Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Discrete Applied Mathematics 156 (2008) 3494–3497
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Note

The complexity of a minimum reload cost diameter problem

Giulia Galbiati

Università degli Studi di Pavia, Facoltà di Scienze M.F.N.– Dipartimento di Informatica e Sistemistica, Via Ferrata 1, I-27100 Pavia, Italy

Received 19 November 2007; received in revised form 15 February 2008; accepted 16 February 2008
Available online 11 April 2008

Abstract

We consider the minimum diameter spanning tree problem under the reload cost model which has been introduced by Wirth and Steffan [H.-C. Wirth, J. Steffan, Reload cost problems: Minimum diameter spanning tree, Discrete Appl. Math. 113 (2001) 73–85]. In this model an undirected edge-coloured graph G is given, together with a nonnegative symmetrical integer matrix R specifying the costs of changing from a colour to another one. The reload cost of a path in G arises at its internal nodes, when passing from the colour of one incident edge to the colour of the other. We prove that, unless P $=$ NP, the problem of finding a spanning tree of G having a minimum diameter with respect to reload costs, when restricted to graphs with maximum degree 4, cannot be approximated within any constant $\alpha < 2$ if the reload costs are unrestricted, and cannot be approximated within any constant $\beta < 5/3$ if the reload costs satisfy the triangle inequality. This solves a problem left open by Wirth and Steffan [H.-C. Wirth, J. Steffan, Reload cost problems: minimum diameter spanning tree, Discrete Appl. Math. 113 (2001) 73–85].

Keywords: Reload cost model; Combinatorial optimization; Minimum diameter spanning tree; Approximation algorithm

1. The problem

In this paper we consider a graph-theoretical optimization problem, introduced in [1], that arises in the context of network design.

Consider a scenario in which a transportation network is divided in subnetworks, such that transportation costs are negligible within each subnetwork, but are significant when moving from one subnetwork to another. This scenario fits networks which use different means of transportation, like overlay networks, peer-to-peer telecommunication networks, and in general complex telecommunication networks, but also fits energy distribution networks, cargo transportation networks, etc. Obviously, the costs that arise when moving from one subnetwork to another depend on the specific network. For instance, in overlay networks the costs may be related to the change of technology, in a cargo transportation network to unloading and reloading goods at different junctions, in large communication networks to data conversion at interchange points, etc. In this scenario the costs at the interchange points between the subnetworks usually dominate the costs within individual subnetworks.

The reload cost model, introduced in [1], models this scenario. In it, we are given an undirected edge coloured graph G, with edges of the same colour constituting a subnetwork, and a "reload cost" matrix R, representing the costs of moving from one subnetwork to another, i.e. of switching between the corresponding colours. The reload

E-mail address: giulia.galbiati@unipv.it.

0166-218X/$ - see front matter
doi:10.1016/j.dam.2008.02.013

cost of a path in G is the sum of the reload costs that arise at its internal nodes from passing from the colour of one incident edge to that of the other. In practical applications it is often assumed that R satisfies a triangle inequality, which enforces the requirement that one cannot save costs by performing more than one reload job at a single vertex.

Despite the usefulness of this setup, to our knowledge only few papers have appeared in the literature dealing with problems formulated in this model [1,2].

In this paper we consider the minimum diameter spanning tree problem under the reload cost model (MIN-DIAM), i.e. the problem of finding a spanning tree of G having a minimum diameter with respect to the reload cost. This problem has been extensively studied in [1], where the authors prove that MIN-DIAM is not approximable at all, even when restricted to graphs of maximum degree 5. If instead the reload costs satisfy the triangle inequality they show that the problem (Δ-MIN-DIAM) is not approximable within a logarithmic bound on general graphs, and not within 3 on graphs with maximum degree 5; they solve the problem exactly on graphs with maximum degree 3. In their seminal paper the authors leave open the natural question of the complexity of the problem on graphs with maximum degree 4.

Here we prove that, on graphs having maximum degree 4, MIN-DIAM cannot be approximated within any constant $\alpha < 2$ and Δ-MIN-DIAM cannot be approximated within any constant $\alpha < 5/3$, unless P = NP. Hence our results are a step towards a more complete understanding of the complexity status of the problem.

2. Definitions and preliminaries

Let us formally introduce the reload cost model and the problems that we study.

In the model an undirected graph $G = (V, E)$ is given, together with an edge colouring taking values in a finite set L of colours and a symmetrical, nonnegative integer matrix $R = \{r_{l\,l'}\}_{l,l' \in L}$, with all diagonal elements equal to zero, called "reload cost" matrix; the integer $r_{l\,l'}$ represents the reload cost of passing from colour l to colour l'. We say that the reload cost matrix R satisfies the triangle inequality if, for any three edges e, e', e'' incident in a node of G, coloured, respectively, with colours l, l', l'', we have that $r_{l\,l'} \leq r_{l\,l''} + r_{l''\,l'}$.

If P is a path in G consisting of k consecutive edges $e_1, \ldots, e_k$, respectively of colours $l_1, \ldots, l_k$, then the reload cost of P is $r(P) = \sum_{j=1}^{k-1} r_{l_j l_{j+1}}$.

If T is a spanning tree of G, then for any two vertices x, y of V, we denote by $r_T(x, y)$ the reload cost of the unique path in T between x and y. The diameter of T under the reload cost model is $\mathrm{diam}^r(T) = \max_{x,y \in V} r_T(x, y)$.

Let us now define the problems we study.

Problem MIN-DIAM:

An *instance* I of MIN-DIAM consists of a graph G, a finite set L and a matrix R as above. The *goal* is to find a spanning tree T^* of G such that $\mathrm{diam}^r(T^*)$ is minimized among all spanning trees of G. As usual, we set $\mathrm{opt}(I) = \mathrm{diam}^r(T^*)$.

Problem Δ-MIN-DIAM is the name for the problem when matrix R satisfies the triangle inequality.

We will prove our results on the hardness of approximating MIN-DIAM and Δ-MIN-DIAM by exhibiting a *gap-introducing* reduction, inspired by [1], from an NP-complete problem to them. For a short introduction to the central notion of *gap-introducing* reduction see, for instance, [3]. The NP-complete problem that we will use is 3-SAT-3, a restricted form of the SAT problem, already presented in [4], that we formulate as follows:

Problem 3-SAT-3:

Instance: a set $X = \{x_1, \ldots, x_n\}$ of boolean variables and a collection $C = \{C_1, \ldots, C_m\}$ of clauses over X, with $|C_h| \leq 3$, for each $C_h \in C$, and with at most three clauses in C that contain either x or $\overline{x}$, for each $x \in X$.

Question: Does there exist a satisfying truth assignment for C?

3. The result

In order to prove our results we will present two polynomial time reductions from 3-SAT-3 to MIN-DIAM. We start presenting the first reduction.

3.1. The main reduction

Given an instance I' of 3-SAT-3 we construct a corresponding instance I of MIN-DIAM. In order to help the reader in understanding the reduction, in Fig. 1 we illustrate it on a specific instance of 3-SAT-3.

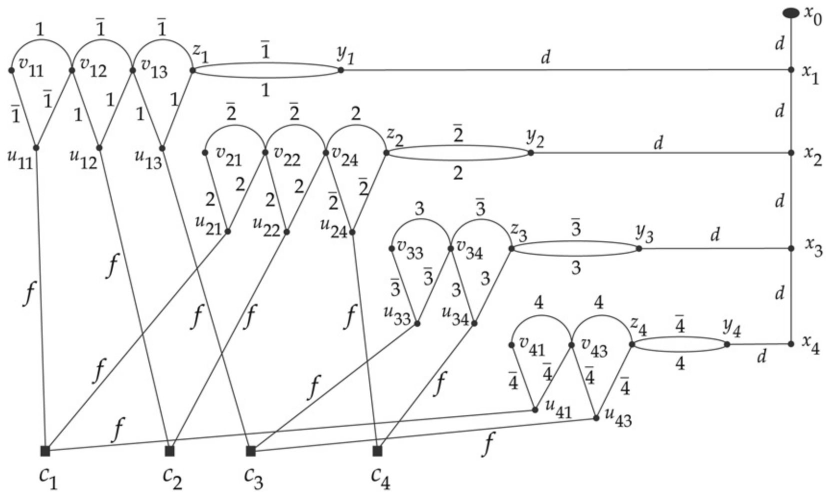

Fig. 1. Graph G corresponding to an instance I of 3-SAT-3 having clauses $c_1 = \overline{x}_1 \vee x_2 \vee \overline{x}_4$, $c_2 = x_1 \vee x_2$, $c_3 = x_1 \vee \overline{x}_3 \vee \overline{x}_4$ and $c_4 = \overline{x}_2 \vee x_3$.

Graph $G = (V, E)$ has set V consisting of a root vertex, called x_0, three nodes x_i, y_i, z_i, for each $i \in \{1, \ldots, n\}$, and m "clause" vertices $c_1, \ldots, c_m$, each one corresponding to a clause in C. Moreover, for each $i \in \{1, \ldots, n\}$, if x_i or $\overline{x_i}$ appears in clause C_h, then set V includes two vertices, called u_{ih}, v_{ih}, for a total of at most six vertices, since each variable appears, possibly negated, in at most three clauses. The set of colours for the edges is the set $L = \{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}, f, d\}$. The set E of edges includes, for each $i = 1, \ldots, n$, the edges $\{x_{i-1}, x_i\}$ and the edges $\{x_i, y_i\}$ of colour d, two parallel edges $\{y_i, z_i\}$, one of colour i, the other of colour $\overline{i}$; additionally, for each $i = 1, \ldots, n$, it includes a chain of "upper" consecutive edges connecting z_i to the $k \leq 3$ vertices v_{ih} and a chain of "lower" edges connecting z_i to the k consecutive pairs of vertices u_{ih}, v_{ih}. Finally, for each $h = 1, \ldots, m$, if clause C_h contains variable x_i or the negated variable $\overline{x}_i$, then E includes the edge $\{u_{ih}, c_h\}$, of colour f. The colours of the "upper" and "lower" edges are set as follows. For any vertex u_{ih}, the two "lower" edges incident to it receive the same colour, which is i if clause C_h contains variable x_i, and is $\overline{i}$ otherwise; additionally, the "corresponding" upper edge receives colour $\overline{i}$ in the first case, colour i in the second case. To complete the reduction we need to define the reload cost matrix. The cost of changing from colour d to any other colour is one, all remaining costs of changing colour are set equal to a fixed value $K > 1$.

Lemma 1. *If there exists a spanning tree T of graph G having $r_T(x_0, v) < 1 + 2K$ for each $v \in V$, then instance I' of* 3-SAT-3 *is satisfiable.*

Proof. The first thing we observe is that T must contain all edges $\{x_{i-1}, x_i\}, i = 1, \ldots, n$. Otherwise, if k is the first integer such that $\{x_{k-1}, x_k\}$ is not in T then $r_T(x_0, x_k) \geq 2 + 2K$. The second observation is that any path in T from root x_0 to any clause node c_h can continue but cannot reach another clause node $c_{h'}$, for otherwise $r_T(x_0, c_{h'}) \geq 1 + 3K$. Moreover, for each $h = 1, \ldots, m$, for some $i \in \{1, \ldots, n\}$ only one of the two colours i, $\overline{i}$ can occur in the path in T from x_0 to c_h and hence $r_T(x_0, c_h) = 1 + K$. This implies that the paths in T from x_0 to all c_h identify an assignment for C that satisfies all clauses. □

3.2. The main result

Now we prove our main result.

Theorem 2. *On graphs of maximum degree* 4, MIN-DIAM *is not approximable within any factor $\alpha < 2$ and* Δ-MIN-DIAM *is not approximable within any factor $\beta < 5/3$, unless* P = NP.

Proof. Consider the following new reduction from 3-SAT-3. Given an instance I' of 3-SAT-3 we construct an instance I of MIN-DIAM as follows. Let G^1 and G^2 be two identical copies of the graph associated to I' by the first reduction, and let G be the graph obtained from their union by identifying the root nodes. Obviously G has

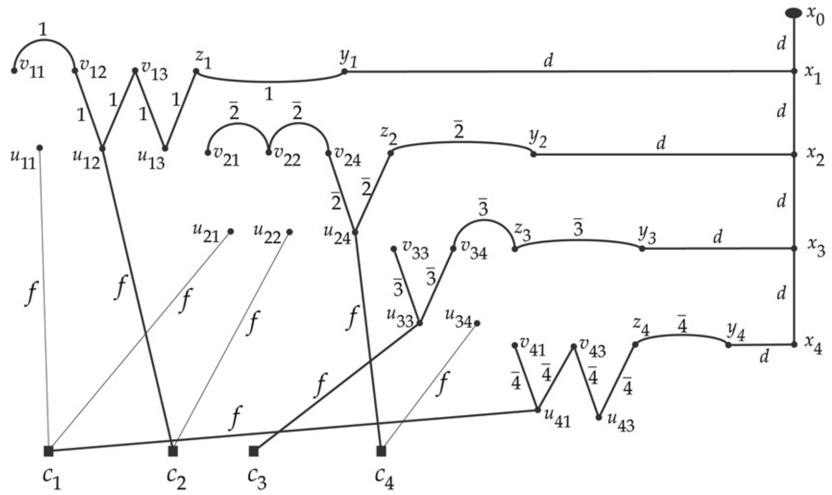

Fig. 2. A tree spanning the graph in Fig. 1 constructed from the assignment $x_1 = 1, x_2 = x_3 = x_4 = 0$ of satisfiable instance I.

maximum degree node equal to four and its reload costs satisfy the triangle inequality if $K \leq 2$. We now show that this new reduction is a gap-introducing reduction such that

i. I' is satisfiable $\Longrightarrow \mathrm{opt}(I) \leq 2K + 2$

ii. I' is not satisfiable $\Longrightarrow \mathrm{opt}(I) \geq 4K + 2$.

To prove i. we observe that, if I' is satisfiable, then a satisfying assignment to the variables in X can be used in a straightforward way to construct a spanning tree T of G with the property that every clause is connected to the root by a path of reload cost equal to $K + 1$; these paths from the root to the clauses touch all x_i, y_i, and z_i vertices, can be extended at no cost to reach all v_{ih} vertices, and touch as many u_{ih} vertices as there are clauses; the remaining u_{ih} vertices can be connected in the tree to the unique c_h vertex without increasing the diameter (See Fig. 2. for an example of a tree spanning the vertices of G^1, with the paths from x_0 to the clauses and their extensions drawn in heavy lines; a similar tree can be drawn for G^2). It follows that $\mathrm{diam}^r(T) \leq 2K + 2$ and hence $\mathrm{opt}(I) \leq 2K + 2$.

To prove ii. suppose that $\mathrm{opt}(I) < 4K + 2$. Then there exists a spanning tree T of G having $\mathrm{diam}^r(T) < 4K + 2$. It follows that every vertex v in G^1 or every vertex v in G^2 must satisfy $r_T(x_0, v) < 2K + 1$, for otherwise the existence of a vertex v_1 in G^1 having $r_T(x_0, v_1) \geq 2K + 1$ and a vertex v_2 in G^2 having $r_T(x_0, v_2) \geq 2K + 1$ would imply $r_T(v_1, v_2) \geq 4K + 2$ and $\mathrm{diam}^r(T)$ could not be less than $4K + 2$. Suppose then, for instance, that every vertex v in G^1 satisfies $r_T(x_0, v) < 2K + 1$. Lemma 1 would imply that I' is satisfiable, a contradiction.

Properties i. and ii. together imply that this reduction is a gap-introducing reduction. It follows that MIN-DIAM is not approximable within $\frac{4K+2-\varepsilon'}{2k+2}$, for any $\varepsilon' > 0$, unless P = NP. Hence we may conclude that, unless P = NP, for any given $\varepsilon > 0$, MIN-DIAM is not approximable within ratio $2 - \varepsilon$, since $2 - \varepsilon \leq \frac{4K+2-\varepsilon'}{2k+2}$ for a sufficiently small ε', positive if K is chosen sufficiently large, i.e., if $1 + K > 1/\varepsilon$.

Using $K = 2$ we may conclude that Δ-MIN-DIAM is not approximable within $\frac{10-\varepsilon'}{6}$, for any $\varepsilon' > 0$, hence not within any factor $\beta < 5/3$, unless P = NP. □

References

[1] H.-C. Wirth, J. Steffan, Reload cost problems: minimum diameter spanning tree, Discrete Appl. Math. 113 (2001) 73–85.

[2] S. Raghavan, I. Gamvros, L. Gouveia, Reload cost trees and network design, in: Proc. International Network Optimization Conference, INOC 2007, Spa, 2007.

[3] V.V. Vazirani, Approximation Algorithms, Springer-Verlag, Berlin, 2001.

[4] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman and Co, San Francisco, California, 1979.