Hadamard products and $Q_K$ spaces

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Abstract

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in the unit disk. The Hadamard product of $f$ and $g$ is defined by $f \ast g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. This paper gives some characterizations of functions in $Q_K$ spaces in terms of the Hadamard products.

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1. Introduction

A special class of Möbius invariant function spaces, the so-called $Q_p$ spaces, has been extensively studied for years in the context of a wide class of function spaces. Let $g(a, z) = \log(1/|\varphi_a(z)|)$ be the Green function on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ is the Möbius transformation of $\mathbb{D}$. For $0 \leq p < \infty$, the space $Q_p$ consists of analytic functions $f$ on $\mathbb{D}$ for which

$$
\|f\|_{Q_p}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (g(a, z))^p \, dA(z) < \infty,
$$

where $dA$ is an area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. It is easy to check that the space $Q_p$ is Möbius invariant in the sense that $\|f \circ \varphi_a\|_{Q_p} = \|f\|_{Q_p}$ for every $f \in Q_p$ and $a \in \mathbb{D}$. We know that $Q_1 = \text{BMOA}$, the space of all analytic functions of bounded mean oscillation [10], and for each $p > 1$, the space $Q_p$ is the Bloch space $B$ consisting of analytic functions $f$ on $\mathbb{D}$ whose derivatives $f'$ are subject to the growth restriction

$$
\|f\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.
$$

When $p = 0$, the space $Q_p$ degenerates to the Dirichlet space $D$. See [13] and [15] for a summary of recent research about $Q_p$ spaces.

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Let $H(D)$ denote the class of all functions analytic in $D$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the Hadamard product of $f$ and $g$ is defined by $f \ast g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Anderson [1] proved that if $f, g \in B$, then $f \ast g \in B$ and $\| f \ast g \|_B \leq C \| f \|_B \cdot \| g \|_B$.

By Fefferman’s duality theorem on Hardy-BMO and a result of Mateljevic and Pavlovic in [11], it is easy to see that if $f \in BMOA$ and $g \in B$, then $f \ast g \in BMOA$ and $\| f \ast g \|_{BMOA} \leq C \| f \|_{BMOA} \cdot \| g \|_B$.

Since $Q_1 = BMOA$ and $Q_p = B$ for all $p > 1$, the above results can be stated in the words of $Q_p$ spaces: for $1 \leq p < \infty$, if $f \in Q_p$ and $g \in B$, then $f \ast g \in Q_p$ and $\| f \ast g \|_{Q_p} \leq C \| f \|_{Q_p} \cdot \| g \|_B$.

Aulaskari, Girela and the first author [2] proved that the above result is still true for $0 < p < 1$ if the Taylor coefficients of functions $f \in Q_p$ are positive. In same paper, they asked whether the condition that the Taylor coefficients of functions $f \in Q_p$ are positive can be dropped. Recently, Pavlovic [12] showed that (1.3) holds for all $f \in Q_p$ and $g \in B$ without any more assumption.

In this paper, we show that the above problem in fact remains true for more general function spaces $Q_K$ and some related spaces.

For a function $K : (0, \infty) \to [0, \infty)$, the space $Q_K$ is the space of functions $f \in H(D)$ for which

$$
\| f \|^2_{Q_K} = \sup_{a \in D} \int_D |f'(z)|^2 K(g(z, a)) dA(z) < \infty.
$$

(1.4)

We say that the spaces $Q_K$ is trivial if $Q_K$ contains only constant functions. We know that if the integral

$$
\int_0^{1/e} K\left(\frac{1}{\rho}\right) \rho d\rho = \int_1^\infty K(t)e^{-2t} dt
$$

(1.5)

is convergent, then $Q_K$ is not trivial and $Q_K \subset B$; see [6].

From now on we always assume the function $K$ is nondecreasing and right-continuous on $(0, \infty)$. Moreover, we assume (1.5) is convergent for $K$ and $K$ has the double condition $K(2t) \approx K(t) > 0$, $t > 0$.

A useful tool in the study of $Q_K$ spaces is the auxiliary function $\varphi_K$ defined by

$$
\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.
$$

It is clear to see that $\varphi_K(s)$ is nondecreasing and right-continuous on $(0, \infty)$ and when $0 < s < 1$, $\varphi_K(s) < 1$; when $s \geq 1$, $\varphi_K(s) \geq 1$.

We mention that the following two conditions are important in the study of $Q_K$ spaces during the last few years:

$$
\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty, \quad (A)
$$

$$
\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty. \quad (B)
$$

For more results on $Q_K$ spaces, see [6] and [7].

For $0 < p \leq \infty$, the Hardy space $H^p$ consists of those functions $f \in H(D)$ for which

$$
\| f \|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty,
$$

for more results on $Q_K$ spaces, see [6] and [7].
where

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{\frac{1}{p}}$$

for $0 < p < \infty$ and

$$M_\infty(r,f) = \sup_{0 \leq t \leq 2\pi} |f(re^{it})|.$$

For $1 \leq q \leq \infty$, $0 \leq \alpha \leq 1$, the mean Lipschitz space $\Lambda(q,\alpha)$ consists of those functions $f \in H(D)$ for which

$$\|f\|_{\Lambda(q,\alpha)} = |f(0)| + \sup_{0 < r < 1} (1-r)^{1-\alpha} M_q(r, f').$$

Note that $\Lambda(q, \frac{1}{q})$ increases with $q \in (1, \infty)$. It is clear that the space $\Lambda(\infty, 0)$ coincides with the Bloch space $B$, and is contained in $\Lambda(q, 0)$ for every $1 < q < \infty$. We refer to [5] for more notations and results of the Hardy and mean Lipschitz spaces.

2. Some inequalities for $Q_K$ spaces

Throughout this paper the letter $C$ denotes a positive constant which may vary at each occurrence.

**Theorem 2.1.** Suppose $f \in H(D)$ and $2 \leq q < \infty$. Let $K$ satisfy

$$A_q = \int_0^1 \frac{\varphi_K(s)}{s^{2-\frac{2}{q}}} \, ds < \infty. \quad (2.1)$$

Then

$$\Lambda\left(q, \frac{1}{q}\right) \subsetneq Q_K$$

and

$$\|f\|_{Q_K} \leq A_q K(1) \sup_{0 < r < 1} (1-r)^{1-\frac{1}{q}} M_q(r, f'). \quad (2.2)$$

**Proof.** Let $f \in A(q, \frac{1}{q})$ and let

$$C = \sup_{0 < r < 1} (1-r)^{1-\frac{1}{q}} M_q(r, f').$$

For $2 \leq q < \infty$, the Hölder inequality gives

$$\theta + \frac{\theta}{2} \int_0^{|I|} |f'(re^{it})|^2 \, dt \leq |I|^{1-\frac{2}{q}} M_q^2(r, f') \leq C |I|^{1-\frac{2}{q}} (1-r)^{2\left(\frac{1}{q}-1\right)},$$

where subarc $I \in \partial \mathbb{D}$ and $|I| \leq 1$. Thus

$$\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) |f'(z)|^2 \, dA(z) \leq C |I|^{1-\frac{2}{q}} \int_{1-|I|}^1 K\left(\frac{1-r}{|I|}\right) (1-r)^{2\left(\frac{1}{q}-1\right)} \, dr$$

$$= C |I|^{1-\frac{2}{q}} \int_0^1 K(s)s^{2\left(\frac{1}{q}-1\right)} |I|^\frac{2}{q} - 1 \, ds$$

$$= C \int_0^1 K(s) \frac{ds}{s^{2\left(1-\frac{1}{q}\right)}} \leq CK(1) \int_0^1 \frac{\varphi_K(s)}{s^{2\left(1-\frac{1}{q}\right)}} \, ds.$$
By Theorem 3.1 of [7] we have that \( f \in Q_K \) and (2.2) holds. To prove the inclusion is strict, for given \( q \in [2, \infty) \) we consider the function \( K(t) = t^m \) with \( 1 - \frac{2}{q} < m < 1 \) and choose the function
\[
f_0(z) = \sum_{k=1}^{\infty} k 2^{-k/q} z^{2^k}.
\]
By Lemma 5.3 in [8], we see that \( f_0 \notin \Lambda(q, \frac{1}{q}) \) since
\[
\sup_k \{ |a_k| n_k^{1/q} \} = \sup_k \{ k 2^{-k/q} 2^k \} = \infty.
\]
On the other hand, we know that (B) holds for \( K(t) = t^m \). By [14] we see that \( f_0 \in Q_K \) since
\[
\sum_{k=1}^{\infty} |a_k|^2 n_k K \left( \frac{1}{n_k} \right) = \sum_{k=1}^{\infty} k^2 2^{(1-m-2/q)k} < \infty.
\]
This completes the proof. \( \square \)

**Remark.** Note that (2.1) becomes (A) for \( q = 2 \). Since \( \Lambda(q, \frac{1}{q}) \) is increasing on \( 1 \leq q < \infty \), we have \( \Lambda(q, \frac{1}{q}) \subseteq Q_K \) for all \( 1 \leq q \leq 2 \) if \( K \) satisfies condition (A). For the special case \( K(t) = t^p \), we have that
\[
\int_0^1 \frac{\varphi_K(s)}{s^{2-\frac{2}{q}}} \, ds < \infty
\]
holds if \( 1 - \frac{2}{q} < p \). Thus, Theorem 2.1 improves Theorem 1.4 in [3].

**Theorem 2.2.** Suppose \( f \in H(D) \) and \( 2 < q \leq \infty \). Let \( K \) satisfy
\[
B_q = \int_1^{+\infty} \frac{\varphi_K(s)}{s^{2-\frac{2}{q}}} \, ds < \infty. \tag{2.3}
\]
Then
\[
\|f\|_{Q_K}^2 \leq B_q \int_0^1 K(1-r) M_q^2(r, f') \, dr. \tag{2.4}
\]

**Proof.** Without loss of generality, we assume \( I \in \partial \mathbb{D} \) with \( |I| \leq 1 \). We consider the first case \( q = \infty \). Note that
\[
\int_{\theta - \frac{|I|}{2}}^{\theta + \frac{|I|}{2}} |f'(r e^{it})|^2 \, dt \leq |I| M_\infty^2(r, f').
\]
Thus we find that
\[
\int_{S(I)} \left( \frac{1 - |z|}{|I|} \right) |f'(z)|^2 \, dA(z) \leq \int_{1-|I|}^{1} |I| K \left( \frac{1-r}{|I|} \right) M_\infty^2(r, f') \, dr
\]
\[
\leq \int_{1-|I|}^{1} |I| \varphi_K \left( \frac{1}{|I|} \right) K(1-r) M_\infty^2(r, f') \, dr
\]
\[
\leq \int_{1-|I|}^{1} \left( \int_{\frac{1}{|I|}}^{\infty} \frac{\varphi_K(s)}{s^2} \, ds \right) K(1-r) M_\infty^2(r, f') \, dr
\]
\[ \leq B_\infty \int_0^1 K(1 - r)M_{2q}^2(r, f') \, dr. \]

Here we used the estimate

\[ K\left(\frac{1 - r}{|I|}\right) \leq K(1 - r)\varphi_K\left(\frac{1}{|I|}\right). \]

For the case \(2 < q < \infty\), using the Hölder inequality,

\[ \frac{\theta + |I|}{2} \int_{\theta - \frac{|I|}{2}} f'(re^{it})^2 \, dt \leq |I|^{1 - \frac{2}{q}} M_q^2(r, f'). \]

It follows that

\[ \int_{S(I)} K\left(\frac{1 - |z|}{|I|}\right) |f'(z)|^2 \, dA(z) \leq \int_{1 - |I|}^1 K(1 - r)\varphi_K\left(\frac{1}{|I|}\right) |I|^{1 - \frac{2}{q}} M_q^2(r, f') \, dr. \]

By condition (2.3), we obtain

\[ \varphi_K\left(\frac{1}{|I|}\right)|I|^{1 - \frac{2}{q}} = \left(1 - \frac{2}{q}\right)\varphi_K\left(\frac{1}{|I|}\right) \int_{\theta}^{\infty} \frac{ds}{s^{2 - \frac{2}{q}}} \leq \left(1 - \frac{2}{q}\right) \int_{\theta}^{\infty} \frac{\varphi_K(s)}{s^{2 - \frac{2}{q}}} \, ds \leq B_q. \]

Thus

\[ \int_{S(I)} K\left(\frac{1 - |z|}{|I|}\right) |f'(z)|^2 \, dA(z) \leq B_q \int_0^1 K(1 - r)M_q^2(r, f') \, dr. \]

**Remark.** Note that the estimate (2.4) is the best possible for \(2 < q \leq \infty\). That is, (2.4) is false for \(q = 2\). In fact, we have

\[ f \in Q_K \quad \Rightarrow \quad \int_0^1 K(1 - r)M_2^2(r, f') \, dr < \infty. \]

### 3. Main results

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \). Then the Hadamard product of \( f \) and \( g \) can be rewritten as

\[ f \ast g(re^{i\theta}) = \sum_{n=0}^{2\pi} a_n (\sqrt{r} e^{i(\theta - \varphi)})^n \cdot \sum_{n=0}^{\infty} b_n (\sqrt{r} e^{i\varphi})^n \, d\varphi. \]

Thus, we have Young’s inequality as follows. See [4].

**Lemma 3.1.** Let \( f, g \in H(\mathbb{D}) \) and \( \frac{1}{s} + \frac{1}{t} = 1 + \frac{1}{q}, \ q, s, t \geq 1 \). We have

\[ M_q(r, f \ast g) \leq M_s(\sqrt{r}, f)M_t(\sqrt{r}, g). \]

**Theorem 3.1.** Let \( 2 \leq q < \infty \) and let \( K \) satisfy condition (2.1). If \( f \in \Lambda(q, \frac{1}{q}) \) and \( g \in \Lambda(1, 0) \), then \( f \ast g \in Q_K \) and

\[ \| f \ast g \|_{Q_K} \leq C \| f \|_{\Lambda(q, \frac{1}{q})} \cdot \| g \|_{\Lambda(1, 0)}. \]

(3.1)
Proof. For $2 \leq q < \infty$, Theorem 5.5 in [5] shows that
\[
\sup_{0<r<1} (1-r)^{1-\frac{1}{q}} M_q(r,f) \approx \sup_{0<r<1} (1-r)^{2-\frac{1}{q}} M_q(r,f').
\]
This together with the following estimate
\[
M_q(r, (f*g)''(x)) \leq CM_q(r,f'')M_1(\sqrt{r},g') \tag{3.2}
\]
gives
\[
\sup_{0<r<1} (1-r)^{1-\frac{1}{q}} M_q(r, (f*g)') \leq C \sup_{0<r<1} (1-r)^{2-\frac{1}{q}} M_q(r, f'g').
\]
It follows from Theorem 2.1 and Lemma 3.1 that
\[
\|f*g\|_{Q_K} \leq A_K (1) \sup_{0<r<1} (1-r)^{1-\frac{1}{q}} M_q(r, (f*g)') \leq C \|f\|_{Q_K} \cdot \|g\|_{A(\frac{2q}{q},0)}.
\]

Theorem 3.2. Let $2 < q \leq \infty$ and let $K$ satisfy (2.3), that is,
\[
B_q = \int_{\frac{1}{\sqrt{2-q}}}^{+\infty} \frac{q_K(s)}{s^{2-\frac{1}{q}}} \, ds < \infty.
\] (3.3)
If $f \in Q_K$ and $g \in A(\frac{2q}{2+q},0)$, then $f*g \in Q_K$ and
\[
\|f*g\|_{Q_K} \leq C \|f\|_{Q_K} \cdot \|g\|_{A(\frac{2q}{2+q},0)}.
\] (3.4)
Before embarking the proof let us state a result which will be used in our proof; see [9].

Lemma 3.2. Let $1 \leq k < \infty$, $\mu > 0$, $\delta > 0$, and let $h: (0,1) \rightarrow [0,\infty)$ be measurable. Then
\[
\int_0^1 (1-r)^{k\mu-1} \left( \int_0^r (r-t)^{\delta-1} h(t) \, dt \right)^k \, dr \leq C \int_0^1 (1-r)^{k\mu+k\delta-1} h(r) \, dr.
\]

Now let us go to the proof of Theorem 3.2. We first show that
\[
\|f\|_{Q_K}^2 \leq C \left( \|f'(0)\|^2 + \int_0^1 (1-r)^2 M_q^2(r,f'')K(1-r) \, dr \right).
\] (3.5)
Since
\[
|f'(re^{i\theta})| \leq |f'(0)| + \int_0^r |f''(se^{i\theta})| \, ds,
\]
we obtain
\[
M_q^2(r,f') \leq C \left( |f'(0)|^2 + \left( \int_0^r M_q(s,f'') \, ds \right)^2 \right).
\]
By Theorem 2.2 we have
\[ \|f\|_{Q_K}^2 \leq B_q \int_0^1 K(1 - r) M_q^2(r, f') \, dr. \]

By Lemma 3.2 and the boundedness of \(K\), the last integral is dominated by
\[ \left| f'(0) \right|^2 + \int_0^1 K(1 - r) \left( \int_0^r M_q(s, f'') \, ds \right)^2 \, dr \leq \left| f'(0) \right|^2 + \int_0^1 (1 - r)^2 M_q^2(r, f'') K(1 - r) \, dr. \]

Here we used the monotonicity of \(K\) and then (3.5) holds. Hence by Lemma 3.1 we have
\[ \|f * g\|_{Q_K}^2 \leq C \left( \left| (f * g)'(0) \right|^2 + \int_0^1 (1 - r)^2 K(1 - r) M_q^2(r, f' * g') \, dr \right) \]
\[ \leq C \left( \left| (f * g)'(0) \right|^2 + \int_0^1 (1 - r) M_q^2(r, f'') M_{2q/(2q + 1)} \, dr \right). \]

For \(f \in Q_K\) and \(g \in \Lambda(2q/(2 + q), 0)\), it follows that
\[ \int_0^1 K(1 - r) (1 - r)^2 M_q^2(\sqrt{r}, f') M_q^2(\sqrt{r}, g') \, dr \leq C \|g\|_{\Lambda(2q/(2 + q), 0)}^2 \int_0^1 K(1 - r) M_q^2(r, f') \, dr \]
\[ \leq C \|f\|_{Q_K}^2 \cdot \|g\|_{\Lambda(2q/(2 + q), 0)}^2. \]

On the other hand,
\[ \int_0^1 K(1 - r) \left| (f * g)'(0) \right|^2 \, dr \leq \|f\|_{Q_K}^2 \cdot \|g\|_{\Lambda(2q/(2 + q), 0)}^2. \]

Consequently,
\[ \|f * g\|_{Q_K} \leq C \|f\|_{Q_K} \cdot \|g\|_{\Lambda(2q/(2 + q), 0)}. \]

Since (3.3) becomes (B) for \(q = \infty\) and the Bloch space \(B\) contained in \(\Lambda(t, 0)\) for all \(1 < t < \infty\), we obtain from Theorem 3.2 the following result.

**Theorem 3.3.** Let \(K\) satisfy (B). If \(f \in Q_K\) and \(g \in B\), then \(f * g \in Q_K\) and
\[ \|f * g\|_{Q_K} \leq C \|f\|_{Q_K} \cdot \|g\|_{B}. \]
Moreover, we have

**Corollary 3.4.** Let $2 < q \leq \infty$ and let $K$ satisfy (3.3). If $f \in Q_K$ and $g \in H^{\frac{2q}{2+q}}$, then $f \ast g \in Q_K$ and

$$\|f \ast g\|_{Q_K} \leq C \|f\|_{Q_K} \cdot \|g\|_{H^{\frac{2q}{2+q}}}.$$

**Proof.** Let $p = \frac{2q}{2+q}$. By the Cauchy formula we have

$$g'(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{g(\rho e^{it})e^{it} dt}{(\rho e^{it} - z)^2},$$

where $\rho = \frac{1+r^2}{2}$. By Minkowski’s inequality, we see that

$$(1-r^2)M_p(r,g') \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)M_p(\rho, g) dt}{\rho^2 - 2\rho r \cos t + r^2} = \frac{1-r^2}{\rho^2 - r^2} M_p(\rho, g) \leq C \|g\|_{H^p}.$$

It means that

$$\|g\|_{\Lambda(q,0)} \leq C \|g\|_{H^p}.$$ 

By Theorem 3.2, the desired result is proved. □

Note that $1 < \frac{2q}{2+q} \leq 2$ when $2 < q \leq \infty$. The following result shows that above theorem in fact holds for all $p > 2$.

**Theorem 3.5.** Let $K$ satisfy (B) and $p \geq 2$. If $f \in Q_K$ and $g \in H^p$, then $f \ast g \in Q_K$ and

$$\|f \ast g\|_{Q_K} \leq C \|f\|_{Q_K} \cdot \|g\|_{H^p}.$$ 

**Proof.** Similar to the proof of Theorem 3.2 we have

$$\|f \ast g\|_{Q_K} \leq C \left( \left| (f \ast g)'(0) \right|^2 + \int_0^1 K(1-r)(1-r)^2 M^2_q(r, f' \ast g') dr \right).$$

We choose $q = \infty$ in Lemma 3.1,

$$M_\infty(r, f \ast g) \leq M_s(\sqrt{r}, f)M_p(\sqrt{r}, g),$$

where $1 < s \leq 2$ and $p \geq 2$. By the proof of Corollary 3.4 we obtain

$$(1-r^2)M_p(r,g') \leq C \|g\|_{H^p}.$$ 

It gives that

$$\|f \ast g\|^2_{Q_K} \leq C \|g\|^2_{H^p} \int_0^1 K(1-r)M^2_s(r, f') dr \leq C \|g\|^2_{H^p} \int_0^1 K(1-r)M^2_2(r, f') dr \leq C \|f\|^2_{Q_K} \cdot \|g\|^2_{H^p}.$$ □

For the Hardy space $H^1$ by using the estimate

$$(1-r^2)M^1_1(r,g') \leq C \|g\|_{H^1},$$

we obtain $H^1 \subset \Lambda(1,0)$, so that

**Corollary 3.6.** Let $2 \leq q < \infty$ and let $K$ satisfy condition (2.1). If $f \in \Lambda(q, \frac{1}{q})$ and $g \in H^1$, then $f \ast g \in Q_K$ and

$$\|f \ast g\|_{Q_K} \leq C \|f\|_{\Lambda(q, \frac{1}{q})} \cdot \|g\|_{H^1}.$$
References