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Applied Mathematics Letters 18 (2005) 739–748

Applied Mathematics Letters

www.elsevier.com/locate/aml

On the representation of fractional Brownian motion as an integral with respect to $(\mathrm{d}t)^a$

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Received 7 May 2004; accepted 13 May 2004

Abstract

Maruyama introduced the notation $\mathrm{d}b(t) = w(t)(\mathrm{d}t)^{1/2}$ where $w(t)$ is a zero-mean Gaussian white noise, in order to represent the Brownian motion $b(t)$. Here, we examine in which way this notation can be extended to Brownian motion of fractional order a (different from $1/2$) defined as the Riemann–Liouville derivative of the Gaussian white noise. The rationale is mainly based upon the Taylor's series of fractional order, and two cases have to be considered: processes with short-range dependence, that is to say with $0 \lhd a \leq 1/2$, and processes with long-range dependence, with $1/2 \lhd a \leq 1$.

Keywords: Fractional Brownian motion; Taylor series of fractional order; Maruyama notation; Stochastic differential equation; Stochastic calculus of fractional order

1. Introduction

Let $w(t)$ denote a Gaussian white noise with the constant variance σ^2 and let $b(t)$ denote its companion Brownian motion. Since the covariance function of $w(t)$ is the derivative of the covariance function of $b(t)$, one is used to considering $w(t)$ as being the derivative of $b(t)$ and to writing

$$\mathrm{d}b(t) = w(t)\mathrm{d}t, \tag{1}$$

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2004.05.014

or, in a like manner, the equality

$$b(t) = \int_0^t w(\tau)\mathrm{d}\tau, \tag{2}$$

which yields the variance

$$\mathrm{Var}\{b(t)\} = \sigma^2 t. \tag{3}$$

This expression (3) can be obtained from (2) but it cannot be thought of as a direct consequence of (1) as a result of the equality

$$\mathrm{Var}\{b(t)\} = \int_0^t \mathrm{Var}\{\mathrm{d}b\}\mathrm{d}\tau.$$

In order to circumvent this pitfall, Maruyama introduced a contrivance, and he used the notation (very successful in engineering mathematics)

$$\mathrm{d}b(t) = w(t)(\mathrm{d}t)^{1/2}, \tag{4}$$

which provides $E\{(\mathrm{d}b)^2\} = \sigma^2\mathrm{d}t$, and therefore (3). There is no inconsistency between (1) and (4): in the former $w(t)$ is a generalized function, clearly a sequence of Dirac delta generalized functions, whilst in (4) $w(t)$ is thought of as a function.

Our purpose in the present short work is to examine in which way we can use the same model to represent Brownian motion of fractional order different from $1/2$. We shall show that such a modeling is still meaningful, but then care must be exercised. Briefly, it remains an excellent tool for a formal manipulation of fractional Brownian motions.

The work is organized as follows. In the next section, we shall give a short background on fractional Brownian motion and then, by using a formal calculus, we shall obtain a Taylor's series of fractional order. We shall use this result to define integrals with respect to $(\mathrm{d}t)^\alpha$, whereby we shall be in a position to clarify the meaning of the generalized use of Maruyama's notation.

2. Fractional Brownian motion and the fractional derivative

2.1. *Brownian motion of order different from* $(1/2)$

The basic properties of the fractional Brownian motion defined as a fractional derivative of Gaussian white noise can be summarized as follows (see the Refs. [8,9] for the physical derivation of this process):

Definition 2.1. Let (Ω, F, P) denote a probability space and $a, 0 \lhd a \leq 1$, be referred to as the Hurst parameter. The stochastic process $\{b(t,a), t \geq 0\}$ defined on this probability space is a fractional Brownian motion $(fBm)_a$ of order a if (see for instance [1])

(i) $\Pr\{b(0,0) = 0\} = 1$;
(ii) for each $t \in \Re_+$, $\beta(t,a)$ is an F-measurable random variable such that $E\{b(t,a)\} = 0$;
(iii) for $t, \tau \in \Re_+$,

$$E\{b(t,a)b(\tau,a)\} = \frac{\sigma^2}{2}(t^{2a} + \tau^{2a} - |t-\tau|^{2a}), \tag{5}$$

where σ is the variance parameter.

It follows from (5) and from Kolmogorov's continuity criterion that, for $a \rhd 1/2$, the sample paths of $b(t, a)$ are continuous with probability one, but nowhere differentiable. □

Further remarks and comments

(i) Unlike the semblance, the equality (5) can be simply derived from the self-similarity equation

$$b(\rho t, a) \underset{law}{=} \rho^a b(t, a), \quad \rho \rhd 0. \tag{6}$$

This yields

$$b(t, a) = t^a b(1, a). \tag{7}$$

(ii) The $(fBm)_a$ can be constructed from the classical Brownian motion $b(t) := b(t, 1/2)$ by a linear transformation of the form (the symbol := means that the left side is defined by the right one)

$$b(t, a) := \int_0^t K_a(t, \tau) \mathrm{d}b(\tau),$$

where $K_a(t, \tau)$ is a kernel dependent on the Hurst parameter a. It has been suggested that one could select the hypergeometric function, but the most useful kernel is the one which yields

$$b(t, a) = \frac{1}{\Gamma(a + (1/2))} \int_0^t (t - \tau)^{a-(1/2)} w(\tau) \mathrm{d}\tau, \tag{8}$$

as proposed by Mandelbrot and van Ness [7] and which directly relates the $(fBm)_a$ to fractional derivatives defined as follows.

2.2. Fractional derivative

Definition 2.2. Let $f : \Re \to \Re, x \to f(x)$ denote a continuous function. Its fractional derivative of order α is defined by the following expression [3–6]:

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} f(\xi) \mathrm{d}\xi, \quad \alpha \lhd 0. \tag{9}$$

For positive α, one sets

$$f^{(\alpha)}(x) := [f^{(\alpha-n)}]^{(n)}, \quad n - 1 \lhd \alpha \lhd n. \quad \square \tag{10}$$

With this notation, Eq. (8) yields

$$b(t, a) = D^{-\left(a+\frac{1}{2}\right)} b(t), \tag{11}$$

where D denotes the derivative operator.

Definition 2.3. Consider the function of Definition 2.2; and let $h \rhd 0$ denote a constant discretization span. Define the forward operator $FW(h)$:

$$FW(h).f(x) := f(x + h); \tag{12}$$

then the fractional difference of order α, $0 \lhd \alpha \lhd 1$, of $f(x)$ is defined by the expression

$$\begin{aligned} \Delta^\alpha . f(x) &:= (FW - 1)^\alpha . f(x) \\ &= \sum_{k=0}^{\infty} (-1)^k (k^\alpha) f[x + (\alpha - k)h]. \quad \square \end{aligned} \tag{13}$$

Lemma 2.1. *The following equality holds:*

$$f^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\Delta^\alpha f(x)}{h^\alpha}. \quad \square \tag{14}$$

The proof can be obtained by using the Laplace transform and Z-transform and then making h tend to zero. See for instance [2].

3. Taylor expansion of fractional order

The Taylor expansion of fractional order is as follows:

Proposition 3.1. *Assume that the function $f : \Re \to \Re, x \to f(x)$ has a fractional derivative of order $k\alpha$, $0 \lhd \alpha \lhd 1$, for any positive integer k; then the following equality holds:*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(1+\alpha k)} f^{(\alpha k)}(x), \quad 0 \lhd \alpha \lhd 1. \tag{15}$$

Here $f^{(\alpha k)}$ is the derivative of order αk of $f(x)$. When, instead, one has $m \lhd \alpha \lhd m+1$, $m \in N - \{0, 1\}$, then

$$f^{(m)}(x+h) = \sum_{k=0}^{\infty} \frac{h^{k(\alpha-m)}}{\Gamma[1+k(\alpha-m)]} D^{k(\alpha-m)} f^{(m)}(x), \quad m \lhd \alpha \leq m+1. \quad \square \tag{16}$$

A formal proof, based on operational calculus, goes as follows.

Proof. Starting from the definition of the operator $FW(h)$, Eq. (12), it is easy to show that the latter has a derivative of order α, and satisfies the operational equation

$$\frac{\mathrm{d}^\alpha FW(h)}{\mathrm{d}h^\alpha} = D_x^\alpha FW(h), \tag{17}$$

where D_x holds for the derivative w.r.t. x. This equality is based on the fact that

$$f_x^{(\alpha)}(x+h) = f_h^{(\alpha)}(x+h).$$

In the Eq. (17), D_x^α can be thought of as a constant, and thus solving it for $FW(h)$ yields (see the Appendix)

$$FW(h) = E_\alpha(h^\alpha D_x^\alpha). \tag{18}$$

where $E_\alpha(x)$ denotes the Mittag–Leffler function defined as

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)}. \quad \square \tag{19}$$

In order to obtain (16), we merely consider the Taylor expansion of order $\alpha - m$ of the derivative $f^{(m)}(x)$. $\square$

As a direct application, one has the following:

Lemma 3.1. *Assume that $f(x)$, in Proposition 3.1, is αth-differentiable; then the following equalities hold:*

$$f^{(\alpha)}(x) = \lim_{h\downarrow 0} \frac{\Delta^\alpha f(x)}{h^\alpha} = \Gamma(1+\alpha) \lim_{h\downarrow 0} \frac{\Delta f(x)}{h^\alpha}, \quad 0 \lhd \alpha \leq 1, \tag{20}$$

and

$$f^{(\alpha)}(x) = \Gamma[1+(\alpha-m)] \lim_{h\downarrow 0} \frac{\Delta f^{(m)}(x)}{h^{\alpha-m}}, \quad m \lhd \alpha \leq m+1. \quad \square \tag{21}$$

A useful relation

Eq. (20) provides the useful relation

$$\mathrm{d}^\alpha f = \Gamma(1+\alpha)\mathrm{d}f, \quad 0 \lhd \alpha \leq 1, \tag{22}$$

between $\mathrm{d}^\alpha f$ and $\mathrm{d}f$.

Assume now that $1 \lhd \alpha \leq 2$. Then Eq. (21) yields (on setting $m = 1$)

$$\mathrm{d}^\alpha f = \Gamma(\alpha)h\mathrm{d}f', \tag{23}$$

where f' denotes the derivative of f.

On substituting the equality $f' = \mathrm{d}f/h$ into (23) we obtain the second equivalence

$$\mathrm{d}^\alpha f = \Gamma(\alpha)\mathrm{d}^2 f, \quad 1 \lhd \alpha \leq 2. \tag{24}$$

4. Integration with respect to $(\mathrm{d}t)^\alpha$

4.1. Fractional order lower than the unit

Our purpose, in this section, is to define the solution of the equation

$$\mathrm{d}x = f(t)(\mathrm{d}t)^\alpha, \tag{25}$$

and, to this end, we shall refer to the following:

Definition 4.1. Let $f(t)$ denote a continuous $\mathfrak{R} \to \mathfrak{R}$ function; then its integral w.r.t. $(\mathrm{d}t)^\alpha$, $0 \lhd \alpha \leq 1$, will be defined by the equality

$$\int_0^t f(\tau)(\mathrm{d}\tau)^\alpha = \alpha \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad 0 \lhd \alpha \leq 1. \quad \square \tag{26}$$

Derivation. This definition can be supported as follows.

(i) Let us consider the fractional differential equation

$$x^{(\alpha)}(t) = f(t), \quad 0 \lhd \alpha \leq 1. \tag{27}$$

In a straightforward manner, its solution is obtained as

$$x(t) = D^{-\alpha} f(t),$$

and, by using the fractional derivative (9), one has

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau. \tag{28}$$

(ii) This being the case, another way to obtain $x(t)$ is as follows. We rewrite (25) in the differential form (see Eq. (13))

$$\mathrm{d}^{\alpha}x = f(t)(\mathrm{d}t)^{\alpha} \tag{29}$$

or again, according to (22),

$$\Gamma(1+\alpha)\mathrm{d}x = f(t)(\mathrm{d}t)^{\alpha}. \tag{30}$$

Formally, integrating (28) yields

$$x(t) = \frac{1}{\Gamma(1+\alpha)}\int_0^t f(\tau)(\mathrm{d}\tau)^{\alpha}. \tag{31}$$

(iii) In order to ascribe a significance to the right hand side term of (31) we shall equate it to (28) to obtain the equality

$$\frac{1}{\Gamma(1+\alpha)}\int_0^t f(\tau)(\mathrm{d}\tau)^{\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau$$

and therefore (26). □

Some examples

On making $f(\tau) = 1$ in (30), one obtains

$$\int_0^t (\mathrm{d}\tau)^{\alpha} = t^{\alpha}, \quad 0 \lhd \alpha \leq 1. \tag{32}$$

Assume now that $f(t)$ is the Dirac delta generalized function $\delta(t)$; then one has

$$\int_0^t \delta(\tau)(\mathrm{d}\tau)^{\alpha} = \alpha t^{\alpha-1}, \quad 0 \lhd \alpha \leq 1. \tag{33}$$

Application to fractional Brownian motion

If we apply formally the relation (25) above with the substitution $f(t) \leftarrow w(t)$, we have an alternative definition for $b(t,a)$, which reads

$$b(t,a) = \left(a+\frac{1}{2}\right)^{-1}\Gamma^{-1}\left(a+\frac{1}{2}\right)\int_0^t w(\tau)(\mathrm{d}\tau)^{a+(1/2)}. \tag{34}$$

This provides (5) when $a = 1/2$.

4.2. Fractional order larger than the unit

Assume now that $1 \lhd \alpha \leq 2$ in Eq. (26), which we rewrite for convenience in the form (on setting $\alpha = 2\gamma$)

$$\mathrm{d}x = f(t)(\mathrm{d}t)^{2\gamma}, \quad 1/2 \lhd \gamma \leq 1, \tag{35}$$

and assume further (provisionally) that $f(t) \geq 0$.

On making the change of variable

$$\mathrm{d}y = (\mathrm{d}x)^{1/2}, \tag{36}$$

Eq. (35) yields

$$\mathrm{d}y = f^{1/2}(t)(\mathrm{d}t)^{\gamma}, \tag{37}$$

and therefore, by using the result of Section 4.2,

$$y = \int_0^t f^{1/2}(\tau)(\mathrm{d}\tau)^{\gamma}, \tag{38}$$

in the sense of the Definition 4.1.

(ii) This being the case, integrating (36) provides (see Eq. (32))

$$y = \int_0^x (\mathrm{d}\xi)^{1/2} = x^{1/2}, \tag{39}$$

and therefore

$$x = \left(\int_0^t f^{1/2}(\tau)(\mathrm{d}\tau)^{\gamma}\right)^2. \tag{40}$$

(iii) We can now readily drop the assumption that $f(t) \geq 0$, and we are thus led to the following:

Definition 4.2. Let $f(t)$ denote a continuous $\mathfrak{R} \to \mathfrak{R}$ function; then its integral with respect to $(\mathrm{d}t)^{2\gamma}$, $1/2 \lhd \gamma \leq 1$, will be defined by the equality

$$\int_0^t f(\tau)(\mathrm{d}\tau)^{2\gamma} = \left(\int_0^t f^{1/2}(\tau)(\mathrm{d}\tau)^{\gamma}\right)^2, \quad 1/2 \lhd \gamma \leq 1, \tag{41}$$

$$= \gamma^2 \left(\int_0^t (t-\tau)^{\gamma-1} f^{1/2}(\tau)\mathrm{d}\tau\right)^2. \quad \square \tag{42}$$

We can now fully define the Maruyama notation of fractional order as follows.

5. Maruyama notation of fractional order

Definition 5.1. Let $b(t, a)$ denote a fractional Brownian motion of order a, as defined by Riemann–Liouville fractional derivative. Then, generalizing the notation $\mathrm{d}b(t) = w(t)(\mathrm{d}t)^{1/2}$, we shall write

$$\mathrm{d}\beta(t, a) = w(t)(\mathrm{d}t)^a, \quad 0 \lhd a \leq 1, \tag{43}$$

where $w(t)$ is a Gaussian white noise with zero mean and the variance σ^2. $\square$

Here we use $\beta(t, a)$ instead of $b(t, a)$ to emphasize the difference between these two processes. Mainly the increments of $\beta(t, a)$ are mutually independent, whilst those of $b(t, a)$ are not.

According to (26), $\beta(t, a)$ is then defined by the expression

$$\beta(t, a) = \int_0^t w(\tau)(\mathrm{d}\tau)^a$$

$$= a\int_0^t (t-\tau)^{a-1} w(\tau)\mathrm{d}\tau. \tag{44}$$

The variance $\sigma_\beta^2(t, a)$ is

$$\sigma_\beta^2(t,a) = \int_0^t \mathrm{Var}\{\mathrm{d}\beta^2(t,a)\} = \sigma^2 \int_0^t (\mathrm{d}t)^{2a}, \tag{45}$$

and therefore (see the Eqs. (32) and (42))

$$\sigma_\beta^2(t.a) = \sigma^2 t^{2a}, \quad 0 \lhd a \leq 1. \tag{46}$$

6. Conclusions

By using a definition of integrals w.r.t $(\mathrm{d}t)^\alpha$ which is fully consistent with the definition of the Riemann–Liouville derivative, we have shown that the so-called Maruyama notation $\mathrm{d}b = w(t)(\mathrm{d}t)^{1/2}$, which is so useful in physics and engineering mathematics, can be meaningfully extended in the form $\mathrm{d}\beta(t,a) = w(t)(\mathrm{d}t)^a$, in order to deal with Brownian motions of fractional order different from $(1/2)$. But care must be exercised: $\beta(t,a)$ has an independent increment whilst $b(t,a)$, defined via the Riemann–Liouville derivative, has increments which are not mutually independent.

But apart from exercising this caution, which prevents us from working on the correlation function and related problems for instance, this modeling could serve at a point of departure for developing a useful formal calculus on fractional Brownian motion, as illustrated, for instance, by the following example.

Illustrative application example

Our purpose is to calculate the state moments

$$m_k(t) := E\{x^k(t)\}, \quad k = 1, 2, \tag{47}$$

of the dynamical system driven by the stochastic differential equation of fractional order

$$\mathrm{d}x = x(c\mathrm{d}t + \sigma w(t)(\mathrm{d}t)^a), \quad \mathrm{pr}\{x(0) = x_0\} = 1, \tag{48}$$

with c, σ = constant, $\mathrm{Var}\{w(t)\} = 1$, $2a \lhd 1$. Taking the mathematical expectation of (48) before and after squaring, one obtains the equations

$$\mathrm{d}m_1 = cm_1\mathrm{d}t, \tag{49}$$
$$\mathrm{d}m_2 = \sigma^2 m_2(\mathrm{d}t)^{2a}. \tag{50}$$

Using (22), we rewrite (50) in the form

$$\mathrm{d}^{2a}m_2 = \Gamma(1+2a)\sigma^2 m_2(\mathrm{d}t)^{2a}, \tag{51}$$

therefore obtaining

$$m_1(t) = x_0 \mathrm{e}^{ct}, \tag{52}$$
$$m_2(t) = x_0^2 E_{2a}\{\Gamma(1+2a)\sigma^2 t^{2a}\}. \tag{53}$$

As a typical application, for instance, in the optimal control of some systems driven by fractional stochastic differential equations, it will be possible to drop the approach via dynamic programming, and to use a variational Lagrangian approach which involves these moments as new state variables.

Appendix. Solution of a linear fractional differential equation

Let us consider the fractional differential equation

$$D_t^\alpha x(t) = -\lambda x(t), \tag{54}$$

with the initial condition

$$x(0) = x_0. \tag{55}$$

Its solution can be obtained as follows.

(i) Using the definition of the fractional derivative, Eqs. (8) and (9), Eq. (54) can be explicitly rewritten as

$$\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau \right) = -\lambda x(t), \tag{56}$$

or

$$\int_0^t (t-\tau)^{-\alpha} x(\tau) \mathrm{d}\tau = -\lambda \Gamma(1-\alpha) \int_0^t x(\tau) \mathrm{d}\tau,$$

or

$$t^{1-\alpha} \int_0^1 (1-u)^{-\alpha} x(tu) \mathrm{d}u = -\lambda \Gamma(1-\alpha) t \int_0^1 x(tu) \mathrm{d}u. \tag{57}$$

(ii) The presence of t^α in Eq. (57) on the one hand, and the solution $\exp\{-\lambda t\}$ of the equation $\mathrm{d}p/\mathrm{d}t = -\lambda p$ on the other hand, suggests looking for a solution in the form

$$x(0,t) = \sum_{k=0}^{\infty} x_k (t^\alpha)^k. \tag{58}$$

Substituting (58) into (57), one obtains

$$x_{k+1} = -\lambda \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k + \alpha + 1)} x_k. \tag{59}$$

(iii) Eq. (60) yields

$$x_k = \frac{(-\lambda)^k}{\Gamma(\alpha k + 1)} x_0, \tag{60}$$

and therefore

$$x(t) = x_0 \sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha)^k}{\Gamma(k\alpha + 1)}, \tag{61}$$

$$= x_0 E_\alpha(-\lambda t^\alpha), \tag{62}$$

where $E_\alpha(z)$ is the Mittag–Lefler function

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(k\alpha + 1)}. \tag{63}$$

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