Applied Mathematics Letters 22 (2009) 590–594

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Oscillation criteria for a certain even order neutral difference equation with an oscillating coefficient

Yaşar Bolat [a,*], Ömer Akin [b], Huseyin Yildirim [a]

[a] Department of Mathematics, Faculty of Science & Literatures, Afyon Kocatepe University, 03200-Afyonkarahisar, Turkey
[b] Department of Mathematics, Faculty of Science, TOBB University, Sögütozü-, Ankara, Turkey

ARTICLE INFO

Article history:
Received 28 August 2006
Received in revised form 3 June 2008
Accepted 5 June 2008

Keywords:
Difference equation
Higher order difference equation
Neutral difference equation
Oscillating coefficient
Oscillatory

ABSTRACT

In this paper we are concerned with the oscillation of solutions of a certain higher order linear neutral type difference equation with an oscillating coefficient. We obtain some sufficient criteria for oscillatory behaviour. In particular, the results are new even when $n = 2$ and there are few results in the case of p is an oscillatory function.

1. Introduction

We consider the higher order linear difference equation of the form

$$\Delta^n\left[y(k)+p(k)y(\tau(k))\right]+q(k)y(\sigma(k))=0, \quad \mathbb{N}\ni n\geq 2,\ k\in\mathbb{N} \tag{1}$$

where $\mathbb{N}$ is a set of natural numbers. Throughout this work, we assume that

(i) p is an oscillating function with $\lim_{k\to\infty} p(k)=0$,
(ii) $q(k)\geq 0$ for $k\geq k_0$,
(iii) $\tau(k) < k$ with $\tau(k)\to+\infty$ as $k\to+\infty$ and $\sigma(k)<k$ with $\sigma(k)\to+\infty$ as $k\to+\infty$.

By a solution of Eq. (1), we mean any function $y(k):\mathbb{Z}\to\mathbb{R}$ ($\mathbb{Z}$ and $\mathbb{R}$ are set of integers and real numbers respectively) which is defined for all $k\geq \min_{i\geq 0}\{\tau(i),\sigma(i)\}$ and satisfies Eq. (1) for sufficiently large k. We consider only such solutions which are nontrivial for all large k. As it is customary, a solution $\{y(k)\}$ is said to be oscillatory if the terms $y(k)$ of the sequence are not eventually positive or not eventually negative. Otherwise, the solution is called nonoscillatory. A difference equation is called oscillatory if all of its solutions oscillate. Otherwise, it is nonoscillatory.

Neutral difference equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. Recently, much researches have been carried out on the oscillatory and asymptotic behaviour of solutions of higher order delay and neutral delay type difference equations (see [1] Theorems 22.2, 22.3, 22.4, and Theorem 22.5, [2–12]) where either the function $p(k)\geq 0$ or $p(k)\leq 0$. But the results on oscillation of Eq. (1) when the coefficient p is an oscillatory function are relatively scarce, see [4].

* Corresponding author.
E-mail addresses: yasarbolat@aku.edu.tr (Y. Bolat), omerakin@etu.edu.tr (Ö. Akin), hyildir@aku.edu.tr (H. Yildirim).

0893-9659/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.06.036

The purpose of this paper is to study oscillatory behaviour of solutions of Eq. (1) and to give some comparison results. For the general theory of difference equations, one can refer to [1–3,6,7,9]. Many references for the applications of the difference equations can be found in [6,7,9].

For the sake of convenience, the function $z(k)$ is defined as

$$z(k) = y(k) + p(k)y(\tau(k)). \tag{2}$$

2. Some auxiliary lemmas

Lemma 2.1 *([1], p. 56, 57, Lemma 1.13.1). Let $y(k)$ be defined for $k \geq k_0 \in \mathbb{N}$, and $y(k) > 0$ with $\Delta^n y(k)$ of constant sign for $k \geq k_0$ and not identically zero. Then, there exists an integer m, $0 \leq m \leq n$ with $(n+m)$ odd for $\Delta^n y(k) \leq 0$ and $(n+m)$ even for $\Delta^n y(k) \geq 0$ such that*

(i) *$m \leq n-1$ implies $(-1)^{m+i}\Delta^i y(k) > 0$ for all $k \geq k_0$, $m \leq i \leq n-1$*
(ii) *$m \geq 1$ implies $\Delta^i y(k) > 0$ for all large $k \geq k_0$, $1 \leq i \leq m-1$.*

Lemma 2.2 *([1], p. 57, Lemma 1.13.2). Let $y(k)$ be defined for $k \geq k_0$, and $y(k) > 0$ with $\Delta^n y(k) \leq 0$ for $k \geq k_0$ and not identically zero. Then, there exists a large integer $k_1 \geq k_0$ such that*

$$y(k) \geq \frac{1}{(n-1)!}(k-k_1)^{n-1}\Delta^{n-1}y(2^{n-m-1}k), \quad k \geq k_1$$

where m is defined as in Lemma 2.1. Further, if $y(k)$ is increasing, then

$$y(k) \geq \frac{1}{(n-1)!}\left(\frac{k}{2^{n-1}}\right)^{n-1}\Delta^{n-1}y(k), \quad k \geq 2^{n-1}k_1.$$

3. Main results

Theorem 3.1. *Assume that n is even and every bounded solution of the first order delay difference equation*

$$\Delta z(k) + \frac{1}{(n-1)!}\frac{1}{2^{(n-1)^2+1}}q(k)\sigma^{n-1}(k)z(\sigma(k)) = 0 \tag{C$_1$}$$

is oscillatory. Then every bounded solution of Eq. (1) is either oscillatory or tends to zero as $k \to +\infty$.

Proof. Assume that Eq. (1) has a bounded nonoscillatory solution $y(k)$. Without loss of generality, assume that $y(k)$ is eventually positive (the proof is similar when $y(k)$ is eventually negative). That is, $y(k) > 0$, $y(\tau(k)) > 0$ and $y(\sigma(k)) > 0$ for all $k \geq k_1 \geq k_0$. Further, suppose that $y(k)$ does not tend to zero as $k \to \infty$. By (1) and (2) we have

$$\Delta^n z(k) = -q(k)y(\sigma(k)) \leq 0, \quad k \geq k_1. \tag{3}$$

It follows that $\Delta^a z(k)$ $(a = 0, 1, 2, \ldots, n-1)$ is strictly monotone and eventually of constant sign. Since $y(k)$ is bounded and does not tend to zero as $k \to \infty$, by virtue of (i), $\lim_{k\to\infty} p(k)y(\tau(k)) = 0$. Then we can find a $k_2 \geq k_1$ such that $z(k) = y(k) + p(k)y(\tau(k)) > 0$ eventually and $z(k)$ is also bounded for sufficiently large $k \geq k_2$. Because n is even and $(n+m)$ odd for $\Delta^n z(k) \leq 0$ and $z(k) > 0$ is bounded, by Lemma 2.1, since $m = 1$ (otherwise, $z(k)$ is not bounded) there exists a $k_3 \geq k_2$ such that for $k \geq k_3$

$$(-1)^{i+1}\Delta^i z(k) > 0 \quad (i = 1, 2, \ldots, n-1). \tag{4}$$

In particular, since $\Delta z(k) > 0$ for $k \geq k_3$, $z(k)$ is increasing. Since $y(k)$ is bounded, $\lim_{k\to\infty} p(k)y(\tau(k)) = 0$ by (i). Then there exists a $k_4 \geq k_3$ by (2)

$$y(k) = z(k) - p(k)y(\tau(k)) \geq \frac{1}{2}z(k) > 0$$

for $k \geq k_4$. We may find a $k_5 \geq k_4$ such that for $k \geq k_5$ we have

$$y(\sigma(k)) \geq \frac{1}{2}z(\sigma(k)) > 0. \tag{5}$$

From (3) and (5), we can obtain the result of

$$\Delta^n z(k) + \frac{1}{2}q(k)z(\sigma(k)) \leq 0, \tag{6}$$

for $k \geq k_5$. Since $z(k)$ defined for $k \geq k_2$, and $z(k) > 0$ with $\Delta^n z(k) \leq 0$ for $k \geq k_2$ and not identically zero, applying directly Lemma 2.2 (second part, since $z(k)$ is increasing) in the inequality (6), we obtain that the above inequality (6) becomes

$$\Delta^n z(k) + \frac{1}{2}\frac{1}{(n-1)!}\left(\frac{\sigma(k)}{2^{n-1}}\right)^{n-1} q(k)\Delta^{n-1} z(\sigma(k)) \leq 0, \quad k \geq k_6 \geq k_5.$$

Let us take $u(k) = \Delta^{n-1} z(k) > 0$. Thus $u(k)$ satisfies that for k, which is large enough,

$$\Delta u(k) + \frac{1}{2}\frac{1}{(n-1)!}\left(\frac{1}{2^{n-1}}\right)^{n-1} q(k)\sigma^{n-1}(k)u(\sigma(k)) \leq 0, \quad k \geq k_6 \geq k_5.$$

By a well-known result (see [5, p. 186, Corollary 7.6.1]), the difference equation

$$\Delta u(k) + \frac{1}{(n-1)!}\frac{1}{2^{(n-1)^2+1}} q(k)\sigma^{n-1}(k)u(\sigma(k)) = 0, \quad k \geq k_6 \geq k_5$$

has an eventually positive solution. This contradicts the fact that (C_1) is oscillatory. The proof of Theorem 3.1 is completed. In the case where $y(k)$ is an eventually negative solution, then $-y(k) = x(k)$ will be an eventually positive solution. Thus the proof can made as same as in the case $y(k) > 0$. Hence the proof is completed. □

It is well known (see [2, p. 423, Theorem 6.20.5], [7]) that if $\{q(k)\}$ is a nonnegative sequence of real numbers and

$$\liminf_{k\to\infty} \sum_{s=\sigma(k)}^{k-1} q(s) > \frac{1}{e} \tag{7}$$

is satisfied, then the equation of

$$\Delta y(k) + q(k)y(\sigma(k)) = 0, \quad k \geq k_0 \tag{8}$$

is oscillatory.

Thus, from Theorem 3.1 we can obtain the following corollary.

Corollary 3.2. *If*

$$\liminf_{k\to\infty} \sum_{s=\sigma(k)}^{k-1} \frac{\sigma^{n-1}(s)}{2^{(n-1)^2+1}} q(s) > \frac{(n-1)!}{e} \quad \text{and} \quad \limsup_{k\to\infty} q(k)\sigma^{n-1}(k) > 0, \tag{9}$$

then every bounded solution of Eq. (1) *is either oscillatory or tends to zero as* $k \to \infty$.

When $p(k) \equiv 0$ and $n = 2$, Corollary 3.2 yields that if

$$\liminf_{k\to\infty} \sum_{s=\sigma(k)}^{k-1} \frac{1}{4}\sigma(s)q(s) > \frac{1}{e} \quad \text{and} \quad \limsup_{k\to\infty} q(k)\sigma(k) > 0,$$

then

$$\Delta^2 y(k) + q(k)y(\sigma(k)) = 0, \quad k \geq k_0 \tag{10}$$

is oscillatory. These results have been established in [9,10] and the references cited therein.

Let

$$\phi(k) = \max_{k_0 \leq s \leq k} \sigma(s) \quad \text{and} \quad \phi^{-1}(k) = \sup\{s \geq k_0 : \phi(s) = k\},$$

$$\phi^{-(m+1)}(k) \overset{\triangle}{=} \phi^{-1}(\phi^{-m}(k)) = \sup\{s \geq \phi^{-m}(k_0) : \phi^{-m}(s) = k\}.$$

Set

$$\Phi(k) = \frac{1}{(n-1)!}\frac{1}{2^{(n-1)^2+1}} q(k)\sigma^{n-1}(k).$$

Define a sequence $\{\Phi_m(k)\}$ of functions as follows:

$$\Phi_1(k) = \sum_{s=\phi(k)}^{k-1} \Phi(s), \quad k \geq \phi^{-1}(k_0),$$

$$\Phi_{m+1}(k) = \sum_{s=\phi(k)}^{k-1} \Phi(s)\Phi_m(s), \quad k \geq \phi^{-(m+1)}(k_0),\ m = 1, 2, \ldots, \tag{11}$$

then there exists s positive integer M such that

$$\Phi_M(k) = \left[\sum_{s=\phi(k)}^{k-1} \Phi(s)\right]^M, \quad k \geq \phi^{-(m+1)}(k_0).$$

Theorem 3.3. *Assume that*

$$\liminf_{k\to\infty} \Phi_M(k) > \frac{1}{e^M} \tag{C_2}$$

satisfied. Then every bounded solution of Eq. (1) is either oscillatory or tends to zero as $k \to \infty$.

Proof. Let $\liminf_{k\to\infty} \Phi_M(k) = c > 0$. In view of ($C_2$) we can write $ce^M > 1$, that is,

$$\liminf_{k\to\infty} \Phi_M(k) > \frac{1}{e^M}. \tag{12}$$

Suppose, for the sake of contradiction, that (1) has an eventually positive solution $y(k)$. Let $z(k) = y(k) + p(k)y(\tau(k))$. For the rest of the proof, we can proceed as in the proof of Theorem 3.1. Thus, we have the following equality

$$\Delta z(k) + \frac{1}{(n-1)!}\frac{1}{2^{(n-1)^2+1}} q(k)\sigma^{n-1}(k) z(\sigma(k)) = 0 \tag{13}$$

which has an eventually positive solution. On the other hand, by [2, p. 421, Theorem 6.20.2] and (12), all the solutions of (13) are oscillatory. This is a contradiction. In the case where $y(k)$ is an eventually negative solution, then $-y(k) = x(k)$ will be an eventually positive solution. Thus the proof can made as same as in the case $y(k) > 0$. Hence the proof of Theorem 3.3 is completed. □

Corollary 3.4. *Assume that the condition (C_2) is satisfied, then the Eq. (10) is oscillatory.*

Example 3.1. Consider equation of the form

$$\Delta^2\left[y(k) + \left(-\frac{1}{2}\right)^k y(k-1)\right] + \left(4 - \frac{1}{4}\left(-\frac{1}{2}\right)^k\right) y(k-3) = 0. \tag{14}$$

Every bounded solution of the first order delay difference equation

$$\Delta u(k) + \left(1 - \frac{1}{16}\left(-\frac{1}{2}\right)^k\right)(k-3)u(k-3) = 0, \quad \text{for all sufficiently large } k$$

is oscillatory.

Because of $\liminf_{k\to\infty}\left[\sum_{s=k-3}^{k-1}\left(1 - \frac{1}{16}\left(-\frac{1}{2}\right)^s\right)(s-3)\right] > \left(\frac{3}{4}\right)^4$ [5, p. 179, Theorem 7.5.1]. Also, for $M = 1$,

$$\liminf_{k\to\infty} \Phi_1 = \liminf_{k\to\infty}\left[\sum_{s=k-3}^{k-1}\left(1 - \frac{1}{16}\left(-\frac{1}{2}\right)^s\right)(s-3)\right] > \frac{1}{e^1}$$

for $k \geq \phi^{-1}(k_0) = k_0 + 3$. Thus all conditions of Theorems 3.1 and 3.3 are satisfied. Hence every bounded solution of the Eq. (14) is oscillatory. Its one of such solution is $y(k) = (-1)^k$.

Acknowledgements

The authors are grateful to two reviewers for the comments and suggestions.

References

[1] R.P. Agarwal, P.J.Y. Wong, Advanced Topics in Difference Equations, Kluwer, Dordrecht, 1997.
[2] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation Theory for Difference and Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 2000.
[3] R.P. Agarwal, Difference Equation and Inequalities, Marcel Dekker, New York, 2000.
[4] Y. Bolat, Ö. Akin, Oscillatory behaviour of a higher order nonlinear neutral type functional difference equation with oscillating coefficients, Applied Mathematics Letters 17 (2004) 1073–1078.
[5] G. Grzegorczyk, J. Werbowski, Oscillation of higher-order linear difference equations, Computers and Mathematics with Applications 42 (2001) 711–717.
[6] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

[7] W.G. Kelley, A.C. Peterson, Difference Equations an Introduction with Applications, Academic Press, Boston, 1991.
[8] G. Ladas, Ch.G. Philos, Y.G. Sficas, Sharp conditions for the oscillation of delay difference equations, Journal of Applied Mathematical Simulation 2 (1989) 101–111.
[9] V. Lakshmikantham, D. Trigiante, Theory of Difference Equations, Numerical Methods and Applications, Academic Press, New York, 1988.
[10] X. Li, J. Jiang, Oscillation of second-order linear difference equations, Mathematical and Computer Modelling 35 (2002) 983–990.
[11] Z. Luo, J. Shen, New results for oscillation of delay difference equations, Computers and Mathematics with Applications 41 (2001) 553–561.
[12] B.G. Zhang, Yong Zhou, Oscillations of difference equations with several delays, Computers and Mathematics with Applications 44 (2002) 817–821.