, Lattice-ordered fields, Order 3 (1986) 179–194.