Certain Bipartite Hypergeometric Transformations and Their Applications

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We have obtained two new transformation formulae with the help of Bailey’s transform, one of which contains series involving two independent bases. It has also been shown that some very interesting new multiple series identities of the Rogers–Ramanujan type can be established.

1. INTRODUCTION

In 1947, Bailey [3] gave the following remarkably simple transformation: If

$$\beta_n = \sum_{r=0}^{n} \alpha_r v_{n-r}^r$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r v_{r-n}^r$$
then, subject to convergence conditions,
\[ \sum_{n=0}^{\infty} \alpha_n z^n = \sum_{n=0}^{\infty} \beta_n \delta_n. \]  
(1)

With the help of this transformation, he developed a technique of obtaining diverse variety of transformations of basic hypergeometric series and identities of the Rogers–Ramanujan type. In a subsequent paper [4], he elucidated his technique by considering five special choices of the sequences \( \{a_n\} \) for which the value of \( \{b_n\} \) could be found in a closed form. This technique was later exploited by various mathematicians, i.e., Slater [11, 12], Andrews [1, 2], Bressoud [6], and Verma and Jain [13, 14] to obtain identities of the Rogers–Ramanujan type with different moduli.

Recently, we [9, 10] have obtained certain very general bibasic hypergeometric transformation formulae with two independent bases and also have discussed a number of special cases involving multiple series Rogers–Ramanujan type of identities. In this paper, we shall derive two new transformation formulae, one of which is bibasic, with the help of Bailey’s transform (1). A number of interesting special and limiting cases of these transformation formulae have also been obtained.

2. NOTATION

We use the following notation throughout this paper. For \(|q| < 1\), let

\[
(a)_n = (a; q)_n = \begin{cases} 
1, & n = 0 \\
(1-a) \cdots (1-aq^{n-1}), & n \geq 1,
\end{cases}
\]

\[(a; q)_n = (a_1, \ldots, a_r; q)_n = (a_1; q)_n \cdots (a_r; q)_n,
\]

\[(a; q)_\infty = \prod_{n=1}^{\infty} [a, q] = \prod_{n=1}^{\infty} (1 - a q^{n-1});
\]

\[\times \prod \left[ a_1; q \right]_b = \prod [a; q]/\prod [b; q].\]

A generalized multibasic hypergeometric series [7, Eqs. (3.9.1) and (3.9.2)]
is defined as

\[ R \phi_S^{(m+1)} \left[ \begin{array}{c} (a_r);(c_{1,r_1});\cdots;(c_{m,r_m}) \\ (b_s);(d_{1,s_1});\cdots;(d_{m,s_m}) \\ q, q_1, \ldots, q_m; z \end{array} \right] \]

\[ \sum_{n=0}^{\infty} \frac{((a_r);q)_n}{(q,(b_s);q)_n} z^n \left\{ (-1)^n q_n^{n-1+r} \right\}^{1+s-r} \times \prod_{j=1}^{m} \frac{((c_{j,r});q_j)_n}{((d_{j,s});q_j)_n} \left\{ (-1)^n q_j^{n-1+r} \right\}^{s_j-r_j}, \]

where \( R = r + r_1 + \cdots + r_m, S = s + s_1 + \cdots + s_m, \) and \((q)_n = n!(n-1)!/2.\)

For \( m, n, k \) positive integers, let

\[ M_i = m_i + m_{i+1} + \cdots + m_{k-1} (1 \leq i < k), \]
\[ N_{k-1} = n + M_1 + \cdots + M_{k-1}, \]
\[ N_{k-1}^2 = n^2 + M_1^2 + \cdots + M_{k-1}^2, \]
\[ g(N,t) = (q^2; q^2)_{2M_1+t-1} / (q^2; q^2)_{n+M_1+t}, \]
\[ h(N,t) = 2N_{k-2}^2 + 2nM_1 - \frac{M_1}{2} (3M_1 - 7) \]
\[ + \frac{t}{2}(5t + 3) + (3M_1 + 2n)t. \]

Further, let

\[ U_{(m_{k-1})}(a, b, c, (a_{2k-3})^2; q^2) \]
\[ = \frac{(a^2q^2/bc; q^2)_{m_{k-1}}(a^2q^2/a_1a_2; q^2)_{m_{k-2}} \cdots (a^2q^2;a_{2k-5}a_{2k-4}; q^2)_{m_1}}{(a^2q^2/b, a^2q^2/c; q^2)_{m_{k-1}}(a^2q^2/a_1, a^2q^2/a_2; q^2)_{m_{k-2}} \cdots} \times \frac{(a_1, a_2; q^2)_{M_{k-1}} \cdots (a_{2k-5}, a_{2k-4}; q^2)_{M_1}(a_{2k-3}; q^2)_{M_1}}{(a^2q^2/a_{2k-5}, a^2q^2/a_{2k-4}, a^2q^2/a_{2k-3}; q^2)_{M_1} \times (a_1a_2)^{-M_{k-1}} \cdots (a_{2k-5}a_{2k-4})^{-M_1}(a_{2k-3})^{-M_1}} \times (a_{2k-5}a_{2k-4})^{-M_1}(a_{2k-3})^{-M_1} \]
\[ V_n(\lambda, (b_{2m}); q_1) = \frac{(b_1, \ldots, b_{2m}; q_1)_n q_1^{m-1}}{(\lambda q_1/b_1, \ldots, \lambda q_1/b_{2m}; q_1)_n (b_1 \cdots b_{2m})^n}, \quad |q_1| < 1, \]
and

\[ B_n(c, (a_{2k-3}); q^2) = \frac{(c, a_1, \ldots, a_{2k-3}; q^2)_n (ca_1 \ldots a_{2k-3})^{-n}}{(a^2q^2/c, a^2q^2/a_1, \ldots, a^2q^2/a_{2k-3}; q^2)_n}. \]

We shall first prove the following transformation formula connecting two series, one of which is "split-poised":

\[
\sum_{n=0}^{\infty} \frac{(a^2/b^3, x^2/b^3, q^3)_n (b^6q^3)^n}{(q^3, a^3q^3/x^3; q^3)_n (x^3)} \times \frac{6\phi_5}{(aq, aq^2, aq^3, x^3, a^3q^{3n}/b^3, q^{-3n}; q^3, q^3)} \\
= \prod \left[ \frac{a^3b^3q^3, b^3q^3; q^3}{a^3q^3, b^6q^3} \right] \\
\times \sum_{n=0}^{\infty} \frac{(a)_n(1 - aq^{2n})(x^3; q^3)_n (a^2/b^3; q^3)_{2n}}{(q)_n(1 - a)(a^2q^3/x^3; q^3)_n (a^2b^2q^3; q^3)_{2n}} \left( \frac{ab^6q^4}{x^3} \right)^n. \tag{2}
\]

**Proof.** In the Bailey's transform (1), we choose \( u_n, v_n, \delta_n, \) and \( \alpha_n \) as

\[
\begin{align*}
\alpha_n &= (a)_n(1 - aq^{2n})(x^3; q^3)_n/(q)_n(1 - a)(a^2q^3/x^3; q^3)_n (aq/x^3)^n.
\end{align*}
\]

Then

\[
\gamma_n = \sum_{i=0}^{\infty} \delta_{n+i} u_{2n+i} v_{2n+i} \\
= \frac{(a^2/b^3; q^3)_{2n}}{(a^3q^3; q^3)_{2n}} \frac{z^n}{a^3q^{3+6n}} \phi_1 \left[ \frac{z^3, a^3q^6n/b^3; q^3; \frac{a^3q^3}{z}/b^3} \right] \\
= \sum_{i=0}^{\infty} \delta_{n+i} u_{2n+i} v_{2n+i} \\
= \prod \left[ \frac{a^3/b^3, x^3/b^3; q^3}{a^3q^3, x^3; q^3} \right] \frac{z^n}{a^3q^{3+6n}} \phi_1 \left[ \frac{z^3, a^3q^6n/b^3; q^3; \frac{a^3q^3}{z}/b^3} \right].
\]
by Heine’s transformation [7, Eq. (1.4.1)]. Also, we have

$$
\beta_n = \sum_{i=0}^{n} \frac{(a, (1 - aq^{2i})(b^{-3}; q^3)_n)}{(q, (1 - a)(q^3; q^3)_n, (a^3/b^3; q^3)_n, (x^3; q^3)_n, (aq^{2i}; q^3)_n, (aq^{2i+1}; q^3)_n)} \left( aq^{2i}, (aq^{2i+1}; q^3)_n \right)
$$

$$
= \frac{(b^{-3}, a^3/b^3; q^3)_n}{(q^3, a^3q^3/x^3; q^3)_n} \left[ a, q\sqrt{a}, -q\sqrt{a}, x, \omega x, \omega^2 x, aq^n/b, \sqrt{a}, -\sqrt{a}, aq/x, aq\omega^2/x, aq\omega/x, bq^{1-3n}, \frac{aq\omega^n/b, a\omega^2 q^n/b, \omega q^{-n}, \omega^2 q^{-n}, q^{-n}; q}{\frac{ab^3q^4}{x^3}} \right]
$$

$$
= \frac{(a^3/b^3, x^3/b^3; q^3)_n}{(q^3, a^3q^3/x^3; q^3)_n, x^{3n}} \times \phi_2 \left[ \frac{aq, aq^2, aq^3, x^3, a^3q^{3n}/b^3, q^{-3n}; q^3}{(aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^2, -a^{3/2}q^2, x^3/b^3} \right]
$$

by the transformation formula [14, Eq. (1.5)] due to Verma and Jain. Thus, on using Bailey’s transform (1), we have

$$
\sum_{n=0}^{\infty} \frac{(a^3/b^3, x^3/b^3; q^3)_n}{(q^3, a^3q^3/x^3; q^3)_n} \left( \frac{z}{x^3} \right)^n \times \phi_2 \left[ \frac{aq, aq^2, aq^3, x^3, a^3q^{3n}/b^3, q^{-3n}; q^3}{(aq)^{3/2}, -(aq)^{3/2}, a^{3/2}q^2, -a^{3/2}q^2, x^3/b^3} \right]
$$

$$
= \prod \left[ a^3/b^3, z/b^3; q^3 \right] \sum_{n=0}^{\infty} \frac{(a, (1 - aq^{2n})(x^3; q^3)_n)}{(q, (1 - a)(a^3q^{3n}/x^3; q^3)_n, (x^3; q^3)_n, (aq^{2n}; q^3)_n)} \left( \frac{aq}{x^3} \right)^n \times \phi_1 \left[ \frac{b^3q^3, z; q^3}{z/b^3}, \frac{a^3q^{6n}}{b^3} \right]
$$

(3)

Now, the argument of the inner series on the right hand side of (3) depends on \( n \), so it is summable only when \( z = 0 \) or \( z = b^6q^3 \). For \( z = 0 \), (3) reduces to the \( q \)-binomial theorem [7, Eq. (1.3.2)]. For \( z = b^6q^3 \), on using the \( q \)-binomial theorem, we easily get the transformation (2) after some simplification.
4. PARTICULAR CASES

We shall now discuss a few interesting special cases of (2).

(i) If we take $x = 1$ in (2), we easily get a particular case of a $q$-analogue of Gauss's theorem [7, Eq. (1.5.1)], while, for $b = x$, we obtain the following summation formula for a “split-poised” series:

\[
\sum_{n=0}^{\infty} \frac{(a)_n(1 - aq^{2n})(x^3; q^3)_n(a^3/x^3; q^3)_n}{(q)_n(1 - a)(a^3 q^3/x^3; q^3)_n(a^3 x^3 q^3; q^3)_n} (ax^4 q^3)^n = \prod \left[ \frac{a^3 q^3, x^6 q^3}{a^3 x^3, x^3 q^3} \right].
\]  

(ii) However, if we make $b, x \to \infty$ in (2), we have the transformation

\[
\sum_{s,t=0}^{\infty} \frac{(aq; q)_s, a^{6s+3t}, a^{6s^2+6st+3t^2}}{(q^3; q^3), (q^3; q^3), (a^3 q^3, a^3 q^4; q^6)} \frac{(aq^{2n})}{(q)_n(1 - a)} (a^7)^n q^{(15/2)n^2 - n/2}.
\]  

Taking $a = 1$ and $a = q$, respectively, in (5) and then using Jacobi's triple product identity [7, Eq. (1.6.1)],

\[
\sum_{n=-\infty}^{\infty} (-1)^n z^n q^n = \prod_{n=1}^{\infty} (1 - zq^{2n-1})(1 - q^{2n-1}/z)(1 - q^{2n}),
\]  

we get on simplification the identities

\[
\sum_{s,t=0}^{\infty} \frac{(q; q)_s, a^{6s^2+6st+3t^2}}{(q^3; q^3), (-q^2; q^3), (q^3; q^3)} = \frac{1}{(\omega q, \omega^2 q; q)_n} \prod_{n=1}^{\infty} (1 - q^n)^{-1}, \quad n \not\equiv 0, \pm 7 \pmod{15}.
\]
and
\[
\sum_{s,t=0}^{\infty} \frac{(q,q^2;q^3)_s(q^{6s^2+6st+3r^2+6s+3t})}{(q^6,q^7,q^8,q^9,s^3,q^3)_t} = \frac{1}{(\omega q^2,\omega^2 q^2;q)_{\infty}} \prod_{n=1}^{\infty} (1-q^n)^{-1}, \quad n \neq 0, \pm 1 \text{ (mod 15).} \quad (8)
\]

Identities (7) and (8) are believed to be new.

(iii) Lastly, let \( b \to \infty \) in (2) first, and then put \( x = -(aq)^{1/2} \). We obtain the transformation
\[
\sum_{s,t=0}^{\infty} \frac{(aq;q)_{3s}a^{(9/2)s^2+3r}q^{(9/2)s^2+3st+3r^2}}{(q^3;q^3)_s(a^3q^3,a^3q^4,q^6),(q^3,-a^{3/2}q^{3/2}+3s,\omega q^3)} = \frac{1}{(a^3q^3;q^3)_s} \sum_{n=0}^{\infty} \frac{(a)_n(1-aq^{2n})}{(q)_n(1-a)} (-1)^n a^{11n/2}q^{6n^2-n/2}. \quad (9)
\]

If we take \( a = 1 \) and \( a = q \) in (9), respectively, and then use the result (6), we get the identities
\[
(-q;q)_s = \sum_{s,t=0}^{\infty} \frac{(q^2;q^6)_s(q^{9s^2+6st+6r^2})}{(q^6;q^12)_s(-q^4,q^6),(q^6,-q^{3+6r},q^6)},
\]
\[
= \frac{1}{(\omega q^2,\omega^2 q^2;q^2)} \prod_{n=1}^{\infty} (1-q^n)^{-1}, \quad n \neq 0, \pm 11 \text{ (mod 24)}, \quad (10)
\]

and
\[
\sum_{s,t=0}^{\infty} \frac{(q,q^2;q^3)_s(q^{(9/2)s^2+3st+3r^2+(9/2)s+3t})}{(q^6,q^7,q^8,q^9),(q^3,-q^{3+3s},q^3)} = \frac{1}{(\omega q^2,\omega^2 q^2;q)} \prod_{n=1}^{\infty} (1-q^n)^{-1}, \quad n \neq 0, \pm 1 \text{ (mod 12)}. \quad (11)
\]

We shall now prove the following general bibasic transformation for-
mula from which identities of the Rogers-Ramanujan type can be deduced:

**Theorem.** Let m, k be positive integers, and r a real positive number, then

\[
\sum_{n,m_1,\ldots,m_{k-1}\geq 0} (-1)^{M_1} U_{(m_{k-1})} \left( \begin{array}{c}
\left( q^2; q^2 \right)_{m_1} \\
\vdots \\
\left( q^2; q^2 \right)_{m_{k-1}} 
\end{array} \right) V_M \left( \begin{array}{c}
(1 \lambda q_1^{2M_1}) \\
(1 \lambda q_1^{2M_1}) \\
\vdots \\
(1 \lambda q_1^{2M_1}) 
\end{array} \right) M_1 \frac{1}{a^{2kM_1-2kM_2-\cdots-M_{k-1}}a^{2n}}
\]

\[
\times \left( \frac{a^2; q^2}{2^2 M_1 \cdots 2^2 M_2 \cdots 2^2 n} \right) \left( \frac{q^2, a^2 q^{2n}; q^2}{2^{2n + 2M_1 + 2M_2 + \cdots + 2M_{k-1}} a^{2n + 2M_1 + 2M_2 + \cdots + 2M_{k-1} \sqrt{\lambda}} \right)
\]

\[
\times \left( \frac{a^2; q^2}{2^2 M_1 \cdots 2^2 M_2 \cdots 2^2 n} \right) \left( \frac{q^2, a^2 q^{2n}; q^2}{2^{2n + 2M_1 + 2M_2 + \cdots + 2M_{k-1}} a^{2n + 2M_1 + 2M_2 + \cdots + 2M_{k-1} \sqrt{\lambda}} \right)
\]

\[
= \left( \frac{a^2; q^2, e^2 q^2; q^2}{2^2 M_1 \cdots 2^2 M_2 \cdots 2^2 n} \right) \sum_{n=0}^{\infty} \left( \begin{array}{c}
(1 \lambda q_1^{2n}) (a^2/e^2; q^2)_{2n} \\
(1 \lambda q_1^{2n}) (a^2/e^2; q^2)_{2n} \\
\vdots \\
(1 \lambda q_1^{2n}) (a^2/e^2; q^2)_{2n}
\end{array} \right) M_1 \frac{1}{a^{2kM_1-2kM_2-\cdots-M_{k-1}}a^{2n}}
\]

\[
\times B_n(c, (a_{2^{k-3}}); q^2) V_n(\lambda, (b_{2m}); q_1) \left( \frac{\lambda^{m+k/q^2} q^2 e^{2k} q_1^{m-1} e^2}{a^2 b} \right)^n.
\]

To prove (12), let us take in (1), \( u_n, v_n, \delta_n \), and \( \alpha_n \) as

\[
u_n = \left( \begin{array}{c}
(a^2; q^2)_{2n} \\
(1 \lambda q_1^{2n}) (a^2/e^2; q^2)_{2n} \\
\vdots \\
(1 \lambda q_1^{2n}) (a^2/e^2; q^2)_{2n}
\end{array} \right)
\]

\[
\delta_n = z^n;
\]

\[
\alpha_n = \left( \frac{a^2; q^2}{2^2 M_1 \cdots 2^2 M_2 \cdots 2^2 n} \right) B_n(c, (a_{2^{k-3}}); q^2) (1 \lambda q_1^{2n}) V_n(\lambda, (b_{2m}); q_1)
\]

\[
\times \left( \frac{\lambda^{m+k/q^2} q^2 e^{2k} q_1^{m-1} e^2}{a^2 b} \right)^n
\]

and use the general bibasic transformation formula ([9, Eq. (15)] with r a positive real number) obtained recently by me, in place of a transformation.
formula [14, Eq. (1.5)] due to Verma and Jain. We get (12) after some simplification.

**Identities of the Rogers–Ramanujan Type**

(i) Let us take \( \lambda = a^2, q = q^2, r = 1 \) in (12), then make

\[
\begin{align*}
b_1, \ldots, b_m &\to \infty, \\
b_{m+1}, \ldots, b_{2m} &\to 0,
\end{align*}
\]

and sum the resulting innermost series on the left by a well-known \( \phi_3 \) summation formula [7, Eq. (2.7.1)] for the well-poised series. We have the transformation

\[
\sum_{n, m_1, \ldots, m_{k-1} \geq 0} (-1)^{M_1} (1/e^2, a^2/e^2; q^2)_n U_{(m_{k-1})} (a, b, c, (a_{2k-3}); q^2) \\
\times \frac{(a^2q^2/a_{2k-3}, a^2q^{2n}/e^2, q^{-2n}; q^2)_M \gamma}{(e^2q^{2-2n}, e^2q^{2-2M_1}/a_{2k-3}; q^2)_m} \\
\times \frac{(e^2q^{2-2n}/a_{2k-3}; q^2)_n q^{-M_1+M_1+2N_{k-1}}}{a^{-2(M_1+\cdots+M_{k-1})q^{-2(M_1+2n)}}} \\
= \frac{(a^2e^2q^2; q^2)_\infty}{(a^2q^2, e^4q^2; q^2)_\infty} \times \sum_{n=0}^{\infty} (a^2, b; q^2)_n B_n (c, (a_{2k-3}); q^2) (a^2/e^2, q^2)_2n \\
\times \frac{(1-a^2q^{4n})}{(1-a^2)} \left( \frac{e^4a^{2k-2}q^{2k}}{b} \right)^n. \tag{13}
\]

If we first make \( e \to 0 \) and then \( b, c, a_1, \ldots, a_{2k-3} \to \infty \) in (13), we get

\[
\sum_{n, m_1, \ldots, m_{k-1} \geq 0} \frac{(q^{-2n}; q^2)_M (-1)^{M_1} a^{2N_{k-1}} q^{2N_{k-1}+1 + (2n+1-M_{k-1})M_1} (q^2; q^2)_n (q^2; q^2)_m \cdots (q^2; q^2)_m}{(q^2; q^2)_n (q^2; q^2)_m ... (q^2; q^2)_m} \\
= \frac{1}{(a^2q^2; q^2)_\infty} \sum_{n \geq 0} \frac{(a^2; q^2)_n (1-a^2q^{4n})}{(a^2; q^2)_n (1-a^2)} (-1)^n a^{(2k+2)n} q^{(2k+3)n^2-n}. \tag{14}
\]
Taking \( a = 1 \) and \( a = q \) in (14) and making use of identity (6), we get the following identities, respectively, which are believed to be new:

\[
(-q; q)_\infty \sum_{n,m_1,\ldots,m_{k-2} \geq 0} \frac{(-1)^M(q^{-2n}; q^2)_M q^{2N+2(2n+1-M)M_1}}{(q^2; q^2)_n(q^2; q^2)_m \cdots (q^2; q^2)_{m_{k-2}}} \\
= \prod_{j=1}^{\infty} (1 - q^j)^{-1}, \quad k > 2, j \not\equiv 0, \pm 2k \mod 4k + 2, \quad (15)
\]

\[
(-q; q)_\infty \sum_{n,m_1,\ldots,m_{k-2} \geq 0} \frac{(-1)^M(q^{-2n}; q^2)_M q^{2N+2(2n+1-M)M_1+2N_{k-2}}}{(q^2; q^2)_n(q^2; q^2)_m \cdots (q^2; q^2)_{m_{k-2}}} \\
= \prod_{j=1}^{\infty} (1 - q^j)^{-1}, \quad k > 2, j \not\equiv 0, \pm 2 \mod 4k + 2. \quad (16)
\]

Let us now take \( b = -aq \) in (13) and then make \( e \to 0, c, a_1, \ldots, a_{2k-3} \to \infty \). We get the transformation

\[
\sum_{n,m_1,\ldots,m_{k-1} \geq 0} \frac{(-1)^M(q^{-2n}; q^2)_M a^{2N_{k-1}} q^{2N_{k-2}+2(2n+1-M)M_1}}{(q^2; q^2)_n(q^2; q^2)_m \cdots (q^2; q^2)_{m_{k-1}}} (-aq; q^2)_{M_{k-1}} \\
= \frac{1}{(a^2 q^2; q^2)_\infty} \sum_{n \geq 0} \frac{(a^2 q^2)_n(1-a^2 q^{2n})}{(q^2; q^2)_n(1-a^2)} (-1)^n a^{2(k+1)n} q^{(2k+2)n^2-n}. \quad (17)
\]

Again, if we take \( a = 1 \) and \( a = q \) in (17) and use the identity (6), we easily get, respectively, the two identities

\[
(-q; q)_\infty \sum_{n,m_1,\ldots,m_{k-2} \geq 0} \frac{(-1)^M(q^{-2n}; q^2)_M q^{2N+2(2n+1-M)M_1}}{(q^2; q^2)_n(q^2; q^2)_m \cdots (q^2; q^2)_{m_{k-2}}} (-q; q^2)_{M_{k-2}} \\
= \prod_{j=1}^{\infty} (1 - q^j)^{-1}, \quad k > 2, j \not\equiv 0, \pm (2k + 1) \mod 4k. \quad (18)
\]

(cf. Bressoud [5, Eq. (3.6)], Paule [8, Eq. (54)])

\[
(-q; q)_\infty \sum_{n,m_1,\ldots,m_{k-2} \geq 0} \frac{(-1)^M(q^{-2n}; q^2)_M q^{2N+2(2n+1-M)M_1+2N_{k-2}}}{(q^2; q^2)_n(q^2; q^2)_m \cdots (q^2; q^2)_{m_{k-2}}} (-q^2; q^2)_{M_{k-2}} \\
= \prod_{j=1}^{\infty} (1 - q^j)^{-1}, \quad k > 2, j \not\equiv 0, \pm 2 \mod 4k. \quad (19)
\]
(ii) Let us now take \( \lambda = a, q_1 = q, r = 1/2, b = q^2, m = 1, b_1 = a \) in (12) and the make \( e \to 0, c, a_1, \ldots, a_{2k-3}, b_2 \to \infty \).

We get

\[
\begin{align*}
\sum_{t,n,m_1,\ldots,m_{k-1} \geq 0} & \frac{(q^{-2n}; q^2)_{M_1+t}(1-q^{2r+2N})q^{h(N,t)+2M_1^2-1}}{(q^2; q^2)(q^2; q^2)_n(q^2; q^2)_{M_1} \cdots (q^2; q^2)_{M_{k-1}}(a^2; q^2)_{M_{k-1}}} \\
& \times (a^2; q^2)_{2M_1+t}(a; q)_{M_1+t}(1-aq^{2M_1+2t}) \\
& \times (a^2q^2; q^2)_{n+M_1+t}(q; q)_{M_1+t} \\
& = \frac{1}{(a^2q^2; q^2)} \sum_{n=0}^{\infty} \frac{(a; q)^n(1-aq^{2n})}{(q; q)^n} (-1)^n \\
& \times a^{2k+2n}q^{qk+5(n^2/2)-n/2}. \tag{20}
\end{align*}
\]

Further, if we take \( a = 1 \) and \( a = q \) in (20) and then make use of identity (6), we get the following identities, respectively, which are believed to be new:

\[
(-q; q) = \sum_{t,n,m_1,\ldots,m_{k-2} \geq 0} \frac{(q^{-2n}; q^2)_{M_1+t}g(N,t)}{(q^2; q^2)(q^2; q^2)_n(q^2; q^2)_{M_1} \cdots} \\
\times (-1)^t(1+aq^{M_1+t})q^{h(N,t)} \\
\times (q^2; q^2)_{m_{k-2}}(q^2; q^2)_{M_{k-1}}^{-1} \\
= \prod_{j=1}^{\infty} (1-q^j)^{-1}, \quad k > 2, j \not\equiv 0, \pm 2k \pmod{4k+1}, \tag{21}
\]

\[
(-q; q) = \sum_{t,n,m_1,\ldots,m_{k-2} \geq 0} \frac{(-1)^t(q^{-2n}; q^2)_{M_1+t}g(N,t+1)}{(q^2; q^2)(q^2; q^2)_n(q^2; q^2)_{M_1} \cdots} \\
\times (1-q^{2M_1+2t+1})q^{h(N,t)+2(N_{k-2}+t)} \\
\times (q^2; q^2)_{m_{k-2}}(q^2; q^2)_{M_{k-2}}^{-1} \\
= \prod_{j=1}^{\infty} (1-q^j)^{-1}, \quad k > 2, j \not\equiv 0, \pm 1 \pmod{4k+1}, \tag{22}
\]

(cf. Paule [8, Eqs. (57) and (58)].)
(iii) Lastly, if we take \( \lambda = a, q_1 = q, r = 1/2, b = q^2, m = 2, b_1 = a \), we thus obtain the result

\[
\sum_{t, n, m_1, \ldots, m_{k-1} \geq 0} \frac{(-1)^t(q^{-2n}; q^2)_{M_1+t}a^{2N_{k-1}+1}M_1+3q^b(N,t)+(M_1+t)^2+2M_{k-1}^2}{(q^2; q^2)_t(q^2; q^2)_n(q^2; q^2)_m \cdots (q^2; q^2)_{m_{k-1}}(a^2; q^2)_{M_{k-1}}} 
\]

\[
\times \frac{(a^2; q^2)_{2M_1+t}(a; q)_{M_1+t}(1 - aq^{2M_1+2t})}{(a^2; q^2)_{n+M_1+t}(q; q)_{M_1+t}} 
\]

\[
= \frac{1}{(a^2q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(a; q)_n(1 - aq^{2n})}{(q; q)_n} (-1)^n a^{(2k+3)n}q^{(4k+7)(n^2/2)-n/2}.
\]

(23)

Let us now take \( a = 1 \) and \( a = q \), respectively, in (23) and then use the identity (6). We thus obtain the following two identities, which are believed to be new:

\[
(-q; q)_t \sum_{t, n, m_1, \ldots, m_{k-2} \geq 0} \frac{(-1)^t(q^{-2n}; q^2)_{M_1+t}g(N, t)}{(q^2; q^2)_t(q^2; q^2)_n(q^2; q^2)_m \cdots} 
\]

\[
\times \frac{(1 + q^{M_1+t})q^{b(N, t) + (M_1+t)^2}}{(q^2; q^2)_{n+M_1+t}(q^2; q^2)_{M_{k-2}-1}} 
\]

\[
= \prod_{j=1}^{\infty} (1 - q^j)^{-1}, \quad k > 2, j \neq 0, \pm (2k + 2) \pmod{4k + 3},
\]

(24)

\[
(-q; q)_t \sum_{t, n, m_1, \ldots, m_{k-2} \geq 0} \frac{(-1)^t(q^{-2n}; q^2)_{M_1+t}g(N, t + 1)}{(q^2; q^2)_t(q^2; q^2)_n(q^2; q^2)_m \cdots} 
\]

\[
\times \frac{(1 - q^{2M_1+2t+1})q^{b(N, t) + (M_1+t)^2 + 2N_{k-2}+M_1+3t}}{(q^2; q^2)_{n+M_1+t}(q^2; q^2)_{M_{k-2}}} 
\]

\[
= \prod_{j=1}^{\infty} (1 - q^j)^{-1}, \quad k > 2, j \neq 0, \pm 1 \pmod{4k + 3}.
\]

(25)

(cf. Paule [8, Eqs. (45) and (46)].)
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