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ScienceDirect
Discrete Mathematics 309 (2009) 363–370
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Bose–Burton type theorems for finite Grassmannians

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Received 27 June 2006; accepted 6 December 2007
Available online 22 January 2008

Dedicated to the memory of Alessandro Bichara

Abstract

In this paper both blocking sets with respect to the s-subspaces and covers with t-subspaces in a finite Grassmannian are investigated, especially focusing on geometric descriptions of blocking sets and covers of minimum size. When such a description exists, it is called a Bose–Burton type theorem. The canonical example of a blocking set with respect to the s-subspaces is the intersection of s linear complexes. In some cases such an intersection is a blocking set of minimum size, that can occasionally be characterized by a Bose–Burton type theorem. In particular, this happens for the 1-blocking sets of the Grassmannian of planes of $\mathrm{PG}(5, q)$.

Keywords: Grassmannian; Blocking set; Cover; Linear complex

1. Introduction

A *semilinear space* is a point–line geometry $\Sigma = (\mathcal{P}, \mathcal{L})$, consisting of a non-empty set $\mathcal{P}$, whose elements are called *points*, and a collection $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, such that the following axioms hold: (i) any two distinct points lie on at most one line, (ii) every line contains at least two points, and (iii) every point lies on at least one line. Two points x and y are *collinear*, if there exists a line containing x and y. In particular every point x is collinear to itself. A *subspace* of $\Sigma = (\mathcal{P}, \mathcal{L})$ is a subset W of $\mathcal{P}$ such that for every two distinct collinear points of W the line joining them is contained in W. Since the intersection of subspaces is a subspace, it is possible to define the *closure* $[X]_\Sigma$ of a subset X of $\mathcal{P}$ as the intersection of all subspaces containing X. A *singular subspace* of Σ is a subspace W such that any two points of W are collinear, and a *prime* of Σ is a proper subspace K such that $L \cap K \neq \emptyset$ for every line $L \in \mathcal{L}$.

Two subspaces S and T of a semilinear space will be called *incident* if and only if $S \subseteq T$ or $T \subseteq S$. Finally, a *full embedding* of a semilinear space $\Sigma = (\mathcal{P}, \mathcal{L})$ into a semilinear space $\Sigma' = (\mathcal{P}', \mathcal{L}')$ is an injective mapping $e : \mathcal{P} \longrightarrow \mathcal{P}'$ such that $[\mathcal{P}^e]_{\Sigma'} = \Sigma'$, and for every line $L \in \mathcal{L}$, $L^e \in \mathcal{L}'$. A full embedding will be also denoted by

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.12.023

$e : \Sigma \longrightarrow \Sigma'$. If Σ' is the point–line geometry of a desarguesian projective space, then e is called a *full projective embedding*. An *isomorphism* between two semilinear spaces is a bijection f between their point sets, such that both f and f^{-1} are full embeddings.

The h-th *Grassmannian* of a projective space $PG(n, q)$, $0 \leq h \leq n-1$, is the semilinear space $\Gamma_q(n, h) = (\mathcal{P}, \mathcal{L})$, whose points are the h-subspaces of $PG(n, q)$, and whose lines are all pencils of h-subspaces, a *pencil* being the set of all h-subspaces through an $(h-1)$-subspace and contained in an $(h+1)$-subspace. In order to avoid ambiguities, the elements of $\mathcal{P}$ and $\mathcal{L}$ will be often called *G-points* and *G-lines*, respectively. When the field is clear from the context, $\Gamma_q(n, h)$ will be replaced by $\Gamma(n, h)$. Note that $\Gamma(n, 0)$ is the projective space $PG(n, q)$ and $\Gamma(n, n-1)$ is the dual projective space $PG^*(n, q)$. Every singular subspace of $\Gamma(n, h)$ consists of h-dimensional projective subspaces of $PG(n, q)$ pairwise intersecting in an $(h-1)$-subspace, and it is a projective space of finite dimension over the finite field $\mathbb{F}_q$. If d is the projective dimension of a singular subspace W of $\Gamma(n, h)$, then W will be called a *d-G-subspace*. In particular, for $0 < h < n-1$ the maximal singular subspaces of $\Gamma(n, h)$ are partitioned into two families $\mathcal{S}$ and $\mathcal{T}$. More precisely, a singular subspace of $\mathcal{S}$ consists of all h-subspaces of $PG(n, h)$ passing through a fixed $(h-1)$-subspace U, and it will be called a *star of center U*. The elements of $\mathcal{T}$ are called *dual stars*, and every one of them consists of all h-subspaces contained in an $(h+1)$-subspace. Every star is an $(n-h)$-G-subspace and every dual star is an $(h+1)$-G-subspace. A dual star of $\Gamma(n, 1)$ will be also called a *ruled plane*. The *duality map* $\delta : PG(n, q) \longrightarrow PG^*(n, q)$ transforms every h-dimensional subspace X of $PG(n, q)$ into the $(n-h-1)$-dimensional subspace X^* of $PG^*(n, q)$ consisting of all the hyperplanes of $PG(n, q)$ passing through X. Such δ is an isomorphism between $\Gamma(n, h)$ and $\Gamma(n, n-h-1)$. The *Plücker embedding* π defines a full projective embedding of $\Gamma(n, h)$ into the projective space $PG(N, q)$, where

$$N = \binom{n+1}{h+1} - 1, \tag{1}$$

and the image $\Gamma(n, h)^\pi = \mathcal{G}_{n,h}$, also denoted by $\mathcal{G}_{n,h,q}$, is an algebraic variety intersection of quadrics of $PG(N, q)$. In particular, $\mathcal{G}_{3,1}$ is the *Klein quadric* $Q^+(5, q)$ of $PG(5, q)$, so the Plücker embedding $\pi : \Gamma(3, 1) \longrightarrow PG(5, q)$ is also called the *Klein correspondence*. The mapping π transforms stars and dual stars of $\Gamma(n, h)$ into $(n-h)$-dimensional and $(h+1)$-dimensional projective subspaces of $PG(N, q)$, respectively, and, more generally, d-G-subspaces are mapped bijectively into d-dimensional subspaces of $PG(N, q)$ contained in $\mathcal{G}_{n,h}$.

If $\Sigma = (\mathcal{P}, \mathcal{L})$ is the point–line geometry of either a projective space $PG(n, q)$, or a Grassmannian $\Gamma(n, h)$, then an *(s, t)-blocking set* of Σ is a set K of t-dimensional singular subspaces, such that each s-dimensional singular subspace is incident with an element of K. The smallest cardinality of such a set K is denoted either by $A_q(n, s, t) = A(n, s, t)$, in the case $\Sigma = PG(n, q)$, or by $A_q(n, h, s, t) = A(n, h, s, t)$, if $\Sigma = \Gamma(n, h)$. In this paper the investigation is restricted to $st = 0$. If $t = 0$, then K is a blocking set in the usual sense, or an *s-blocking set*, whereas a $(0, t)$-blocking set will be also called a *t-cover*.

Various and interesting geometrical structures can be characterized in terms of blocking sets. From this point of view the starting result is the theorem of Bose and Burton [4], characterizing the $(n-d)$-dimensional subspaces of a finite projective space $PG(n, q)$ as d-blocking sets of minimum cardinality. Each characterization similar to the preceding one will be called a *Bose–Burton type theorem*. Many Bose–Burton type theorems are known in the literature, for instance the characterization of the Baer subplanes of a finite projective plane of order q^2 as blocking sets of minimum size q^2+q+1, due to Bruen (1970), and several characterizations of blocking sets of minimum size of quadrics [5,8–12]. From the isomorphism between $\Gamma_q(3, 1)$ and $Q^+(5, q)$, Bose–Burton type theorems for the first non-trivial Grassmannian easily follow from the theory of blocking sets and ovoids on a quadric [6,10]. The known cases concerning $\Gamma(3, 1)$ are summarized as follows:

Result 1. (i) [8] $A(3, 1, 1, 0) = (q+1)(q^2+1)$*, and the only point sets that meet the bound are the non-tangent hyperplane sections of* $Q^+(5, q)$.
(ii) $A(3, 1, 2, 0) = q^2 + 1$*, and the point sets that meet the bound are precisely the ovoids of* $Q^+(5, q)$.
(iii) [6] $A(3, 1, 0, 1) = q^3 + 2q + 1$.
(iv) [6] $A(3, 1, 0, 2) = q^2 + q$*, and the* 2*-covers attaining the bound are completely characterized.*

As regards (iii), in [6] some properties of 1-covers of size $q^3 + 2q + 1$ are described, and examples of 1-covers reaching that bound are given.

In [17], Bose–Burton type theorems for 1-blocking sets of $\Gamma(n, 1)$, n odd, and $\Gamma(4, 1)$ are proved. Here the goal is to extend to general (s, t)-blocking sets the results of [17], finding Bose–Burton type theorems for (s, t)-blocking sets of Grassmannians. Unfortunately, the Grassmannians, with the only exception of the Klein quadric, are not polar spaces, therefore a major algebraic tool is missing, and the methods used in [6] seemingly cannot be extended to arbitrary indices n, h. However, it is possible to obtain some general properties for $A(n, h, s, t)$ and to prove some results that could be of interest, and among them some Bose–Burton type theorems.

2. General properties of $A(n, h, s, t)$

In this section, general lower and upper bounds for $A(n, h, s, t)$ will be proved. In the previous section it has been observed that $\Gamma(n, h)$ is isomorphic to $\Gamma(n, n - h - 1)$. Hence, the following equality holds.

$$A(n, h, s, t) = A(n, n - h - 1, s, t). \tag{2}$$

This allows one to restrict the investigation to the case $h \leq (n - 1)/2$. Let $\theta_i = (q^{i+1} - 1)/(q - 1)$, where $i \in \mathbb{Z}$, $i \geq -1$. If $\gamma_{n,h}$ denotes the number of the G-points of $\Gamma(n, h)$, then

$$\gamma_{n,h} = \prod_{i=0}^{h} \frac{\theta_{n-i}}{\theta_i}. \tag{3}$$

Proposition 2. *The following lower bounds hold:*

$$A(n, h, 0, t) \geq \frac{1}{\theta_t} \prod_{i=0}^{h} \frac{\theta_{n-i}}{\theta_i}; \tag{4}$$

$$A(n, h, s, 0) \geq \frac{\theta_{n-h-s}}{\theta_{n-h}} \prod_{i=0}^{h} \frac{\theta_{n-i}}{\theta_i}. \tag{5}$$

In (5) the equality holds if, and only if, there is a set of G-points meeting each star in an $(n - h - s)$-G-subspace.

Proof. Inequality (4) is trivial. In order to prove (5), let K be an s-blocking set of $\Gamma(n, h)$. Let M be the number of pairs (X, S), where S is a star, and $X \in K \cap S$. Since each $X \in K$ belongs to exactly θ_h stars, it holds $M = |K|\theta_h$. From the theorem of Bose and Burton [4], every set of points that intersects every s-subspace of $\mathrm{PG}(d, q)$ has size greater than or equal to θ_{d-s}. By applying this result to a star S, one obtains $|K \cap S| \geq \theta_{n-h-s}$. So, $M \geq \gamma_{n,h-1}\theta_{n-h-s}$, and $|K| \geq \gamma_{n,h-1}\theta_{n-h-s}/\theta_h$. □

For particular values, in (5) the equality holds (cf. Section 3).

Now assume that K is an s-blocking set in $\Gamma(n, h)$, E a hyperplane of $\mathrm{PG}(n, q)$, $P \notin E$ a point. By intersecting the elements of K through P with E, a set $K_{E,P}$ of $(h - 1)$-subspaces in E arises which is an s-blocking set in $\Gamma(n - 1, h - 1)$. This implies, by a double counting argument,

$$A(n, h, s, 0) \geq \frac{\theta_n}{\theta_h} A(n - 1, h - 1, s, 0). \tag{6}$$

The following statement is then straightforward:

Proposition 3. *If K is an s-blocking set in $\Gamma(n, h)$ of size $(\theta_n/\theta_h)A(n - 1, h - 1, s, 0)$, then the size of each $K_{E,P}$ is equal to $A(n - 1, h - 1, s, 0)$.* □

In (6) the equality holds at least in the following cases: (i) n odd, $h = s = 1$ (cf. (15)), (ii) $n = 5$, $h = 2$, $s = 1$, (20). Finally, $A(4, 1, 1, 0) > \lceil(\theta_4/\theta_1)A(3, 0, 1, 0)\rceil$ (cf. Theorem 9).

Proposition 4. *For $1 \leq h < (n - 1)/2$ the following equalities hold:*

$$A(n, h, n - h, 0) = A(n, h - 1, h); \tag{7}$$

$$A(n, h, 0, n - h) = A(n, h, h - 1). \tag{8}$$

Proof. Let K be an $(n-h)$-blocking set of $\Gamma(n,h)$. The inequality $h < (n-1)/2$ implies $n-h > h+1$, hence every $(n-h)$-G-subspace of $\Gamma(n,h)$ is a star with center an $(h-1)$-dimensional subspace of PG(n,q). It follows that K consists of h-subspaces of PG(n,q) such that every $(h-1)$-subspace of PG(n,q) is contained in an element of K. Thus, K is an $(h-1,h)$-blocking set of PG(n,q). Conversely each $(h-1,h)$-blocking set of PG(n,q) is also an $(n-h)$-blocking set of $\Gamma(n,h)$. This implies (7). Similarly to the above case, each $(n-h)$-cover of $\Gamma(n,h)$ can be associated with an $(h,h-1)$-blocking set of PG(n,q). This yields (8). □

Proposition 5. *For* $1 \le h < (n-1)/2$*, it holds*

$$A(n,h,n-h,0) \ge A(n,h-1,0,h). \tag{9}$$

Proof. Since $h < (n-1)/2$, an $(n-h)$-blocking set K of $\Gamma(n,h)$ is an $(h-1,h)$-blocking set of PG(n,q). Taking for each $X \in K$ the related dual star of $\Gamma(n,h-1)$, an h-cover of $\Gamma(n,h-1)$ results, having the same size of K. This implies (9). □

In [2] it was proved that the smallest cardinality of a t-cover of PG(n,q), with $t \le n$, is

$$A(n,0,t) = \left\lceil \frac{\theta_n}{\theta_t} \right\rceil. \tag{10}$$

It is known that PG(n,q) contains a t-spread if, and only if, $t+1$ divides $n+1$, i. e. if, and only if, θ_t divides θ_n. If this is the case, a t-spread contains exactly θ_n/θ_t elements. For $h=1$, $A(n,1,n-1,0) = A(n,0,1) = \lceil \theta_n/\theta_1 \rceil$, and in (9) the equality holds. It is an open question, whether in (9) the inequality can occur. From the above arguments a trivial Bose–Burton type theorem follows:

Theorem 6. $A(n,1,n-1,0) = \theta_n/\theta_1$*, for every odd n, and the* $(n-1)$*-blocking set that meet the bound correspond precisely to the line spreads of* PG(n,q).

The set Θ of all stars with center in a hyperplane of PG(n,q) is a cover of $\Gamma(n,h)$. The following inequality can be obtained by covering each such star with $\lceil \theta_{n-h}/\theta_t \rceil t$-$G$-subspaces:

$$A(n,h,0,t) \le \gamma_{n-1,h-1} \left\lceil \frac{\theta_{n-h}}{\theta_t} \right\rceil. \tag{11}$$

Similarly, the set of the dual stars determined by all $(h+1)$-subspaces through a point of PG(n,q) is an $(h+1)$-cover of $\Gamma(n,h)$. So,

$$A(n,h,0,t) \le \gamma_{n-1,h} \left\lceil \frac{\theta_{h+1}}{\theta_t} \right\rceil. \tag{12}$$

The smallest cardinalities $A(3,1,0,t)$, $t=1,2$, given in Result 1 improve (11) and (12) for the related parameters. Finally, two canonical examples of s-blocking sets will be described. If $h+1 < s \le (n-1)/2$, then every s-G-subspace W is contained in a star. Such W is the set of all h-subspaces in PG(n,q) containing an $(h-1)$-subspace and contained in an $(h+s)$-subspace. Therefore, the collection of all h-subspaces having non-empty intersection with a fixed $(n-h-s)$-subspace is an s-blocking set. In PG(n,q) the set of all h-subspaces skew with a given d-subspace $(d < n-h)$ has size

$$\delta_{n,h,d} = q^{(d+1)(h+1)} \prod_{i=0}^{h} \frac{\theta_{n-(d+1+i)}}{\theta_i},$$

whence

$$A(n,h,s,0) \le \gamma_{n,h} - \delta_{n,h,n-h-s} \quad \text{for } h+1 < s \le (n-1)/2. \tag{13}$$

For $1 \le s \le n-h$, let $U_1, U_2, \ldots, U_s$ be not necessarily distinct $(n-h-1)$-subspaces of PG(n,q). The set K of h-subspaces of PG(n,q) meeting all U_i's is an s-blocking set of $\Gamma(n,h)$. More precisely, K is the intersection of s linear complexes (cf. Section 3). In particular,

$$A(n,h,1,0) \le \gamma_{n,h} - q^{(n-h)(h+1)}. \tag{14}$$

3. Blocking sets of $\Gamma(n, h)$ and linear complexes of PG(n, q)

For every s-blocking set B of PG(N, q) (cf. (1)) the set $(B \cap \mathcal{G}_{n,h})^{\pi^{-1}}$ is an s-blocking set of $\Gamma(n, h)$. In particular, from the theorem of Bose and Burton [4], the 1-blocking sets of minimum cardinality of PG(N, q) are precisely the hyperplanes. If B is a hyperplane H of PG(N, q), then $K = (H \cap \mathcal{G}_{n,h})^{\pi^{-1}}$ is called a *linear complex of h-subspaces* of PG(n, q).

Proposition 7. *The intersection of s linear complexes of h-subspaces of* PG(n, q) *is an s-blocking set of $\Gamma(n, h)$* $(1 \leq s \leq \max\{n - h, h + 1\})$. □

Although a hyperplane of PG(N, q) is a minimal blocking set, a linear complex does not need to be a minimal 1-blocking set of $\Gamma(n, h)$. For example, the set K of all h-subspaces in PG$(2h + 1, q)$ having non-empty intersection with a fixed h-subspace X is a linear complex, but not a minimal 1-blocking set in $\Gamma(2h + 1, h)$, because $K \setminus \{X\}$ is a 1-blocking set.

Let K be a linear complex of h-subspaces of PG(n, q). If S is a star of $\Gamma(n, h)$ and U is the center of S, then one of the following holds:

(i) $S \subseteq K$;
(ii) there exists an hyperplane E of PG(n, q) such that, for every $X \in S$, there holds $X \in K$ if, and only if, $X \subseteq E$.

If (i) holds, then U is called a *singular* $(h-1)$*-subspace* of K; otherwise, E is the *polar hyperplane* of U. If $h = 1$, and K does not contain singular points, then K is called a *general linear complex of lines*. The following properties of linear complexes will be freely used in the paper.

- Every prime of $\Gamma(n, h)$ is a linear complex of h-subspaces of PG(n, q), and conversely [7,16].
- The singular $(h - 1)$-subspaces of a linear complex K form a (possibly non-singular) subspace V of $\Gamma(n, h - 1)$; if $h = 1$, then $\dim V \equiv n \pmod 2$.
- A general linear complex of lines in PG(n, q) exist if, and only if, n is odd. It contains $\theta_n\theta_{n-2}/\theta_1$ lines of PG(n, q).
- If $h = 1$, then every linear complex with exactly one singular point contains precisely θ_{n-1}^2/θ_1 lines of PG(n, q) [17].

Proposition 8. *Let K be a linear complex of h-subspaces in* PG(n, F)*. If $1 \leq h \leq n - 1$ and $n + h \equiv 1 \pmod 2$, then each $(h - 2)$-subspace in* PG(n, F) *is contained in a singular $(h - 1)$-subspace.*

Proof. The assertion is true if $n = 3$ or $h = 1$. Now use induction on n, assuming $n > 3$ and $h > 1$. Let Y be an $(h - 2)$-subspace in PG(n, F). If P is any point of Y, and E is a hyperplane not through P, then $K_{E,P}$ is a linear complex of h'-subspaces in an n'-dimensional projective space, where $h' = h - 1$, $n' = n - 1$, $1 \leq h' \leq n' - 1$ and $n' + h' \equiv 1 \pmod 2$. Therefore, there is a singular $(h' - 1)$-subspace of $K_{E,P}$, and a singular $(h - 1)$-subspace of K containing Y. □

The existence of total subspaces, which are the dual objects of singular subspaces, in the classical Grassmannians has been investigated in [1,13].

In [17] it is proved that

$$A(n, 1, 1, 0) = \frac{\theta_n\theta_{n-2}}{\theta_1} \quad \text{for odd } n, \tag{15}$$

and the blocking sets of minimum size are precisely the general linear complexes. For $n = 3$ this is Result 1 (i). For even n, there exists a linear complex of lines with exactly one singular point. So,

$$\frac{\theta_n\theta_{n-2}}{\theta_1} < A(n, 1, 1, 0) \leq \frac{\theta_{n-1}^2}{\theta_1} \quad \text{for even } n. \tag{16}$$

Theorem 9 (*[17]*). (i) $A(4, 1, 1, 0) = {\theta_3}^2/\theta_1$.(ii) *The* 1*-blocking sets of minimum size in* $\Gamma(4, 1)$ *are precisely the linear complexes having exactly one singular point.* □

Proposition 10. *Let K be a 1-blocking set of $\Gamma(n,h)$. Then*

$$|K| \geq \frac{\theta_{n-h-1}}{\theta_{n-h}} \prod_{i=0}^{h} \frac{\theta_{n-i}}{\theta_i}. \tag{17}$$

The equality holds if, and only if, K is a linear complex of h-subspaces of $\mathrm{PG}(n,q)$ without singular $(h-1)$-subspaces.

Proof. The inequality (17) is just a particular case of (5). By Proposition 2 the equality holds if, and only if, for every star $S \in \mathcal{S}$, $K \cap S$ is a hyperplane of S. For every G-line L of $\Gamma(n,h)$, let S_L be the unique star containing L. It follows that $L \cap K = L \cap (K \cap S_L)$ is either equal to L or is a point. Therefore, K is a prime of $\Gamma(n,h)$ i.e. a linear complex of h-subspaces of $\mathrm{PG}(n,q)$. Finally, since for every star S, $K \cap S$ is a hyperplane of S, there are no singular $(h-1)$-subspaces. □

Next, it will be proved that for $h = 1$, $s = 2$ and some values of n, the bound (5) is sharp.

Theorem 11. *The equality*

$$A(n,1,2,0) = \frac{\theta_n \theta_{n-3}}{\theta_1} \tag{18}$$

holds in at least the following cases:

(i) $n = 3$;
(ii) *q odd and* $n \equiv 3 \pmod 4$.

Proof. By (5), $A(n,1,2,0) \geq \theta_n \theta_{n-3}/\theta_1$.

(i) Consider a line spread of $\mathrm{PG}(3,q)$.
(ii) Assume $n > 3$.

By Proposition 2, it is enough to find two hyperplanes H_1 and H_2 of $\mathrm{PG}(N,q)$, satisfying the following property:

$$H_1 \cap H_2 \text{ intersects every } (n-1)\text{-subspace contained in } \mathcal{G}_{n,1}, \text{ which is associated with a star, in an } (n-3)\text{-subspace.} \tag{19}$$

Each hyperplane H_i is associated with a null polarity and an $(n+1) \times (n+1)$ skew-symmetric matrix, say A_i. If A_i is singular, then H_i contains one of the above $(n-1)$-subspaces. So, take into account only non-singular matrices with odd n. Let $P(\mathbf{y})$ denote the point of $\mathrm{PG}(n,q)$ of coordinates $\mathbf{y} \in \mathbb{F}_q^{n+1}$. The skew-symmetric matrix A_i induces a map ω_i, mapping $P(\mathbf{y})$ into the hyperplane P^{ω_i} of equation $\mathbf{y}^T A_i \mathbf{x} = 0$. A line of $\mathrm{PG}(n,q)$ passing through $P(\mathbf{y})$ is represented by a G-point of $H_1 \cap H_2$ if, and only if, it is contained in $P^{\omega_1} \cap P^{\omega_2}$, that is if and only if its points satisfy the equations

$$\mathbf{y}^T A_1 \mathbf{x} = \mathbf{y}^T A_2 \mathbf{x} = 0.$$

Therefore, H_1 and H_2 satisfy (19) if, and only if, for every $\mathbf{y} \in \mathbb{F}_q^{n+1} \setminus \{0\}$, $\mathbf{y}^T A_1 \mathbf{x} = 0$ and $\mathbf{y}^T A_2 \mathbf{x} = 0$ represent two distinct hyperplanes. This is equivalent to the non-existence of eigenvalues of $A_2^{-1} A_1$ rational over $\mathbb{F}_q$.

Now assume that q is odd and $n \equiv 3 \pmod 4$. Let $m = (n+1)/4$. Denote by I_m the identity matrix of order m, by σ a non-square element of $\mathbb{F}_q$, and by J_m the following matrix

$$J_m = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & & & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathrm{GL}(m, \mathbb{F}_q).$$

If

$$A_1 = \begin{pmatrix} O & O & -I_m & O \\ O & O & O & I_m \\ I_m & O & O & O \\ O & -I_m & O & O \end{pmatrix}, \qquad A_2 = \begin{pmatrix} O & -J_m & O & O \\ J_m & O & O & O \\ O & O & O & \sigma J_m \\ O & O & -\sigma J_m & O \end{pmatrix},$$

then $A_2^{-1} A_1$ has no eigenvalue in $\mathbb{F}_q$, and the proof is complete. □

It is an open question, whether (18) holds for $n \equiv 1 \pmod 4$ and/or q even.

4. A Bose–Burton type theorem in $\Gamma(5, 2)$

The linear complexes of planes in the five-dimensional projective space over $\mathbb{C}$ were described in [15,3]. In this section, the properties of a special linear complex of planes in $\mathrm{PG}(5, q)$ are dealt with. In order to define such a complex, embed $\mathrm{PG}(5, q)$ in $\mathrm{PG}(5, q^2)$. The *real points* are the points of $\mathrm{PG}(5, q)$, whereas the points of $\mathrm{PG}(5, q^2) \setminus \mathrm{PG}(5, q)$ are *imaginary*. More generally, a subspace of $\mathrm{PG}(5, q^2)$ is *real* if it is the solution set of some simultaneous linear equations with coefficients in $\mathbb{F}_q$. A real subspace U contains imaginary points as well, and if the point $P = \mathbb{F}_{q^2}(x_0, x_1, \dots, x_5)$ belongs to U, so does its *conjugate* point $\overline{P} = \mathbb{F}_{q^2}(x_0^q, x_1^q, \dots, x_5^q)$. An *imaginary subspace* V is a subspace satisfying the condition $V \cap \overline{V} = \emptyset$. So, there are subspaces that are neither real, nor imaginary. Take an imaginary plane ε. The set $\mathcal{F}$ of all real lines which meet ε is a spread of $\mathrm{PG}(5, q)$. In the following proposition, $\pi\colon \Gamma(5, 2) \to \mathrm{PG}(19, q)$ denotes the Plücker embedding.

Proposition 12. *Let J be the set of all real planes having non-empty intersection with ε. Then* (i) *each $\rho \in J$ contains exactly one line of $\mathcal{F}$, and* (ii) *there exists a 17-subspace, say G, of the real space* $\mathrm{PG}(19, q)$ *such that $J = (\mathcal{G}_{5,2,q} \cap G)^{\pi^{-1}}$.*

Proof. (i) If $\rho \in J$, then there exists a point $P \in \varepsilon \cap \rho$. This implies $\overline{P} \in \rho$, and the line $P\overline{P}$ belongs to $\mathcal{F}$. (ii) The set of all planes in $\mathrm{PG}(5, q^2)$ meeting ε is a linear complex K_ε, and $K_\varepsilon = (\mathcal{G}_{5,2,q^2} \cap H)^{\pi^{-1}}$, where H is a hyperplane in $\mathrm{PG}(19, q^2)$. Similarly, let $\overline{H}$ be the hyperplane associated with $\overline{\varepsilon}$. Define $G = H \cap \overline{H}$, which is a real 17-subspace of $\mathrm{PG}(19, q^2)$. A real plane ρ meets ε if and only if it meets $\overline{\varepsilon}$. This implies that $\rho \in J$ if, and only if, ρ^π is a real point in G. □

Now let $H_0 = G \vee \{X\}$, where $X \in \mathcal{G}_{5,2,q} \setminus G$, and $K_0 = \left(\mathcal{G}_{5,2,q} \cap H_0\right)^{\pi^{-1}}$.

Proposition 13. *The singular lines of K_0 are exactly the lines of $\mathcal{F}$.*

Proof. If $\ell \in \mathcal{F}$, then each real plane through ℓ, say ρ, meets ε, whence $\rho \in J \subseteq K_0$. If ℓ' is a real line not in $\mathcal{F}$, then each plane of J through ℓ' is contained in the three-dimensional projective space $(\ell' \vee \varepsilon) \cap (\ell' \vee \overline{\varepsilon})$. This implies that, if W denotes the set of all planes of J through ℓ', and S is the star with center ℓ', then W^π is a line contained in S^π. Since $W^\pi \subseteq G$ and $\dim H_0 = \dim G + 1$, the subspace S^π is not contained in H_0. □

Now it is possible to find the size of K_0. This can be done by a double counting of the pairs (ℓ, ϵ) where ℓ is a line of $\mathrm{PG}(5, q)$, and $\ell \subseteq \epsilon \in K_0$:

$$|K_0| = \theta_3(q^3 + 1)(q^2 + 1). \tag{20}$$

The linear complex K_0 can be characterized by the following Bose–Burton type theorem.

Theorem 14. *The minimum size of a* $(1, 0)$*-blocking set in* $\Gamma(5, 2)$ *is*

$$A(5, 2, 1, 0) = \theta_3(q^3 + 1)(q^2 + 1). \tag{21}$$

In $\Gamma(5, 2)$ *there exists exactly one 1-blocking set of size* $A(5, 2, 1, 0)$ *up to collineations.*

Proof. Theorem 9, (6) and (20) imply (21). Now assume that K_1 is a 1-blocking set of size $A(5, 2, 1, 0)$. By Proposition 3 and Theorem 9, each $(K_1)_{E,P}$ is a linear complex of lines with exactly one singular point. Since each $(K_1)_{E,P}$ is a prime, so is K_1. Therefore, K_1 is a linear complex. On the other hand, the singular lines of K_1 form a spread of PG$(5, q)$, say $\mathcal{F}_1$. Assume that there are precisely m solids in PG$(5, q)$ containing at least two lines of $\mathcal{F}_1$. By a double counting, $m \geq |\mathcal{F}_1|$, with equality if, and only if, $\mathcal{F}_1$ is a normal spread, i.e. the lines of $\mathcal{F}_1$ in each solid D containing two lines of $\mathcal{F}_1$ are a spread of D. The computation of the size of K_1 by means of $m \geq \theta_5/\theta_1$, once again by a double counting of the incident pairs given by an element of K_1 and a dual star, gives

$$|K_1| \geq \frac{\theta_5}{\theta_1\theta_2}(\theta_4\theta_2 + q^3) = A(5, 2, 1, 0),$$

and the equality holds if, and only if, $m = |\mathcal{F}_1|$. So, $\mathcal{F}_1$ is a normal spread. There exists a collineation χ of PG$(5, q)$ such that $\mathcal{F}_1^{\chi} = \mathcal{F}$ [14], with $\mathcal{F}$ as above. Both linear complexes K_0 and K_1^{χ} contain the set J of planes. Let $\delta \in K_1^{\chi} \setminus J$ and $\delta' \in K_0$ be such that three lines of $\mathcal{F}$ not belonging to a common solid meet both δ and δ'. There exists a projectivity of PG$(5, q)$, say χ', stabilizing ε as a set, and such that $\delta^{\chi'} = \delta'$. Since both K_0^{π} and $K_1^{\chi\chi'\pi}$ contain G and have a common point outside G, it holds $K_0^{\pi} = K_1^{\chi\chi'\pi}$, whence $K_0 = K_1^{\chi\chi'}$. □

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