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On properties of special magic square matrices

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ABSTRACT

By treating regular (or associative), pandiagonal, and most-perfect (MP) magic squares as matrices, we find a number of interesting properties and relationships. In addition, we introduce a new class of quasi-regular (QR) magic squares which includes regular and MP magic squares. These four classes of magic squares are called "special."

We prove that QR magic squares have signed pairs of eigenvalues just as do regular magic squares according to a well-known theorem of Mattingly. This leads to the fact that odd powers of QR magic squares are magic squares which also can be established directly from the QR condition. Since all pandiagonal magic squares of order 4 are MP, they are QR. Also, we show that all pandiagonal magic squares of order 5 are QR but higher-order ones may or may not be. In addition, we prove that odd powers of MP magic squares are MP. A simple proof is given of the known result that natural (or classic) pandiagonal and regular magic squares of singly-even order do not exist.

We consider the reflection of a regular magic square about its horizontal or vertical centerline and prove that signed pairs of eigenvalues of the reflected square differ from those of the original square by the factor i. A similar result is found for MP magic squares and a subclass of QR magic squares.

The paper begins with mathematical definitions of the special magic squares. Then, a number of useful matrix transformations between them are presented. Next, following a brief summary of the spectral analysis of matrices, the spectra of these special magic squares are considered and the results mentioned above are established. A few numerical examples are presented to illustrate our results.

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1. Introduction

Magic squares have a long and fascinating history as detailed by Pickover [17], Pasles [16], Ollerenshaw and Brée [15], and Cammann [4]. The treatment of magic squares as matrices leads to many interesting new results and insights as seen from nearly all of our references. An historical account of the matrix-theoretic approach is given by Loly et al. [10], where references to other work may be found. Magic squares have a continuing appeal to both professional and recreational mathematicians and they have found application in physics, computer science, image processing, and cryptography [8,10,15].

The present paper is concerned with magic squares that have special properties, namely, regular (or associative), pandiagonal, and most-perfect magic squares, plus a new class of quasi-regular magic squares which includes regular and most-perfect magic squares. These four classes of magic squares (defined in the next section) are called **special**. We employ methods of matrix algebra to study transformations between these special magic square matrices, their spectra, and their matrix products and powers. Many of our results are new and others confirm known results obtained by other methods.

2. Definitions

We begin by defining a number of special matrices and several classes of special magic squares that will be studied in what follows. All matrices considered here are square. Let u be the **unity column vector** with all elements 1, U – the **unity matrix** with all elements 1, I – the **identity matrix**, and R – the **reflection matrix** (or counteridentity matrix) with 1's on the cross (or dexter) diagonal and all other elements 0. It should be noted that other authors use various other symbols for our U, u, and R. In matrix notation R and U satisfy the following identities:

$$R^T = R^{-1} = R, \quad RU = UR = U, \quad \mathrm{tr}\,[U] = n, \quad U^i = n^{i-1}U, \quad i = 1, 2, \ldots, \tag{1}$$

where R^T denotes the transpose of R and $\mathrm{tr}\,[U]$ denotes the trace of U. The matrix product RA reflects the elements of a matrix A about its horizontal centerline, AR reflects the elements of A about its vertical centerline, A^TR rotates the elements of A a quarter turn clockwise about its center, RA^T rotates the elements of A a quarter turn counter-clockwise, and RAR rotates the elements of A a half turn. These matrices together with A^T, RA^TR, and A itself constitute the eight **phases** (or variants) of A as discussed in [10].

The matrix M is **semi-magic** if the sums of all its rows and all its columns equal the same **index** m, i.e., if

$$Mu = \left(u^TM\right)^T = mu \quad \text{or} \quad MU = UM = mU. \tag{2}$$

The matrix M is **magic** if, in addition to (2), its main diagonal and cross diagonal also sum to m, i.e., if

$$\mathrm{tr}\,[M] = \mathrm{tr}\,[RM] = m. \tag{3}$$

By these definitions, a magic square also is semi-magic and a semi-magic square may or may not be magic. In what follows, M denotes a magic square matrix unless otherwise noted and A denotes a generic matrix, both having integer elements (for simplicity). Subscripts are used to denote special classes of M. The n by n (**order**-n) matrix A is **natural** (or classic) if its elements are in the numerical sequence $1, 2, \ldots, n^2$. Some authors use the sequence $0, 1, \ldots n^2 - 1$ which would slightly change some of our results. Various methods of constructing magic squares are presented in [7,12–20]. The index for a natural magic square is given by

$$m = n(n^2 + 1)/2. \tag{4}$$

The eight phases of M also are magic as is easily verified. Some authors, e.g., Pickover [17], restrict magic squares to those having distinct elements while many others do not enforce this restriction. In this paper we allow magic squares with nondistinct elements, but magic squares with distinct elements are called **strictly magic**. A condition for a matrix to have distinct elements is given at the end of this section.

The matrix A_C is **centrosymmetric** if

$$RA_CR = A_C. \tag{5}$$

A **permutation matrix** P has a single 1 in all rows and columns and 0 for all other elements. It has the following properties:

$$UP = PU = U, \quad P^{-1} = P^T, \quad \text{and} \quad \det P = \pm 1. \tag{6}$$

The matrix operation PA interchanges rows of A and AP interchanges columns of A.

A magic square M_R is **regular** (or associative) if pairs of its elements that are symmetrically positioned with respect to its center add to the same **regularity index** r, i.e., if

$$M_R + RM_RR = rU \quad \text{or} \quad RM_R + M_RR = rU. \tag{7}$$

This requires M_R of odd order to have $r/2$ as its center element. On taking the trace of (7) and noting (3) and (4), we have

$$r = 2mn^{-1} \quad \text{or} \quad r = n^2 + 1 \text{ if } M_R \text{ is natural.} \tag{8}$$

It is known that all order-3 magic squares are regular [10].

A magic square M_P is **pandiagonal** (Nasik, perfect, or diabolic) if all its broken diagonals (of n elements) in both directions sum to the magic index m. These conditions, including (3), can be expressed as

$$\operatorname{tr}\left[K^iM_P\right] = \operatorname{tr}\left[K^iRM_P\right] = m, \quad i = 1, 2, \ldots, n, \tag{9}$$

where K is the order-n permutation matrix that has all elements 0 except $K_{1n} = 1$ (upper right corner) and $K_{i,i-1} = 1, \ i = 2, 3, \ldots, n$ (diagonal below the main diagonal). The operation KM shifts rows of M down one (and bottom row to top) while MK shifts columns of M one to the left (and first column to last). Power operations K^iM and MK^i give rise to repeated shifts. The following identities can be easily verified:

$$\begin{aligned} &K^n = I, \quad K^{-i} = K^{n-i} = RK^iR, \quad i = 1, 2, \ldots, n \\ &\sum_{i=1}^{n} K^i = \sum_{i=1}^{n} K^{-i} = U. \end{aligned} \tag{10}$$

It follows from the shifting properties of K that M_P satisfies the conditions

$$\sum_{i=1}^{n} K^iM_PK^i = mU \quad \text{and} \quad \sum_{i=1}^{n} K^{-i}M_PK^i = mU \tag{11}$$

which provide a convenient numerical test for pandiagonality.

The term **panmagic** is applied to a pandiagonal magic square. An **ultra-magic** square M_U is both panmagic and regular. It is known that other than U there are no order-2 magic squares, no order-3 panmagic squares, and no order-4 ultra-magic squares [17]. Furthermore, Rosser and Walker [19, Theorem 5.2] proved that there are no natural panmagic squares of singly-even order ($n = 6, 10, \ldots$). A simple derivation of the identity used in their proof is given in Appendix A. Also, it follows from a known transformation (to be discussed in the next section) that there are no natural regular squares of singly-even order either.

As an extension of regular magic squares, we define a **quasi-regular (QR)** magic square M_Q as one that satisfies

$$M_Q + Q^{-1}M_Q Q = rU \quad \text{or} \quad QM_Q + M_Q Q = rU, \tag{12}$$

where r is given by (8) and the **operator** Q is a nonsingular matrix of the same order as M_Q that satisfies

$$UQ = QU = U \quad \text{and} \quad \det Q \neq 0. \tag{13}$$

A quasi-regular magic square M_Q with a centrosymmetric operator Q_C is called **QRC**.

Lemma 1. A QRC semi-magic square M_Q is a magic square.

It follows from (12), (5), and (8) that M_Q satisfies the diagonal sum condition (3). This lemma will be used in a later section to prove that odd powers of a QRC magic square are QRC magic squares. Also, if M_Q is QR with operator Q, then it follows from (12) that the phases RM_Q, $M_Q R$, and $RM_Q R$ are QR with operator Q, whereas the phases $(M_Q)^T$, $R\,(M_Q)^T$, $(M_Q)^T R$, and $R\,(M_Q)^T R$ are QR with operator Q^T.

On comparing (7) with (12), we see that a regular magic square is QRC with operator $Q_C = R$. This case was studied extensively by Mattingly [12], who also gave a condition of the form (12) but with a different right-hand side and without (13). The case $Q^{-1} = Q$ was studied by Chu et al. [5] with application to order-4 magic squares and their Moore–Penrose inverses. Also, Staab et al. [20] considered the case where $Q = Q^{-1} = P = P^T$ is a symmetric permutation matrix of various special forms. In the present paper Q need not be a permutation matrix nor must $Q = Q^{-1}$.

An important subclass of QR magic squares follows from noting that, by (12) and (13), we have

$$Q^2 M_Q = M_Q Q^2 \tag{14}$$

which is satisfied if

$$Q^2 = Q_S^2 = (1 - \beta n)\, I + \beta U, \quad \beta \neq n^{-1}. \tag{15}$$

Here, $\beta = 0$ corresponds to $Q^{-1} = Q$ which includes the case where Q is a symmetric permutation matrix. However, other forms of Q are possible. A quasi-regular magic square M_Q with operator Q_S that satisfies (14) is called **QRS** and it need not be QRC.

A useful special form for the QR operator is

$$Q_K = \sum_{i=0}^{\hat{n}} a_i \left(K^i + K^{-i}\right), \quad \text{where } \hat{n} = \lfloor n/2 \rfloor \tag{16}$$

and, in order to satisfy (13), the constants a_i must satisfy

$$2\sum_{i=0}^{\hat{n}} a_i = 1 \quad \text{and} \quad \det Q_K \neq 0. \tag{17}$$

A QR magic squares M_Q with operator Q_K is called **QRK** and it also is QRC.

The matrix $Q = n^{-1}U$ also satisfies (12) but not (13) since U is singular. For a given operator Q, the following **associated operator** $\hat{Q}$ satisfies (12) and (13):

$$\hat{Q} = (1 - \alpha n)\, Q + \alpha U, \quad \alpha \neq n^{-1}, \tag{18}$$

where $\alpha \neq n^{-1}$ ensures that $\hat{Q}$ is nonsingular as seen from the identity

$$\det\left[\hat{Q}\right] = \det[Q]\det[(1-\alpha n)I + \alpha U] = (1-\alpha n)^{n-1}\det[Q]. \tag{19}$$

Often, α can be chosen so that $\hat{Q}$ has a simple form. In addition, it can be shown that Q^{-1} and $\hat{Q}^{-1}$ are suitable QR operators and that $\hat{Q}$, Q^{-1}, and $\hat{Q}^{-1}$ retain the special property of Q_C, Q_S, or Q_K.

For a given magic square or a class of magic squares of a particular form, one may attempt to construct Q by solution of (12) and (13). However, for most magic squares this turns out to be impossible and they are not QR. Nevertheless, the study of QR magic squares leads to interesting results as will be seen in what follows.

The following theorem generalizes a result given by Ollerenshaw and Brée [15] for the special case $Q_K = K^{\frac{n}{2}}$ (n even):

Theorem 2. *A QRK magic square is pandiagonal.*

Proof. From (12) and (10), we have

$$K^iM_Q + K^iQ_K^{-1}M_QQ_K = rU$$

$$K^iRM_Q + K^iRQ_K^{-1}M_QQ_K = rU. \tag{20}$$

On taking the trace of these two equations and noting that $Q_KK^i = K^iQ_K$ and $Q_KR = RQ_K$ follow from (16) and (10), we see that the pandiagonal conditions (9) are satisfied. □

A **most-perfect (MP)** magic square M_M of doubly-even order ($n = 4, 8, \ldots$) has the following additional properties:

(1) Two elements that are $n/2$ elements apart along all diagonals (including broken ones in both directions) sum to r, i.e.,

$$M_M + K^{\frac{n}{2}}M_MK^{\frac{n}{2}} = rU. \tag{21}$$

(2) The elements of all 2 by 2 subsquares (including broken top–bottom, broken left–right, and the four corners) sum to the constant $2r$, i.e.,

$$(I+K)\,M_M\,(I+K) = 2rU. \tag{22}$$

On comparing (21) with (12) and (13), we see that a MP matrix is QRK with operator $Q_K = Q_K^{-1} = K^{\frac{n}{2}}$. Thus, in view of Theorem 2, a MP matrix must be pandiagonal as noted by Ollerenshaw and Brée [15]. For an order-4 panmagic square, the pandiagonality conditions (11) and Eq. (21) lead to the known fact [17] that all such squares are MP and QRK with operator $Q_K = K^2$. A method of constructing and counting the number of MP natural magic squares of doubly-even order is given in [15]. Next, we derive the matrix form of identities for M_M given in subscript form in [15].

Theorem 3. *A MP matrix M_M satisfies the following identities:*

$$\left.\begin{aligned} M_M(I+K) &= K^iM_M(I+K) \\ (I+K)M_M &= (I+K)M_MK^i \end{aligned}\right\}, \quad i \text{ even} \tag{23}$$

$$\left.\begin{aligned} M_M(I+K)\left(I+K^{\frac{n}{2}}\right) &= 2rU \\ \left(I+K^{\frac{n}{2}}\right)(I+K)M_M &= 2rU \end{aligned}\right\}, \quad n \text{ doubly-even.} \tag{24}$$

Proof. Eq. (22) may be written as

$$\begin{aligned} &M_M(I+K)+KM_M(I+K)=2rU,\\ &KM_M(I+K)+K^2M_M(I+K)=2rU,\\ &\therefore M_M(I+K)=K^2M_M(I+K) \end{aligned} \tag{25}$$

and $(23)_1$ follows by induction. Since n is doubly-even, by $(23)_1$ and (21), for $(24)_1$ we have

$$\begin{aligned} M_M(I+K)\left(I+K^{\frac{n}{2}}\right) &= M_M(I+K)+K^{\frac{n}{2}}M_MK^{\frac{n}{2}}(I+K)\\ &= M_M(I+K)+(rU-M_M)(I+K)=2rU. \end{aligned} \tag{26}$$

The proofs of $(23)_2$ and $(24)_2$ are similar. □

On multiplying $(24)_1$ and $(24)_2$ by U and noting (10), it follows that (2) is satisfied and M_M is semi-magic. Since M_M is QRK, by Lemma 1, it is magic as stated in its definition. In a later section (24) will be used to prove that odd powers of MP magic squares are MP magic squares. Next, we present a general result for order-5 panmagic squares.

Theorem 4. *Any order-5 panmagic square M_{P5} is QRKS with operators*

$$Q_{K5}=\left(I+2K+2K^{-1}\right)/5 \quad \textit{and} \quad \hat{Q}_{K5}=\left(I+2K^2+2K^{-2}\right)/5 \tag{27}$$

with $\beta=4/25$ in (14).

Proof. It is easy to show that Q_{K5} and $\hat{Q}_{K5}$ satisfy (13), (16), and (17). On adding the two pandiagonal conditions (11) on M_{P5} and noting that $K^3=K^{-2}$ and $K^4=K^{-1}$, we find that

$$2M_{P5}+\left(K+K^{-1}\right)M_{P5}\left(K+K^{-1}\right)+\left(K^2+K^{-2}\right)M_{P5}\left(K^2+K^{-2}\right)=2mU \tag{28}$$

which, by the identity $(10)_2$, becomes

$$Q_{K5}\left(Q_{K5}M_{P5}+M_{P5}Q_{K5}\right)=rU. \tag{29}$$

Since Q_{K5} is not singular and $U=Q_{K5}U$, it follows that M_{P5} satisfies the QR condition (12) with operator Q_{K5}. The associated operator $\hat{Q}_{K5}$ in (27) comes from (18) with $\alpha=2/5$ and (10). The relation (14) with $\beta=4/25$ is easily verified using (10). □

The QR equation (12) on M_{P5} with operators Q_{K5} and $\hat{Q}_{K5}$ given by (27) shows that the elements corresponding to those marked with ★ in the following matrices sum to m:

$$\Psi_1=\begin{bmatrix} 0&0&0&0&0\\ 0&0&\bigstar&0&0\\ 0&\bigstar&\bigstar&\bigstar&0\\ 0&0&\bigstar&0&0\\ 0&0&0&0&0 \end{bmatrix},\quad \Psi_2=\begin{bmatrix} 0&0&\bigstar&0&0\\ 0&0&0&0&0\\ \bigstar&0&\bigstar&0&\bigstar\\ 0&0&0&0&0\\ 0&0&\bigstar&0&0 \end{bmatrix}, \tag{30}$$

and these patterns can be shifted up/down and left/right. Other sum invariants also can be derived from (12) and (27). The Ψ_1 sum invariant was obtained in a different manner by Andress [2] who found similar sum invariants for other panmagic squares. Sum invariants also follow from the QR condition (12) for other magic squares which need not be pandiagonal.

A necessary and sufficient condition for an order-n matrix A to have distinct elements is

$$A - K^i A K^j = \Theta_{ij}, \quad i, j = 1, 2, \ldots, n, \quad (i = j \neq n), \tag{31}$$

where Θ_{ij} is an order-n matrix that has no zero elements. Also, a check for distinct elements and naturalness follows from sorting the elements of A into numerical order.

3. Transformations

In this section we examine matrix transformations between regular, QR, pandiagonal, and MP magic square matrices. The phase transformations mentioned after (1) preserve each of these special properties as is easily verified. First, we note a basic transformation for semi-magic squares.

Lemma 5. A semi-magic square M_1 can be transformed to a semi-magic square M_2 by the transformation

$$M_2 = P_1 M_1 P_2, \tag{32}$$

where P_1 and P_2 are perturbation matrices. If M_1 is natural, then so is M_2 .

This follows from (2) and the fact that row/column interchanges do not effect naturalness. The next four theorems establish sufficient conditions for the transformed square to be magic and retain any special properties. The straightforward proofs of the first three of these are left as exercises.

Theorem 6. *A magic square M_1 can be transformed to a magic square M_2 by the transformation*

$$M_2 = P M_1 P^T, \quad \text{where } RPR = P. \tag{33}$$

If M_1 is QR with operator Q_1, then M_2 is QR with operator

$$Q_2 = P Q_1 P^T. \tag{34}$$

If M_1 is regular, QRC, or QRS, then so is M_2.

Theorem 7. *If M_Q is a QR magic square with operator Q, then so are QM_Q and $M_Q Q$. If M_Q is QRK (hence pandiagonal) or MP, then so are $Q_K M_Q$ and $M_Q Q_K$.*

Theorem 8. *A panmagic square M_{P1} can be transformed to a panmagic square M_{P2} by the transformation*

$$M_{P2} = K^i M_{P1} K^j, \quad i, j = 0, 1, 2, \ldots, \text{ or } n. \tag{35}$$

Here, (35) is the well-known row/column shift transformation [17]. Next, we present the matrix form of a known transformation.

Theorem 9. *A regular magic square M_R of even order n can be transformed to a panmagic square M_P by the transformation*

$$M_P = W M_R W, \tag{36}$$

where the permutation matrix W is expressed in terms of order-$n/2$ submatrices $\hat{R}$, $\hat{I}$, and $\hat{O}$ as

$$W = W_1 = \begin{bmatrix} \hat{R} & \hat{O} \\ \hat{O} & \hat{I} \end{bmatrix} \quad \text{or} \quad W = W_2 = RW_1R = \begin{bmatrix} \hat{I} & \hat{O} \\ \hat{O} & \hat{R} \end{bmatrix}, \tag{37}$$

where $\hat{O}$ is the order-$n/2$ matrix with all elements zero. Furthermore, M_P is QRK with operator $Q_K = K^{\frac{n}{2}}$.

Proof. From (37), we have the identities

$$WRW = K^{\frac{n}{2}} \quad \text{and} \quad W^{-1} = W \tag{38}$$

from which the regularity condition (7) with (36) leads to (21). Thus M_P is QRK with operator $Q_K = K^{\frac{n}{2}}$ and, by Theorem 2, M_P is pandiagonal. □

The transformation (36) was known to Planck [18] who called it the "A–D method." Also, for order-8 magic squares, this transformation was given by Setsuda as posted on the Suzuki website [21]. However, the convenient matrix formulation (36) and the QRK condition on M_P are believed to be new.

Corollary 10. *A QRK panmagic square M_P of even order n with operator $Q_K = K^{\frac{n}{2}}$ can be transformed to a regular magic square M_R by the transformation*

$$M_R = WM_PW. \tag{39}$$

The proof follows in reverse the proof of Theorem 9. In view of (21), this corollary applies to all MP magic squares. Since all order-4 panmagic squares are MP, they can be transformed to a regular magic square by (39). In addition, some other panmagic squares of higher even order can be transformed to a regular magic square M_R by transformations similar to (39). However, the author has been unable to establish such a transformation for all panmagic squares of even order $n > 4$ and it appears that this is impossible (conjecture) since there are more elemental regular conditions from (7) than elemental pandiagonal conditions from (9) for such squares. Also, we note that the foregoing transformations, including the phase transformations, can be combined sequentially where appropriate, e.g., $M_{P2} = RK^3\,(M_{P1})^T\,RK^2$.

Some natural panmagic squares M_P of odd order can be transformed to a natural regular square M_R by the shift transformation (35) with i and j chosen such that the element $\left(n^2+1\right)/2$ of M_P is shifted to the center of M_R as required for regularity. In some cases a further permutation of rows and/or columns results in a regular magic square. For example, the following natural order-9 panmagic square given by Pickover [17, p. 255] (attributed to Gakuho Abe):

$$M_{P9} = \begin{bmatrix} 1 & 16 & 51 & 30 & 45 & 77 & 56 & 71 & 22 \\ 41 & 47 & 61 & 67 & 73 & 9 & 15 & 21 & 35 \\ 69 & 75 & 8 & 14 & 20 & 34 & 40 & 46 & 63 \\ 13 & 19 & 36 & 42 & 48 & 62 & 68 & 74 & 7 \\ 53 & 32 & 64 & 79 & 58 & 12 & 27 & 6 & 38 \\ 81 & 60 & 11 & 26 & 5 & 37 & 52 & 31 & 66 \\ 25 & 4 & 39 & 54 & 33 & 65 & 80 & 59 & 10 \\ 29 & 44 & 76 & 55 & 70 & 24 & 3 & 18 & 50 \\ 57 & 72 & 23 & 2 & 17 & 49 & 28 & 43 & 78 \end{bmatrix} \tag{40}$$

is QRKS with operator

$$Q_{K9} = (Q_{K9})^{-1} = \left(2K^3 + 2K^{-3} - I\right)/3 \tag{41}$$

and M_{P9} can be transformed to the following natural ultra-magic square by row/column shifts and permutations:

$$M_{U9} = \begin{bmatrix} 77 & 56 & 71 & 22 & 1 & 16 & 51 & 30 & 45 \\ 24 & 3 & 18 & 50 & 29 & 44 & 76 & 55 & 70 \\ 34 & 40 & 46 & 63 & 69 & 75 & 8 & 14 & 20 \\ 65 & 80 & 59 & 10 & 25 & 4 & 39 & 54 & 33 \\ 9 & 15 & 21 & 35 & 41 & 47 & 61 & 67 & 73 \\ 49 & 28 & 43 & 78 & 57 & 72 & 23 & 2 & 17 \\ 62 & 68 & 74 & 7 & 13 & 19 & 36 & 42 & 48 \\ 12 & 27 & 6 & 38 & 53 & 32 & 64 & 79 & 58 \\ 37 & 52 & 31 & 66 & 81 & 60 & 11 & 26 & 5 \end{bmatrix}. \quad (42)$$

Again, M_{U9} is QRKS with operator Q_{K9}, thus ensuring that M_{U9} is pandiagonal according to Theorem 2. In addition, the elements of all 3 by 3 subsquares of M_{P9} and M_{U9} sum to $m = 369$, i.e.,

$$\left(I + K + K^{-1}\right) M_{U9} \left(I + K + K^{-1}\right) = mU, \quad (43)$$

a property similar to (22) for MP squares. Thus, M_{U9} might be called **ultra-perfect**. Furthermore, the QR condition (12) with the operator Q_{K9} leads to the following sum invariant: for any element, the sum of its four neighboring elements (with wrap around) three squares right, left, above, and below minus the element itself is 123. An order-9, ultra-perfect magic square also was constructed by Frost [7]. His square also is QRK with operator Q_{K9} and he noted the sum condition given by (43).

The author doubts (conjecture) that the above transformation method can be applied to all odd-order panmagic squares since here again there are more elemental regular conditions from (7) than elemental pandiagonal conditions from (9) for $n \geqslant 5$. Also, I have been unable to transform a generic regular magic square M_R of odd order to a panmagic square M_P by a matrix transformation and this appears to be impossible (conjecture). Furthermore, not all panmagic squares of order $n > 5$ are QR, e.g.,

$$M_{P7} = \begin{bmatrix} 4 & 20 & 29 & 45 & 12 & 28 & 37 \\ 10 & 26 & 42 & 2 & 18 & 34 & 43 \\ 16 & 32 & 48 & 8 & 24 & 40 & 7 \\ 22 & 38 & 5 & 21 & 30 & 46 & 13 \\ 35 & 44 & 11 & 27 & 36 & 3 & 19 \\ 41 & 1 & 17 & 33 & 49 & 9 & 25 \\ 47 & 14 & 23 & 39 & 6 & 15 & 31 \end{bmatrix} \quad (44)$$

is a natural order-7 panmagic square that is not QR as will be verified in the next two sections. However, we have encountered many QR panmagic squares of order $n \geqslant 6$ with various Q operators, some of which are given in [14]. Use will be made of the QR property in the next two sections.

4. Spectra

First, the main concepts of the spectral analysis of matrices are briefly reviewed. A fuller presentation of this subject is available in standard texts on matrices.

For an order-n matrix M, the **characteristic equation** is obtained from

$$\det\left[M - \lambda I\right] = 0 \quad (45)$$

which leads to a polynomial equation of degree n having roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ called the **eigenvalues** of M. The eigenvalues may be real or complex and they need not be distinct. Corresponding to each distinct eigenvalue λ_j there is a **simple eigenvector** S_{j1} such that

$$(M - \lambda_j I)\, S_{j1} = 0, \tag{46}$$

where S_{j1} is determined only to within a scalar multiple. If an eigenvalue is repeated k times it is said to have **algebraic multiplicity** k. Repeated eigenvalues λ_j may have **generalized eigenvectors** S_{jk} for which

$$\left.\begin{array}{r} (M - \lambda_j I)^k\, S_{jk} = 0 \\ (M - \lambda_j I)^{k-1}\, S_{jk} \neq 0 \end{array}\right\}, \quad k = 2, 3, \ldots, \hat{k}_j \leqslant k_j < n, \tag{47}$$

where k is the **rank** of S_{jk} and k_j is the algebraic multiplicity of λ_j. There may be several sets of such S_{jk}, the number of which is the **geometric multiplicity** of λ_j. In some cases there also may be several simple (rank-1) eigenvectors corresponding to the repeated eigenvalue. For a matrix with real elements, if there is a complex eigenvalue λ_j with eigenvector S_{jk}, then, by (46) or (47), there also must be a complex conjugate eigenvalue $\bar{\lambda}_j$ with eigenvector $\bar{S}_{jk}$. By (47), S_{jk} satisfies the recursion relation

$$(M - \lambda_j I)\, S_{jk} = S_{j,k-1}, \quad k = 2, 3, \ldots, \hat{k}_j. \tag{48}$$

The simple and generalized eigenvectors can be assembled into a matrix S such that

$$M = SJS^{-1}, \tag{49}$$

where the columns of S are the eigenvectors and J is a matrix with their corresponding eigenvalues on the main diagonal and zeros elsewhere if all eigenvectors are simple (**diagonable** case). If there are generalized eigenvectors (**nondiagonable** case), then J has ones on the diagonal above or below the main diagonal (depending on the ordering of S_{jk} in S) corresponding to the repeated eigenvalues. Equation (49) is called the **Jordan form** of M. The character of the eigenvalues and eigenvectors for various special magic squares is examined next.

Lemma 11. A semi-magic square M with positive real elements has an eigenvalue m of algebraic multiplicity one with a simple eigenvector $S_1 = u$, i.e.,

$$MS_1 = mS_1, \quad \text{where } S_1 = u = [1, 1, \ldots, 1]^T. \tag{50}$$

Furthermore, the magnitude of all other eigenvalues is less than m.

Equation (50) follows from the summation condition (2). As noted by Mattingly [12], the last part of this lemma follows from Perron's theorem.

Lemma 12. A simple or rank-k generalized eigenvector S_k of a semi-magic square with eigenvalue λ satisfies

$$US_k = 0 \quad \text{for } \lambda \neq m. \tag{51}$$

Proof. Multiply (46) or (47) by U to obtain

$$(\lambda - m)^k\, US_k = 0 \tag{52}$$

which establishes (51). □

The next theorem extends to QR magic squares a result established by Mattingly [12] for regular magic squares.

Theorem 13. *If a QR magic square M_Q with operator Q has a nonzero eigenvalue $\lambda \neq m$ with a rank-k generalized eigenvector S_k, then M_Q also has a companion eigenvalue $-\lambda$ with a rank-k generalized eigenvector QS_k.*

Proof. From the QR condition (12), (51), and (46), we have

$$Q^{-1} M_Q Q S_1 = -M_Q S_1 = -\lambda S_1, \tag{53}$$

whence $QS_1 \neq 0$ and

$$M_Q \left(QS_1\right) = -\lambda \left(QS_1\right). \tag{54}$$

Therefore QS_1 is a simple eigenvector with eigenvalue $-\lambda$. For the generalized eigenvector S_k, on multiplying (48) by Q and using (12) and (51), we find that

$$(M + \lambda I)\, QS_k = -QS_{k-1}, \quad k = 2, 3, \ldots, \hat{k}. \tag{55}$$

Then, by induction, starting with (54) for $k = 2$, it follows from (55) and (47) that

$$\left.\begin{array}{r}(M + \lambda I)^k\, QS_k = 0\\ (M + \lambda I)^{k-1}\, QS_k \neq 0\end{array}\right\}, \quad k = 2, 3, \ldots, \hat{k}. \tag{56}$$

Therefore, QS_k satisfies (47) and is a rank-k generalized eigenvector with eigenvalue $-\lambda$. □

The foregoing proof does not require the introduction of an auxiliary matrix as in Mattingly's proof [12] for $Q = R$. It is essential that Q be nonsingular so that $QS_1 \neq 0$ follows from (53), for we have found specific cases where $QS_1 = 0$ when Q is singular. However, our proof does not require that Q be centrosymmetric. Recently, Staab et al. [20] proved Theorem 13 when Q is a special symmetric permutation matrix $Q = P = P^T$. Previously, Thompson [22] proved the existence of signed pairs of eigenvalues for panmagic squares of orders 4 and 5 using vector space methodology. Since we have already shown that all panmagic squares of orders 4 and 5 are QRK, Theorem 13 confirms Thompson's result.

It follows from Theorem 13 that if a magic square M has nonzero eigenvalues that do not come in signed pairs, then M is not QR. For example, the natural panmagic square M_{P7} of (44) has nonzero eigenvalues that are not signed pairs and therefore it is not QR.

The converse of Theorem 13 would imply that a magic square having only signed pairs of eigenvalues (except 0 and m) is QR. However, a counterexample to this converse is furnished by the following order-6 natural magic square given by Mattingly [12]:

$$M_6 = \begin{bmatrix} 35 & 1 & 6 & 26 & 19 & 24 \\ 3 & 32 & 7 & 21 & 23 & 25 \\ 31 & 9 & 2 & 22 & 27 & 20 \\ 8 & 28 & 33 & 17 & 10 & 15 \\ 30 & 5 & 34 & 12 & 14 & 16 \\ 4 & 36 & 29 & 13 & 18 & 11 \end{bmatrix} \tag{57}$$

which has eigenvalues 0, 111, ± 27, $\pm 4\sqrt{6}$. Computations in [14] attempting to determine Q lead to the conclusion that M_6 is not QR. Therefore, the QR property is a sufficient condition, but not a necessary condition, for the existence of signed pairs of eigenvalues. We note that M_6 is generated by MATLB® and it can be constructed by the Strachey method for singly-even order, natural magic squares as described by Pasles [16]. Also, we have constructed an order-10 Strachey magic square that has signed pairs of eigenvalues [14] and we conjecture that this is a property of all such singly-even-order squares. For the doubly-even order, regular, rank-3, natural magic squares generated by MATLB®, Kirkland and Neumann [9] derived explicit equations for their three nonzero eigenvalues, two of which are signed pairs.

Next, we generalize to QR magic squares Mattingly's [12] results on the number of zero eigenvalues of regular magic squares.

Theorem 14. *An odd-order QR magic square has an even number of zero eigenvalues, whereas an even-order QR magic square has an odd number of zero eigenvalues and is singular.*

Proof. From (49), the trace of a matrix M equals the sum of the eigenvalues of M and it follows from (3) that

$$\mathrm{tr}\,[M] = \sum \lambda_i = m. \tag{58}$$

For an odd-order M_Q, according to Lemma 11 and Theorem 13, (58) is satisfied by the signed pairs of eigenvalues, $\lambda = m$, and an even number of zero eigenvalues. For an even-order M_Q, by (58) together with the signed pairs of eigenvalues and $\lambda = m$, there must be an odd number of zero eigenvalues. Thus, an even-order M_Q is singular. □

As an example of Theorems 13 and 14, the eigenvalues of the panmagic square M_{P9} of (40) are $0, 0, 0, 0, 369, \pm 3(369 \pm 3i\sqrt{2343})^{\frac{1}{2}}$, where the four zero eigenvalues all have simple eigenvectors, i.e., M_{P9} is diagonable. The ultra-magic square M_{U9} of (42) has eigenvalues 0, 0, 0, 0, 0, 0, 369, $\pm 12\sqrt{39}$, where the six zero eigenvalues correspond to four simple eigenvectors and two generalized eigenvectors of ranks 2 and 3, so M_{U9} is nondiagonable. Next, we consider the effect the transformation (33) on the spectra of magic squares.

Theorem 15. *The transformations*

$$M_2 = PM_1P^T \quad \text{and} \quad M_3 = P\,(M_1)^T\,P^T \tag{59}$$

leave eigenvalues unchanged and the eigenvector matrices S in their Jordan forms are related by

$$S_2 = PS_1 \quad \text{and} \quad S_3 = P\left(S_1^{-1}\right)^T. \tag{60}$$

The proof follows from the Jordan forms (49) of M_1, M_2, and M_3. Since $W^T = W$ in (37), Theorem 15 applies to the transformation (36) between regular and pandiagonal magic squares of even order. For a magic square M, Theorem 15 shows that the **normal phases** M, M^T, RMR, and RM^TR all have the same eigenvalues and J matrix as do the **reflected phases** RM, MR, RM^T, and M^TR as noted previously by Loly et al. [11]. The next theorem connects the eigenvalues of these two phase groups for regular magic squares.

Theorem 16. *If a QRS magic square M_Q with operator Q_S has nonzero eigenvalues λ and $-\lambda$ with corresponding rank-k generalized eigenvectors S_k and Q_SS_k, then the QRS magic squares Q_SM_Q and M_QQ_S have eigenvalues $\pm i\gamma\lambda$ and $\mp i\gamma\lambda$, respectively, with corresponding rank-k generalized eigenvectors $\hat{S}_k$ and $Q_S\hat{S}_k$, where*

$$\hat{S}_k = Q_SS_k + i\gamma S_k \quad \text{and} \quad \gamma = \sqrt{1 - \beta n}. \tag{61}$$

Proof. Theorem 7 ensures the magic property of Q_SM_Q and M_QQ_S, and Theorem 13 ensures that nonzero eigenvalues occur as signed pairs. On applying Q_SM_Q to (61), noting (14), and appealing to Theorem 13, we find that

$$(Q_SM_Q - i\gamma\lambda)\,\hat{S}_1 = 0. \tag{62}$$

Thus, $\hat{S}_1$ is a simple eigenvector of $Q_S M$ with eigenvalue $i\lambda$ and, by Theorem 13, $Q_S\hat{S}_1$ is a simple eigenvector of $Q_S M_Q$ with eigenvalue $-i\gamma\lambda$. The proof for generalized eigenvectors of rank k depends on the recursion relation

$$(Q_S M_Q - i\gamma\lambda I)\,\hat{S}_k = i\hat{S}_{k-1}, \quad k = 2, 3, \ldots, \hat{k} \tag{63}$$

which follows from (48), (61), (12), and (51). Then, by induction, from (63) starting with (62) for $k = 2$, it follows from (63) and (47) that

$$\left.\begin{array}{r} (Q_S M_Q - i\gamma\lambda I)^k\,\hat{S}_k = 0 \\ (Q_S M_Q - i\gamma\lambda I)^{k-1}\,\hat{S}_k \neq 0 \end{array}\right\}, \quad k = 2, 3, \ldots, \hat{k}. \tag{64}$$

Thus, $\hat{S}_k$ satisfies (47) and is a rank-k generalized eigenvector of $Q_S M_Q$ with eigenvalue $i\gamma\lambda$. By Theorem 13, $Q_S\hat{S}_k$ is a rank-k generalized eigenvector of $Q_S M_Q$ with eigenvalue $-i\gamma\lambda$. The proof for $M_Q Q_S$ is similar. □

There are two important special cases of Theorem 16. First, for regular magic squares $\left(Q_S = Q_S^{-1} = R,\ \beta = 0,\ \gamma = 1\right)$ the result of this theorem regarding eigenvalues was proved by Thompson [22] for order-3 magic squares and order-4 panmagic squares and a general proof (different from ours) was given by Abu-Jeib [1].

Second, in view of (21), the case $Q_S = Q_S^{-1} = K^{\frac{n}{2}}$ (n even) applies to all MP magic squares, including all order-4 panmagic squares. In addition, all order-5 panmagic squares are covered by Theorem 16 since their operators Q_{K5} and $\hat{Q}_{K5}$ from (27) satisfy (14). Theorem 16 also applies to the squares M_{P9} and M_{U9} of (40) and (42) since $Q_{K9} = (Q_{K9})^{-1}$ in (41). For example, $Q_{K9}M_{P9}$ has eigenvalues 0, 0, 0, 0, 369, $\pm 3i\left(369 \pm 3i\sqrt{2343}\right)^{\frac{1}{2}}$, where the signed pairs are i times the signed pairs of M_{P9} given above. Also, by Theorem 7, $Q_{K9}M_{P9}$ is QRK and panmagic. Furthermore, $Q_{K9}M_{P9}$ is natural (for no apparent reason) and it can be formed by row/column interchanges using (32). However, for other M_P we have found that $Q_K M_P$ is not strictly-magic [14].

5. Products and powers

In this section we study the matrix products and powers[1] of various special magic squares. Our results are stated as theorems for matrix products of magic squares. These theorems can easily be specialized to matrix powers. First, we give a general result that is applicable to all semi-magic squares.

Lemma 17. The matrix product of any number of semi-magic squares $M_1, M_2, \ldots, M_k$ of the same order is a semi-magic square, i.e.,

$$U\hat{M} = \hat{M}U = \hat{m}U, \quad \text{where } \hat{M} = M_1 M_2 \ldots M_k,\ \hat{m} = m_1 m_2 \ldots m_k. \tag{65}$$

This result follows directly from (2). Next, we establish sufficient conditions for $\hat{M}$ to satisfy the diagonal sum conditions (3) and be magic.

Theorem 18. *The matrix product of an odd number k of QR magic squares $M_{Q1}, M_{Q2}, \ldots, M_{Qk}$ of the same order and with the same operator Q is a QR magic square $\hat{M}_Q$ with operator Q, i.e.,*

$$\hat{M}_Q + Q^{-1}\hat{M}_Q Q = \hat{r}U, \quad \text{where } \hat{M}_Q = M_{Q1}M_{Q2}\ldots M_{Qk},\ \hat{r} = 2\hat{m}n^{-1}. \tag{66}$$

If M_{Qi} are QRK (hence pandiagonal) with operator Q_K, then so is $\hat{M}_Q$.

[1] Matrix powers (formed by matrix multiplication) are not to be confused with matrices formed from powers of the individual elements of a matrix as discussed in [17].

Proof. By Lemma 17 $\hat{M}_Q$ is semi-magic. Each QR square M_i satisfies (12) which may be written as

$$M_{Qi} = r_i U - Q^{-1} M_{Qi} Q, \quad \text{where } r_i = 2m_i n^{-1}. \tag{67}$$

On forming $M_1 M_2 M_3$ from (67) for M_2 and using (67) for M_1 and M_3, by (2), (13), and (8), we obtain (66) for $k = 3$. Thus, by Lemma 1, $\hat{M}_Q$ is magic. The QRK property of $\hat{M}_Q$ follows from (66) for $Q = Q_K$. Induction extends these results to higher odd k. □

Theorem 19. *The matrix product of an odd number k of most-perfect magic squares $M_{M1}, M_{M2}, \ldots, M_{Mk}$ is a most-perfect magic square, i.e.,*

$$\hat{M}_M + K^{\frac{n}{2}} \hat{M}_M K^{\frac{n}{2}} = \hat{r} U, \quad \textit{where } \hat{M}_M = M_{M1} M_{M2} \ldots M_{Mk}$$
$$(I + K)\, \hat{M}_M\, (I + K) = 2\hat{r} U, \quad \textit{and} \quad \hat{r} = 2\hat{m} n^{-1}. \tag{68}$$

Proof. The first equation follows from (21) and Theorem 18 with $Q = K^{\frac{n}{2}}$. For the second equation, from (24), we form

$$(I + K)\, \hat{M}_M\, (I + K) \left(I + K^{\frac{n}{2}}\right) = 4\hat{r} U$$
$$\left(I + K^{\frac{n}{2}}\right) (I + K)\, \hat{M}_M\, (I + K) = 4\hat{r} U. \tag{69}$$

Addition of these, together with $(68)_1$, yields $(68)_2$. □

The foregoing lemma and two theorems have counterparts for odd powers of special magic squares of the same type. They follow from setting $M_i = M$ and $m_i = m$ so that $\hat{M} = M^k$ and $\hat{m} = m^k$ in each case. Furthermore, if M_Q is nonsingular with operator Q, then it can be shown that odd negative powers of M_Q also are QR magic squares with operator Q and index m^{-k}. In line with these results, the cube of the QR magic square M_{P9} is magic, whereas the cube of M_{P7} (44) is only semi-magic which again shows that it is not QR. Theorem 18 applies to all order-3 magic squares since they are regular and to all panmagic squares of orders 4 and 5 since they are QRK. This confirms the results of Thompson [22] based on vector space methodology. In addition, Theorem 19 applies to all order-4 panmagic squares since they also are MP.

There is an alternate proof of Theorem 18 for powers of M_Q based on its spectrum. From Lemma 11 and Theorem 13, M_Q has an eigenvalue m and signed pairs of eigenvalues and/or eigenvalues 0. Thus, it follows from (49) that

$$\operatorname{tr}\left[(M_Q)^k\right] = \operatorname{tr}\left[S J^k S^{-1}\right] = \sum_{i=1}^{n} \lambda_i^k = m^k, \quad k \text{ odd}. \tag{70}$$

Similarly, using (66), we have

$$\operatorname{tr}\left[R\, (M_Q)^k\right] = \operatorname{tr}\left[(R M_Q)^k\right] = m^k, \quad k \text{ odd}. \tag{71}$$

Therefore, $(M_Q)^k$ (k odd) satisfies (3) and, by Lemma 17, it is magic. This result provides insight on the connection between the signed pairs of eigenvalues of M_Q and odd matrix powers of M_Q.

Next, let us consider the odd powers of Mattingly's order-6 magic square M_6 of (57). By the Cayley–Hamilton theorem, M_6 must satisfy its characteristic equation which gives

$$M_6\, (M_6 - 111I) \left([M_6]^2 - 729I\right) \left([M_6]^2 - 96I\right) = 0\,. \tag{72}$$

Since M_6 is magic, by (2), this becomes

$$[M_6]^5 = 825\, [M_6]^3 - 69984 M_6 + 2621675700 U \tag{73}$$

and recursion leads to

$$[M_6]^{2k+1} = \frac{1}{633}\left(27^{2k} - 96^k\right)[M_6]^3 + \frac{1}{211}\left(243 \times 96^k - 32 \times 27^{2k}\right) M_6 + \frac{1}{422}\left(7807 \times 111^{2k} - 150775 \times 27^{2k} + 142968 \times 96^k\right) U, \quad k \geqslant 2\,. \tag{74}$$

It can be verified that $[M_6]^{2k+1}$ ($k \geqslant 2$) satisfies the magic square conditions (2) and (3) with $m_{2k+1} = 111^{2k+1}$. Also, $[M_6]^3$ is strictly-magic and it follows that $[M_6]^{2k+1}$ is strictly-magic since the $[M_6]^3$ term in (74) is greater than the M_6 term. A formula similar to (74) for powers of an order-3 natural magic square was given as a problem with solution by Brillhart [3]. Furthermore, our result generalizes a result of van den Essen [6] who used the Cayley–Hamilton theorem to show that odd powers of all order-3 magic squares are magic but odd powers of higher order ones may not be. Formulas similar to (74) can be derived for other magic square matrices; see [14] for examples.

6. Conclusions

The results presented here have furthered our knowledge of transformations, spectra, products, and powers of special magic squares. In particular, the introduction of the quasi-regular property has led to new insights on these topics. Numerical examples in this paper illustrate some of our theoretical results and more numerical examples are presented in [13,14]. As noted in the body of the paper, there are unresolved issues regarding special magic squares.

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Appendix A. Panmagic squares of even order

Here we give a short proof of a basic identity for even-order panmagic squares due to Rosser and Walker [19]. As noted by them, this identity leads to the fact that natural panmagic squares of singly-even order do not exist.

With M_{ij} denoting the elements of an even-order panmagic square M_P, the Rosser–Walker identity can be written as

$$\sum_{x,y=0}^{n/2-1} M_{i+2x,j+2y} = mn/4, \quad i,j = 1 \text{ or } 2. \tag{75}$$

The proof of (75) given by Rosser and Walker [19] is rather complicated. A simpler proof follows from forming (75) by subtracting alternate rows, alternate columns, and alternate diagonals from $2M_P$ and summing over all elements of each matrix to obtain (75). A formal proof can be developed using the **lattice matrix** with elements

$$L_{ij} = \left[1 - (-1)^i\right]\left[1 - (-1)^j\right]/4. \tag{76}$$

The column, row, and pandiagonal sum conditions on M_P can be expressed as

$$\begin{aligned} &(L + LK)\, M_P = m\,(L + LK)\,, \\ &M_P\,(L + KL) = m\,(L + KL)\,, \\ &LKM_PL + LM_PKL = \frac{mn}{2} L, \end{aligned} \tag{77}$$

respectively. These identities lead to

$$(L + LK)\, M_P L + L M_P\, (L + KL) = 2LM_P L + LKM_P L + LM_P KL = mnL,$$
$$\text{whence} \quad LM_P L = \frac{mn}{4} L \tag{78}$$

which is equivalent to (75) for $i = j = 1$. The other three cases of $i, j = 1$ or 2 follow from Theorem 8.

As noted by Rosser and Walker [19], for natural panmagic squares of singly-even order ($n = 6, 10, \ldots$), by (75) and (4), $mn/4$ is a fraction whereas the left-hand-side of (75) is an integer. Therefore, (75) is violated and such squares do not exist. Furthermore, by Theorem 9, a regular magic square of even order can be transformed to a panmagic square. Thus, natural regular magic squares of singly-even order do not exist either.

Unfortunately, the oft-cited proof of nonexistence of natural panmagic squares of singly-even order given by Planck [18] is faulty since his assumed element pattern does not apply to all even-order panmagic squares. For example, in certain natural panmagic squares of order-8 from [21] the sum of the 16 main 2×2 subsquares is not 130 as Planck assumes. Thus, he proved only that natural *most-perfect* squares of singly-even order do not exist.

At History Museum, Xi′an, China (photo by the author).
Another application of magic squares!

References

[1] I.T. Abu-Jeib, Centrosymmetric and skew-centrosymmetric matrices and regular magic squares, Aust. N. Z. J. Stat. 33 (2004) 105–112.
[2] W.R. Andress, Basic properties of pandiagonal magic squares, Amer. Math. Monthly 67 (1960) 143–152.
[3] J. Brillhart, Magic powers, J. Recreational Math., problem: 3 (1970) 104, solution: 6 (1973) 159–160.
[4] S. Cammann, The evolution of magic squares in China, J. Am. Oriental Soc. 80 (1960) 116–124.
[5] K.L. Chu, S.W. Drury, G.P.H. Styan, G. Trenkler, Magic Moore–Penrose inverses and philatelic magic square with special emphasis on the Daniels–Zlobec magic square, Croatian Oper. Res. Rev. 2 (2011) 4–13.
[6] A. van den Essen, Magic squares and linear algebra, Amer. Math. Monthly 97 (1990) 60–62.
[7] A.H. Frost, On the general properties of Nasik squares, Quart. J. Pure Appl. Math. 15 (1878) 34–49.
[8] G. Ganapathy, K. Mani, Add-on security model for public-key cryptosystem based on magic square implementation, in: Proc. World Congress on Engineering and Computer Science 1 WCECS, 2009.
[9] S.J. Kirkland, M. Neumann, Group inverses of M-matrices associated with nonnegative matrices having few eigenvalues, Linear Algebra Appl. 220 (1995) 181–213.
[10] P.D. Loly, Franklin squares: a chapter in the scientific studies of magical squares, Complex Systems 17 (2007) 143–161.
[11] P. Loly, I. Cameron, W. Trump, D. Schindel, Magic square spectra, Linear Algebra Appl. 430 (2009) 2659–2680.
[12] R.B. Mattingly, Even order regular magic squares are singular, Amer. Math. Monthly 107 (2000) 777–782.
[13] R.P. Nordgren, New constructions for special magic squares, Int. J. Pure Appl. Math., in press.
[14] R.P. Nordgren, Examples of special magic squares, in preparation. Available from: <http://arxiv.org>.
[15] K. Ollerenshaw, D.S. Brée, Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration, The Institute of Mathematics and its Applications, Southend-on-Sea, UK, 1998.
[16] P.C. Pasles, Benjamin Franklin's Numbers, Princeton University Press, Princeton, NJ, 2008.
[17] C.A. Pickover, The Zen of Magic Squares, Circles, and Stars, second printing and first paperback printing, Princeton University Press, Princeton, NJ, 2003 (original printing and e-book: 2002).
[18] C. Planck, Pandiagonal magic squares of orders 6 and 10 without minimal numbers, Monist 29 (1919) 307–316.
[19] B. Rosser, R.J. Walker, The algebraic theory of diabolic magic squares, Duke Math. J. 5 (1939) 705–728.
[20] P. Staab, C. Fisher, M. Maggio, M. Andrade, E. Farrell, H. Schilling, The magic of permutation matrices: categorizing, counting and eigenspectra of magic squares, 2010. Available from: <http://arxiv.org/PS_cache/arxiv/pdf/1007/1007.3239v1.pdf>.
[21] M. Suzuki, Magic squares, 2012. Available from: <http://mathforum.org/te/exchange/hosted/suzuki/MagicSquare.htmls>.
[22] A.C. Thompson, Odd magic powers, Amer. Math. Monthly 101 (1994) 339–342.