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FRDTM for numerical simulation of multi-dimensional, time-fractional model of Navier–Stokes equation

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Abstract In this paper, a new approximate solution of time-fractional order multi-dimensional Navier–Stokes equation is obtained by adopting a semi-analytical scheme: "Fractional Reduced Differential Transformation Method (FRDTM)". Three test problems are carried out in order to validate and illustrate the efficiency of the method. The scheme is found to be very reliable, effective and efficient powerful technique to solve wide range of problems arising in engineering and sciences. The small size of computation contrary to the other schemes, is its strength.

1. Introduction

The idea of fractional derivative was first given by a great mathematician Leibniz, in 1695, in a letter to L'Hospital. Fractional calculus deals with the differential and integral operators with non-integral powers. Noting that the integer-order differential operator is a local operator while the fractional-order differential operator is non-local, it means that the next state of a system depends not only upon its current state but also upon all of its previous states. It is more realistic and is one of the main reasons why the fractional calculus has become so popular. In the recent years, advances of fractional differential equations have a great attention due to their numerous applications in a wide range of nonlinear complex systems arising in fluid mechanics, viscoelasticity, mathematical biology, life sciences, electrochemistry and physics [1–8]. For instance, the non-linear oscillation of earthquake can be modeled with fractional derivatives [9], and the fluid-dynamic traffic model with fractional derivatives [10] can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in [11]. Fractional differential equations have created attention among the researcher due to exact description of non-linear phenomena, especially in nano-hydrodynamics where continuum assumption does not well, and fractional model can be considered to be a best candidate. These findings

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invoked the growing interest of studies of the fractal calculus in many branches of science and engineering.

In the recent various analytical techniques such as Homotopy perturbation method (HPM) [10], homotopy perturbation Sumudu transform method [12,13], homotopy analysis method (HAM) [14,15] and Adomian decomposition method (ADM) [16,17] have been developed to solve the fractional partial differential equations. By coupling of HPM and Laplace transform algorithm (LTA), Kumar et al. solved analytically the nonlinear fractional Zakharov–Kuznetsov equation in [18]. At first, Keskin and Oturanc [19] introduce reduced differential transform method (RDTM) as a reduced form of differential transform method, and implement it to find the approximate solutions of partial (and factional partial) differential equations [19,20]. Fractional reduced differential transform method (FRDTM) has been adopted in many articles to solve the differential equations prevailing in mathematics, physics and engineering [21–36].

A famous governing equation of motion of viscus fluid flow called Navier–Stokes (NS) equation has been derived in 1822 [37]. The equation can be regarded as Newton's second law of motion for fluid substances, and is a combination of Momentum equation, continuity equation and the energy equation. This equation describes many physical things such as ocean currents, liquid flow in pipes, blood flow and air flow around the wings of an aircraft. The fractional modeling of NS equations was first done in 2005 by El-Shahed and Salem [38]. The authors [38] generalized the classical NS equations using Laplace transform, finite Hankel transforms and finite Fourier Sine transform. By coupling of HPM and LTA, Kumar et al. [39] solved analytically a nonlinear fractional model of NS equation. Ragab et al. [14] and Ganji et al. [15] solved nonlinear time-fractional NS equation by adopting HAM. Birajdar [16] and Momani and Odibat [17] adopted ADM for numerical computation of time-fractional NS equation. Analytical solution of time-fractional NS equation is obtained using coupling of ADM and LTA by Kumar et al.[40] while Chaurasia and Kumar [41] solved the same equation by coupling of Laplace transform and finite Hankel transform. This paper presents an approximate analytic solution of multi-dimensional, time-fractional model of NS equation by adopting FRDTM.

The rest of the paper is organized as follows: some basic definitions and notations on fractional calculus are revisited in Section 2 while the preliminary on FRDTM is presented in Section 2.1. In Section 3.1, the approximate analytic solutions of three test problems of time-fractional order NS equation are obtained. Section 4 concludes the study.

2. Fractional calculus theory: basic definitions and notations

In this section, among several definitions of fractional integrals or fractional derivatives, available in the literature due to Riemann-Liouville, Grunwald-Letnikov, Caputo, etc., only those basic definitions and preliminaries are revisited, which we need to complete our study.

Definition 1 ([1,2]). Let $\mu \in \mathbb{R}$ and $m \in \mathbb{N}$. A real valued function $f : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathbb{C}_\mu$ if there exists $k \in \mathbb{R},\ k > \mu$ and $g \in C[0,\infty)$ such that $f(x) = x^k g(x)$, for all $x \in \mathbb{R}^+$. Moreover, $f \in \mathbb{C}_\mu^m$ if $f^{(m)} \in \mathbb{C}_\mu$.

Definition 2 ([1,2]). The Riemann–Liouville fractional integral of $f \in \mathbb{C}_\mu$ of the order $\alpha \geqslant 0$ is defined as

$$J_t^\alpha f(t) = \begin{cases} f(t) & \text{if } \alpha = 0, \\ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, & \text{if } \alpha > 0, \end{cases} \tag{1}$$

where Γ *denotes* gamma function: $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, z \in \mathbb{C}$.

In their work, Caputo and Mainardi [3] proposed a modified fractional differentiation operator D_t^α to describe the theory of viscoelasticity in order to overcome the discrepancy of Riemann–Liouville derivative [1,2]. It is mentioned that the proposed Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives.

Definition 3 ([1,3]). The fractional derivative of $f \in \mathbb{C}_\mu$ of the order $\alpha \geqslant 0$, in Caputo sense, is defined as

$$\begin{aligned} D_t^\alpha f(t) &= J_t^{m-\alpha} D_t^m f(t) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, \end{aligned} \tag{2}$$

for $m-1 < \alpha \leqslant m,\ m \in \mathbb{N},\ t > 0,\ f \in \mathbb{C}_\mu^m,\ \mu \geqslant -1$.

The basic properties of Caputo fractional derivative are given as follows:

Lemma 1 ([1–4]). *Let* $m-1 < \alpha \leqslant m, m \in \mathbb{N},$ *and* $f \in \mathbb{C}_\mu^m$, $\mu \geqslant -1$, *then*

$$D_t^\alpha J_t^\alpha f(t) = f(t)$$
$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m} f^{(k)}(0^+)\frac{t^k}{k!}, \quad \text{for } t > 0.$$

In the present work, Caputo fractional derivative is considered because it includes traditional initial and boundary conditions in the formulation of the physical problems. For more details on fractional derivatives, one can refer [1–5].

2.1. Fractional reduced differential transform method (FRDTM)

This section describes the basic properties of fractional reduced differential transform method [25,26]. Let $\psi(x,t)$ be a function of two variables such that $\psi(x,t) = f(x)g(t)$, then from the properties of one-dimensional differential transform (DT) method, we have

$$\psi(x,t) = \sum_{i=0}^{\infty} f(i)x^i \sum_{j=0}^{\infty} g(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Psi(i,j)x^i t^j, \tag{3}$$

where $\psi(i,j) = f(i)g(j)$ is referred as the spectrum of $\psi(x,t)$. Throughout the paper R_D and R_D^{-1} denote the operators for fractional reduced differential transform (FRDT) and inverse FRDT, respectively. Further, the lowercase $\psi(x,t)$ is used for the original function whereas its fractional reduced transformed function is represented by the uppercase $\Psi_k(x)$.

The basic definitions and properties of FRDTM are described below.

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Definition 4 ([25,26]). Let $\psi(x,t)$ be an analytic and continuously differentiable with respect to space variable x and time variable t in the domain of interest, then

(a) FRDT of ψ is given by

$$\Psi_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[D_t^{\alpha k}(\psi(x,t))\right]_{t=t_0}, \quad k = 0,1,2,\dots$$

where α describes the order of time-fractional derivative.

(b) The inverse FRDT of $\Psi_k(x)$ is defined by

$$\psi(x,t) = \sum_{k=0}^{\infty}\Psi_k(x)(t-t_0)^{k\alpha}.$$

(c) From (a) and (b), we have

$$\psi(x,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[D_t^{\alpha k}(\psi(x,t))\right]_{t=t_0}(t-t_0)^{k\alpha}.$$

In particular, for $t_0 = 0$, above equation becomes

$$\psi(x,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[D_t^{\alpha k}(\psi(x,t))\right]_{t=0}t^{k\alpha}.$$

It shows that FRDTM is a generalization of the power series expansion.

Theorem 1 ([24–26]). *Let $u(x,t)$ and $v(x,t)$ be any two analytic and continuously differentiable functions with respect to space variable x and time t such that $u(x,t) = R_D^{-1}[U_k(x)]$ and $v(x,t) = R_D^{-1}[V_k(x)]$, then*

(a) $R_D\{u(x,t)v(x,t)\} = U_k(x)\otimes V_k(x) = \sum_{r=0}^{k}U_r(x)V_{k-r}(x)$;
(b) $R_D\{a_1u(x,t) \pm a_2v(x,t)\} = a_1U_k(x) \pm a_2V_k(x)$;
(c) $R_D\{x^mt^nu(x,t)\} = \begin{cases} x^mU_{k-n}(x) & \text{if } k \geqslant n \\ 0, & \text{else}\end{cases}$;
(d) $R_D\{D_t^{N\alpha}(u(x,t))\} = \frac{\Gamma(1+(k+N)\alpha)}{\Gamma(1+k\alpha)}U_{k+N}(x)$;
(e) $R_D\{D_x^lu(x,t)\} = D_x^lU_k(x)$; $R_D\{x^m\} = x^m\delta(k)$; & $R_D\{e^{\lambda t}\} = \frac{\lambda^k}{k!}$,

where the convolution $\otimes$ denotes the fractional reduced differential transform version of multiplication and the function δ is defined by $\delta(k) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{otherwise}\end{cases}$.

3. Implementation of FRDTM on Navier–Stokes equation

In this section, the numerical study of time-fractional model of NS equation of order $\alpha(\alpha \leqslant 1)$ is presented. The time-fractional model of NS equation for an incompressible fluid flow of kinematic viscosity $v = \eta/\rho$ and constant density ρ is given as follows [16,37]:

$$\begin{cases} D_t^\alpha U + (U\cdot\nabla)U = \rho_0\nabla^2U - \frac{1}{\rho}\nabla p, & \text{on } \Omega\times(0,T) \\ \nabla\cdot U = 0, & \text{on } \Omega\times(0,T) \\ U = 0, & \text{on } \partial\Omega\times(0,T) \end{cases} \tag{4}$$

where $U = (u,v,w)$, t, p denote the fluid vector, time and the pressure, respectively. (x,y,z) are spatial components in Ω and $\partial\Omega$ is the boundary of Ω, η denotes dynamic viscosity and ρ is the density while the ratio $\rho_0 = \eta/\rho$ denotes the kinematic viscosity of the flow. In Cartesian co-ordinates, the above equation becomes

$$\begin{cases} D_t^\alpha u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \rho_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x}, \\ D_t^\alpha v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \rho_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial y}, \\ D_t^\alpha w + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \rho_0\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial z}, \end{cases} \tag{5}$$

Further, if p is known, then $g_1 = -\frac{1}{\rho}\frac{\partial p}{\partial x}$, $g_2 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$, $g_3 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$ can be determined. Applying FRDTM on Eq. (5), we have

$$\begin{cases} \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)}U_{k+1} + \sum_{\ell=0}^{k}\left(\frac{\partial U_\ell}{\partial x}U_{k-\ell} + \frac{\partial U_\ell}{\partial y}V_{k-\ell} + \frac{\partial U_\ell}{\partial z}W_{k-\ell}\right) \\ \quad = \rho_0\nabla^2(U_k) + g_1\delta(k), \\ \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)}V_{k+1} + \sum_{\ell=0}^{k}\left(\frac{\partial V_\ell}{\partial x}U_{k-\ell} + \frac{\partial V_\ell}{\partial y}V_{k-\ell} + \frac{\partial V_\ell}{\partial z}W_{k-\ell}\right) \\ \quad = \rho_0\nabla^2(V_k) + g_2\delta(k), \\ \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)}W_{k+1} + \sum_{\ell=0}^{k}\left(\frac{\partial W_\ell}{\partial x}U_{k-\ell} + \frac{\partial W_\ell}{\partial y}V_{k-\ell} + \frac{\partial W_\ell}{\partial z}W_{k-\ell}\right) \\ \quad = \rho_0\nabla^2(W_k) + g_3\delta(k), \end{cases} \tag{6}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, and $U_k = U_k(x,y,z)$, etc. One can obtain the recursive values of U_k, V_k, W_k by solving above equation simultaneously once the values U_0, V_0, W_0 are known.

3.1. Illustrative examples

Example 1. Consider time-fractional order 2-dimensional NS equation with $g_1 = -g_2 = g$ as

$$\begin{aligned} D_t^\alpha u + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} &= \rho_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g, \\ D_t^\alpha v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \rho_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - g, \end{aligned} \tag{7}$$

subject to the initial condition

$$u(x,y,0) = -\sin(x+y), \quad v(x,y,0) = \sin(x+y), \tag{8}$$

Using FRDTM on the above two equations, we obtained the following recurrence relation:

$$\begin{cases} \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)}U_{k+1} + \sum_{\ell=0}^{k}\left(\frac{\partial U_\ell}{\partial x}U_{k-\ell} + \frac{\partial U_\ell}{\partial y}V_{k-\ell}\right) = \rho_0\nabla^2(U_k) + g\delta(k), \\ \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)}V_{k+1} + \sum_{\ell=0}^{k}\left(\frac{\partial V_\ell}{\partial x}U_{k-\ell} + \frac{\partial V_\ell}{\partial y}V_{k-\ell}\right) = \rho_0\nabla^2(V_k) - g\delta(k), \\ U_0 = -\sin(x+y), \quad V_0 = \sin(x+y) \end{cases} \tag{9}$$

On solving the system (9), we have

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$$U_1(x,y)=\tfrac{2\rho_0}{\Gamma(1+\alpha)}\sin(x+y)+\tfrac{g}{\Gamma(1+\alpha)};$$
$$V_1(x,y)=-\tfrac{2\rho_0}{\Gamma(1+\alpha)}\sin(x+y)-\tfrac{g}{\Gamma(1+\alpha)}$$
$$U_2(x,y)=-\tfrac{(2\rho_0)^2}{\Gamma(1+2\alpha)}\sin(x+y);\quad V_2(x,y)=\tfrac{(2\rho_0)^2}{\Gamma(1+2\alpha)}\sin(x+y)$$
$$U_3(x,y)=\tfrac{(2\rho_0)^3}{\Gamma(1+3\alpha)}\sin(x+y);\quad V_3(x,y)=-\tfrac{(2\rho_0)^3}{\Gamma(1+3\alpha)}\sin(x+y)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$U_k(x,y)=-\tfrac{(-2\rho_0)^k}{\Gamma(1+k\alpha)}\sin(x+y);\quad V_k(x,y)=\tfrac{(-2\rho_0)^k}{\Gamma(1+k\alpha)}\sin(x+y),$$

for any integer $k\geqslant 2$, (10)

By using inverse FRDT, we have

$$\begin{aligned}u(x,y,t)&=\sum_{k=0}^{\infty}U_k(x,y)t^{\alpha k}=-\sin(x+y)\sum_{k=0}^{\infty}\tfrac{(-2\rho_0t^\alpha)^k}{\Gamma(1+k\alpha)}+\tfrac{gt^\alpha}{\Gamma(1+\alpha)},\\&=-\sin(x+y)E_{\alpha,1}(-2\rho_0t^\alpha)+\tfrac{gt^\alpha}{\Gamma(1+\alpha)},\end{aligned}\tag{11}$$

$$\begin{aligned}v(x,y,t)&=\sum_{k=0}^{\infty}V_k(x,y)t^{\alpha k}=\sin(x+y)\sum_{k=0}^{\infty}\tfrac{(-2\rho_0t^\alpha)^k}{\Gamma(1+k\alpha)}-\tfrac{gt^\alpha}{\Gamma(1+\alpha)},\\&=\sin(x+y)E_{\alpha,1}(-2\rho_0t^\alpha)-\tfrac{gt^\alpha}{\Gamma(1+\alpha)}.\end{aligned}\tag{12}$$

where $E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+k\alpha)}$, for $\alpha,\beta>0$ denotes the Mittag–Leffler function with two parameters [1], and notice that $E_{1,1}(z)=e^z$. For $g=0$ and $\alpha=1$ Eqs. (11), (12) reduces to

$$u(x,y,t)=-e^{-2\rho_0t}\sin(x+y);$$
$$v(x,y,t)=e^{-2\rho_0t}\sin(x+y),\tag{13}$$

which is the exact solution of classical NS equation for the velocity field. The behavior of velocity field of the classical NS equation is depicted in Fig. 1, and the behavior of NS equation with time-fraction order $\alpha=0.1$, 0.5 and 0.8 is depicted in Figs. 2–4, respectively.

Example 2. Consider time-fractional order two dimensional NS Eq. (7) subject to the initial condition

$$u(x,y,0)=-e^{x+y},\quad v(x,y,0)=e^{x+y},\tag{14}$$

Using FRDTM on Eqs. (7) and (14), we obtained the following recurrence relation:

$$\begin{cases}\frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)}U_{k+1}+\sum_{\ell=0}^{k}\left(\frac{\partial U_\ell}{\partial x}U_{k-\ell}+\frac{\partial U_\ell}{\partial y}V_{k-\ell}\right)=\rho_0\nabla^2(U_k)+g_1\delta(k),\\ \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)}V_{k+1}+\sum_{\ell=0}^{k}\left(\frac{\partial V_\ell}{\partial x}U_{k-\ell}+\frac{\partial V_\ell}{\partial y}V_{k-\ell}\right)=\rho_0\nabla^2(V_k)-g_1\delta(k),\\ U_0=-e^{x+y},\quad V_0=e^{x+y}.\end{cases}\tag{15}$$

On solving the system (15), we have

$$U_1(x,y)=-\tfrac{2\rho_0}{\Gamma(1+\alpha)}e^{x+y}+\tfrac{g}{\Gamma(1+\alpha)};\quad V_1(x,y)=\tfrac{2\rho_0}{\Gamma(1+\alpha)}e^{x+y}-\tfrac{g}{\Gamma(1+\alpha)}$$
$$U_2(x,y)=-\tfrac{(2\rho_0)^2}{\Gamma(1+2\alpha)}e^{x+y};\quad V_2(x,y)=\tfrac{(2\rho_0)^2}{\Gamma(1+2\alpha)}e^{x+y}$$
$$U_3(x,y)=-\tfrac{(2\rho_0)^3}{\Gamma(1+3\alpha)}e^{x+y};\quad V_3(x,y)=\tfrac{(2\rho_0)^3}{\Gamma(1+3\alpha)}e^{x+y}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$U_k(x,y)=-\tfrac{(2\rho_0)^k}{\Gamma(1+k\alpha)}e^{x+y};\quad V_k(x,y)=\tfrac{(2\rho_0)^k}{\Gamma(1+k\alpha)}e^{x+y},\ \forall k\geqslant 2\tag{16}$$

By using inverse FRDT, we have

$$\begin{aligned}u(x,y,t)&=\sum_{k=0}^{\infty}U_k(x,y)t^{\alpha k}=-e^{x+y}\sum_{k=0}^{\infty}\tfrac{(2\rho_0t^\alpha)^k}{\Gamma(1+k\alpha)}+\tfrac{gt^\alpha}{\Gamma(1+\alpha)},\\&=-e^{x+y}E_{\alpha,1}(2\rho_0t^\alpha)+\tfrac{gt^\alpha}{\Gamma(1+\alpha)},\end{aligned}\tag{17}$$

$$\begin{aligned}v(x,y,t)&=\sum_{k=0}^{\infty}V_k(x,y)t^{\alpha k}=e^{x+y}\sum_{k=0}^{\infty}\tfrac{(2\rho_0t^\alpha)^k}{\Gamma(1+k\alpha)}-\tfrac{gt^\alpha}{\Gamma(1+\alpha)},\\&=e^{x+y}E_{\alpha,1}(2\rho_0t^\alpha)-\tfrac{gt^\alpha}{\Gamma(1+\alpha)}.\end{aligned}\tag{18}$$

With $g=0$, the solution of the problem is obtained by Birajdar [16]. For $g=0$ and $\alpha=1$, we have

$$u(x,y,t)=-e^{x+y+2\rho_0t},\quad v(x,y,t)=e^{x+y+2\rho_0t},\tag{19}$$

which is the exact solution of classical NS equation for the velocity field. The behavior of velocity field of the NS equation is depicted for $\alpha=1$ and 0.5 in Figs. 5 and 6, respectively.

Example 3. Consider time-fractional order three dimensional NS Eq. (5) with $g_1=g_2=g_3=0$ subject to the initial condition

$$u(x,y,z,0)=-0.5x+y+z,\quad v(x,y,z,0)=x-0.5y+z,$$
$$w(x,y,z,0)=x+y-0.5z.\tag{20}$$

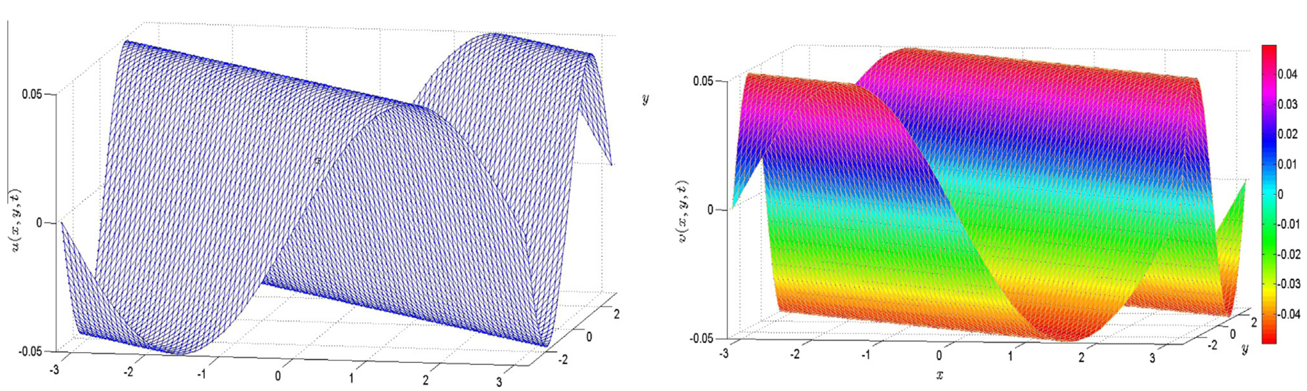

Figure 1 The behavior of u and v of NS equation in Example 1 at $t=3$ with the parameters $\alpha=1,\ g=0,\ \rho_0=0.5$.

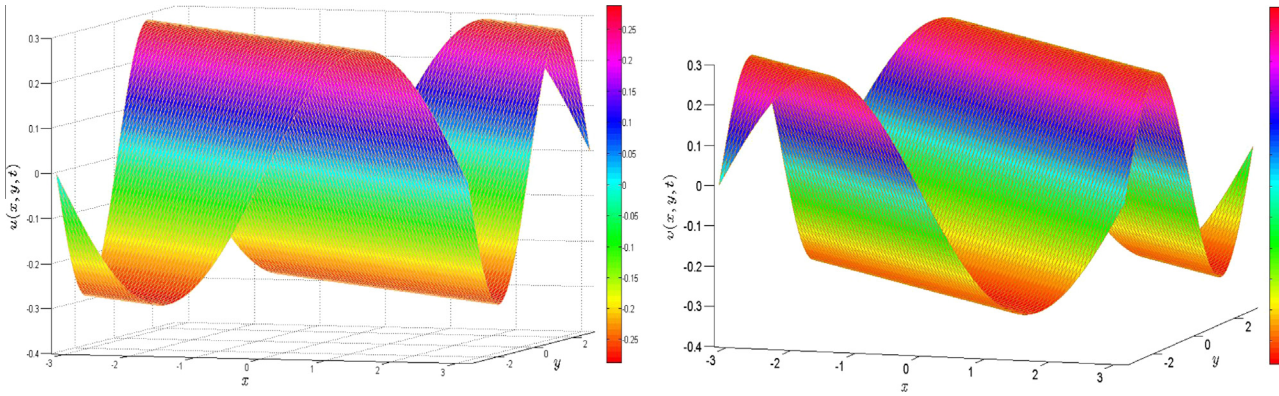

Figure 2 The behavior of u and v of NS equation in Example 1 at $t = 3$ with the parameters $\alpha = 0.5,\ g = 0,\ \rho_0 = 0.5$.

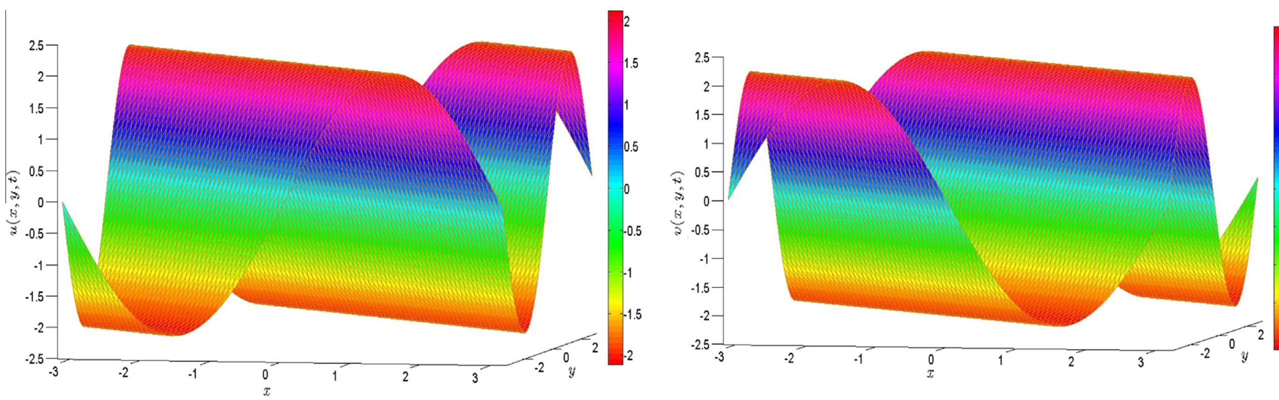

Figure 3 The behavior of u and v of NS equation in Example 1 at $t = 3$ with the parameters $\alpha = 0.1,\ g = 0,\ \rho_0 = 0.5$.

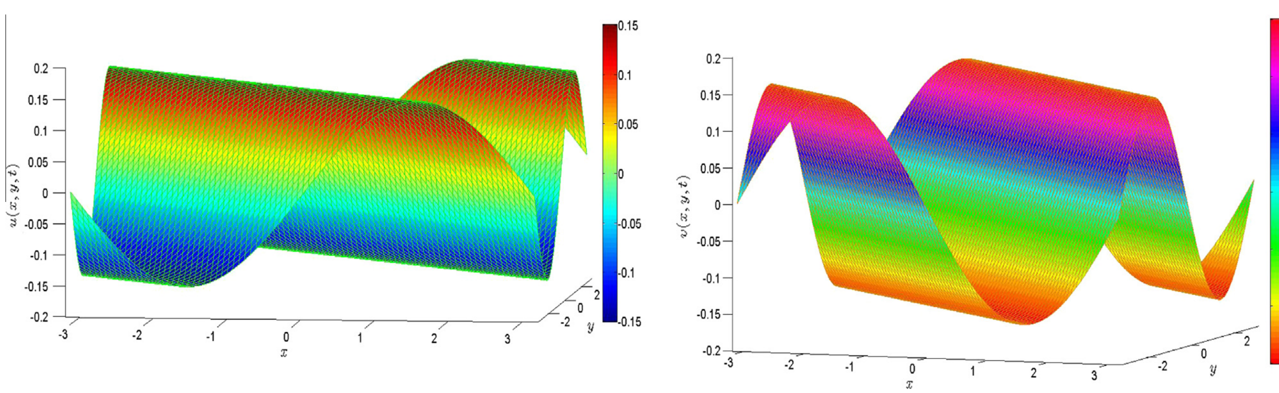

Figure 4 The behavior of u and v of NS equation in Example 1 at $t = 3$ with the parameters $\alpha = 0.8,\ g = 0,\ \rho_0 = 0.5$.

Using FRDTM on these equations, we obtained the following recurrence relation:

$$\begin{cases} \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)} U_{k+1} + \sum_{\ell=0}^{k} \left(\frac{\partial U_\ell}{\partial x} U_{k-\ell} + \frac{\partial U_\ell}{\partial y} V_{k-\ell} + \frac{\partial U_\ell}{\partial z} W_{k-\ell} \right) = \rho_0 \nabla^2(U_k), \\ \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)} V_{k+1} + \sum_{\ell=0}^{k} \left(\frac{\partial V_\ell}{\partial x} U_{k-\ell} + \frac{\partial V_\ell}{\partial y} V_{k-\ell} + \frac{\partial V_\ell}{\partial z} W_{k-\ell} \right) = \rho_0 \nabla^2(V_k), \\ \frac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)} W_{k+1} + \sum_{\ell=0}^{k} \left(\frac{\partial W_\ell}{\partial x} U_{k-\ell} + \frac{\partial W_\ell}{\partial y} V_{k-\ell} + \frac{\partial W_\ell}{\partial z} W_{k-\ell} \right) = \rho_0 \nabla^2(W_k), \\ U_0(x,y,z) = -0.5x + y + z, \quad V_0(x,y,z) = x - 0.5y + z, \quad W_0(x,y,z) = x + y - 0.5z. \end{cases} \tag{21}$$

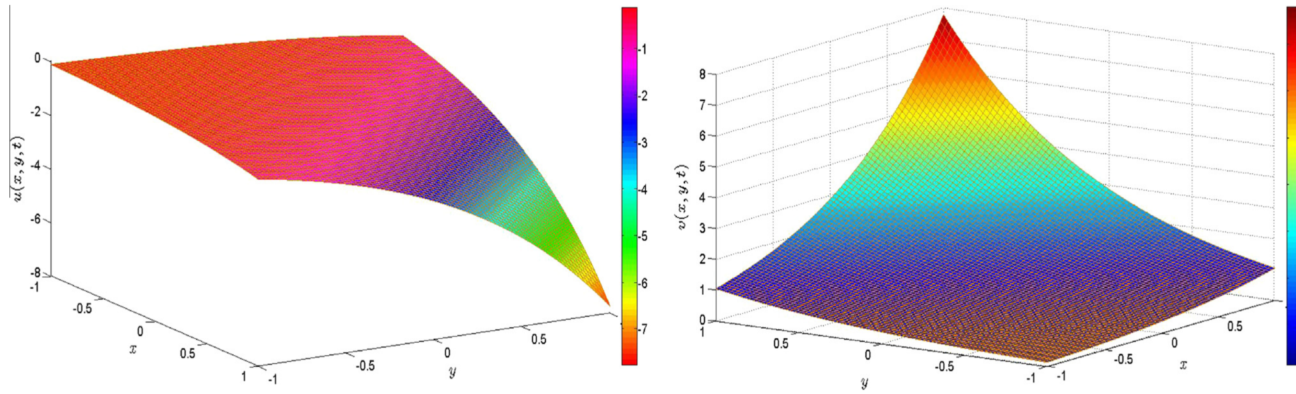

Figure 5 The behavior of u and v of NS equation in Example 2 with the parameters $\alpha = 1,\ g = 0,\ \rho_0 = 0.5$ at $t = 0.05$.

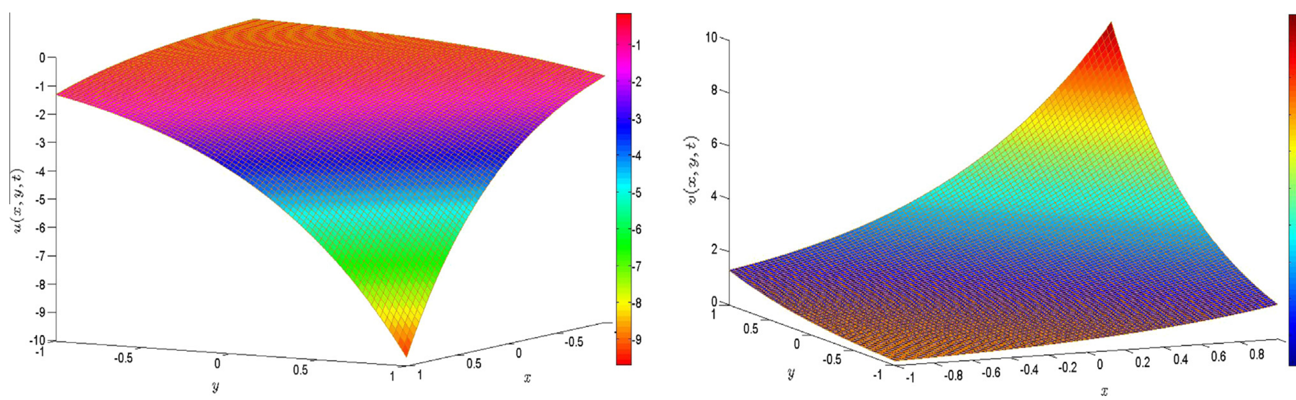

Figure 6 The behavior of u and v of NS equation in Example 2 with the parameters $\alpha = 0.5,\ g = 0,\ \rho_0 = 0.5$ at $t = 0.05$.

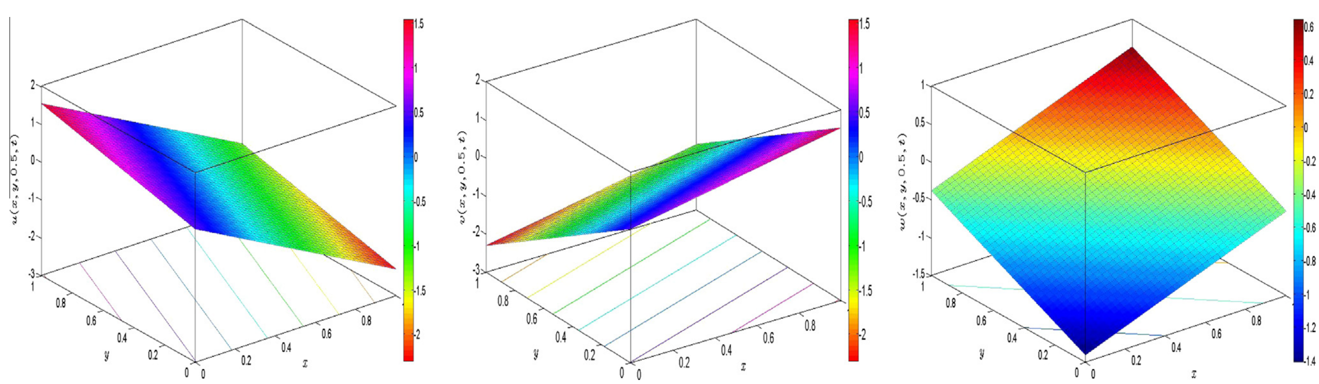

Figure 7 The velocity profile (u, v, w) of NS equation in Example 3 at $t = 0.1$ with $\alpha = 1$.

On solving the simultaneous equations in (21) and denoting $U_k(x,y,z) \equiv U_k$, etc., we have

$$\begin{aligned}
&U_1 = -\tfrac{2.25}{\Gamma(1+\alpha)}x;\quad U_2 = \tfrac{2(2.25)}{\Gamma(1+2\alpha)}U_0;\quad U_3 = -\tfrac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\tfrac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\right)x;\, U_4 = \tfrac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\tfrac{2\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}+\tfrac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)U_0;\ldots\\
&V_1 = -\tfrac{2.25}{\Gamma(1+\alpha)}y;\quad V_2 = \tfrac{2(2.25)}{\Gamma(1+2\alpha)}V_0;\quad V_3 = -\tfrac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\tfrac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\right)y;\, V_4 = \tfrac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\tfrac{2\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}+\tfrac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)V_0;\ldots\\
&W_1 = -\tfrac{2.25}{\Gamma(1+\alpha)}z;\quad W_2 = \tfrac{2(2.25)}{\Gamma(1+2\alpha)}W_0;\quad W_3 = -\tfrac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\tfrac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\right)z;\, W_4 = \tfrac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\tfrac{2\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}+\tfrac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)W_0;\ldots
\end{aligned} \tag{22}$$

By using inverse FRDT, we have

$$
\begin{aligned}
u(x,y,z,t) &= U_0 + U_1 t^\alpha + U_2 t^{2\alpha} + U_3 t^{3\alpha} + U_4 t^{4\alpha} + \ldots \\
&= -0.5x + y + z - \tfrac{2.25}{\Gamma(1+\alpha)} x t^\alpha + \tfrac{2(2.25)}{\Gamma(1+2\alpha)}(-0.5x + y + z)t^{2\alpha} - \tfrac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4 + \tfrac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\right) x t^{3\alpha} \\
&\quad + \tfrac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8 + \tfrac{2\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} + \tfrac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(-0.5x + y + z)t^{4\alpha} + \ldots \\
v(x,y,z,t) &= V_0 + V_1 t^\alpha + V_2 t^{2\alpha} + V_3 t^{3\alpha} + V_4 t^{4\alpha} + \ldots \\
&= x - 0.5y + z - \tfrac{2.25}{\Gamma(1+\alpha)} y t^\alpha + \tfrac{2(2.25)}{\Gamma(1+2\alpha)}(x - 0.5y + z)t^{2\alpha} - \tfrac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4 + \tfrac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\right) y t^{3\alpha} \\
&\quad + \tfrac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8 + \tfrac{2\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} + \tfrac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(x - 0.5y + z)t^{4\alpha} + \ldots \\
w(x,y,z,t) &= W_0 + W_1 t^\alpha + W_2 t^{2\alpha} + W_3 t^{3\alpha} + W_4 t^{4\alpha} + \ldots \\
&= x + y - 0.5z - \tfrac{2.25}{\Gamma(1+\alpha)} z t^\alpha + \tfrac{2(2.25)}{\Gamma(1+2\alpha)}(x + y - 0.5z)t^{2\alpha} - \tfrac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4 + \tfrac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\right) z t^{3\alpha} \\
&\quad + \tfrac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8 + \tfrac{2\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} + \tfrac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(x + y - 0.5z)t^{4\alpha} + \ldots
\end{aligned}
$$

which is the required exact solution. For $\alpha = 1$, we have

$$
\left.
\begin{aligned}
u(x,y,z,t) &= (-0.5x + y + z)(1 + 2.25t^2 + 2.25^2 t^4 + \ldots) - 2.25xt(1 + 2.25t^2 + \ldots) \\
&= \tfrac{-0.5x + y + z - 2.25xt}{1 - 2.25t^2}. \\
v(x,y,z,t) &= (x - 0.5y + z)(1 + 2.25t^2 + 2.25^2 t^4 + \ldots) - 2.25yt(1 + 2.25t^2 + \ldots) \\
&= \tfrac{x - 0.5y + z - 2.25yt}{1 - 2.25t^2}. \\
w(x,y,z,t) &= (x + y - 0.5z)(1 + 2.25t^2 + 2.25^2 t^4 + \ldots) - 2.25zt(1 + 2.25t^2 + \ldots) \\
&= \tfrac{x + y - 0.5z - 2.25zt}{1 - 2.25t^2}.
\end{aligned}
\right\} \tag{23}
$$

which is the exact solution of the associated classical NS equation for the velocity field which is same as reported in [42]. The velocity profile (u, v, w) of the Navier–Stokes equation for $\alpha = 1$ is depicted in Fig. 7.

4. Conclusion

In this paper, fractional reduced differential transformation method is adopted for the numerical simulation of time-fractional model of Navier–Stokes equations with initial conditions. The fractional derivative is considered in the Caputo sense. The analytical results have been given in terms of a power series. Three test problems are carried out in order to validate and illustrate the efficiency of the method. The proposed solutions agree excellently with HPM [15] and ADM [16], and are approximated without any discretization, transformation, perturbation, or restrictive conditions. However, the performed calculations show that the described method needs very small size of computation in comparison with HPM [15] and ADM [16]. Small size of computation contrary to the other schemes, is the strength of the scheme.

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