ACADEMIC PRESS
Available online at www.sciencedirect.com
SCIENCE DIRECT
Advances in Mathematics 180 (2003) 64–86
ADVANCES IN Mathematics
http://www.elsevier.com/locate/aim

Independence of ℓ in Lafforgue's theorem

CheeWhye Chin[1]

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Received 30 November 2001; accepted 24 June 2002

Communicated by Johan De Jong

Abstract

Let X be a smooth curve over a finite field of characteristic p, let $\ell \neq p$ be a prime number, and let $\mathscr{L}$ be an irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X whose determinant is of finite order. By a theorem of L. Lafforgue, for each prime number $\ell' \neq p$, there exists an irreducible lisse $\bar{\mathbb{Q}}_{\ell'}$-sheaf $\mathscr{L}'$ on X which is compatible with $\mathscr{L}$, in the sense that at every closed point x of X, the characteristic polynomials of Frobenius at x for $\mathscr{L}$ and $\mathscr{L}'$ are equal. We prove an "independence of ℓ" assertion on the fields of definition of these irreducible ℓ'-adic sheaves $\mathscr{L}'$: namely, that there exists a number field F such that for any prime number $\ell' \neq p$, the $\bar{\mathbb{Q}}_{\ell'}$-sheaf $\mathscr{L}'$ above is defined over the completion of F at one of its ℓ'-adic places.

MSC: 14G10; 14F20; 14G13; 14G15

0. Introduction

In the recent spectacular work [L], L. Lafforgue has proved the Langlands Correspondence and the Ramanujan–Petersson conjecture for GL_r over function fields. As a consequence, he has also established the following fundamental result concerning irreducible lisse ℓ-adic sheaves on curves over finite fields.

Theorem (L. Lafforgue [L, Théorème VII.6]). *Let X be a smooth curve over a finite field of characteristic p. Let $\ell \neq p$ be a prime number, and let $\mathscr{L}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X, which is irreducible, of rank r, and whose determinant is of finite order.*

E-mail address: cchin@math.princeton.edu, cheewhye@math.berkeley.edu.
[1] Present address: Department of Mathematics, University of California, Berkeley, CA 94720, USA

doi:10.1016/S0001-8708(02)00082-8

(1) *There exists a number field* $E \subset \bar{\mathbb{Q}}_\ell$ *such that for every closed point* x *of* X, *the polynomial*

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{L})$$

has coefficients in E.

(2) *Let* x *be a closed point of* X, *and let* $\alpha \in \bar{\mathbb{Q}}_\ell$ *be an eigenvalue of Frobenius at* x *acting on* $\mathscr{L}$, *i.e.* $1/\alpha$ *is a root of the polynomial*

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{L}).$$

Then:

(a) α *is an algebraic number*;

(b) *for every archimedean absolute value* $|\cdot|$ *of* $E(\alpha)$, *one has*

$$|\alpha| = 1;$$

(c) *for every non-archimedean valuation* λ *of* $E(\alpha)$ *not lying over* p, α *is a* λ*-adic unit*, *i.e. one has*

$$\lambda(\alpha) = 0;$$

(d) *for every non-archimedean valuation* v *of* $E(\alpha)$ *lying over* p, *one has*

$$\left|\frac{v(\alpha)}{v(\#\kappa(x))}\right| \leqslant \frac{(r-1)^2}{r}.$$

(3) *For any place* λ' *of* E *lying over a prime number* $\ell' \neq p$, *and for any algebraic closure* $\bar{\mathbb{Q}}_{\ell'}$ *of the completion* $E_{\lambda'}$ *of* E *at* λ', *there exists a lisse* $\bar{\mathbb{Q}}_{\ell'}$*-sheaf* $\mathscr{L}'$ *on* X, *which is irreducible*, *of rank* r, *such that for every closed point* x *of* X, *one has*

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{L}') = \det(1 - T\,\mathrm{Frob}_x, \mathscr{L}) \quad (\text{equality in } E[T]).$$

Moreover, *the sheaf* $\mathscr{L}'$ *is defined over a finite extension of* $E_{\lambda'}$.

In part (3) of Lafforgue's theorem, it is not a priori clear that the number field E may be replaced by a finite extension (in $\bar{\mathbb{Q}}_\ell$) so that the various $\bar{\mathbb{Q}}_{\ell'}$-sheaves $\mathscr{L}'$ form an (E, Λ)-*compatible system* in the sense of Katz (cf. [K, pp. 202–203, "The notion of (E, Λ)-compatibility"]), or equivalently, that they form an E-*rational system of* λ-*adic representations* in the sense of Serre (cf. [Se, Sections 2.3 and 2.5]). The existence of a number field with this property may be interpreted as an "independence of ℓ" assertion on the fields of definition of these irreducible ℓ'-adic sheaves $\mathscr{L}'$. We shall prove that this is indeed the case.

Theorem. *With the notation and hypotheses of Lafforgue's Theorem, the following assertion holds.*

(3′) *There exists a finite extension F of E in $\bar{\mathbb{Q}}_\ell$ such that for any place λ' of the number field F lying over a prime number $\ell' \neq p$, there exists a lisse $F_{\lambda'}$-sheaf $\mathscr{L}'$ on X (i.e. a lisse $\bar{\mathbb{Q}}_{\ell'}$-sheaf defined over $F_{\lambda'}$), which is absolutely irreducible, of rank r, such that for every closed point x of X, one has*

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{L}') = \det(1 - T\,\mathrm{Frob}_x, \mathscr{L}) \quad \text{(equality in } E[T]).$$

According to a conjecture of Deligne (cf. [D, Conjecture (1.2.10)]), all four assertions (1), (2), (3), (3′) should also hold in the general case when X is a normal variety of arbitrary dimension over a finite field. Our proof of assertion (3′) uses assertions (1) and (3) of Lafforgue's Theorem only as "black boxes"; so assertion (3′) will hold for higher-dimensional varieties *if* parts (1) and (3) of Lafforgue's Theorem hold for these varieties. To state this more precisely, we make assertions (1) and (3) into hypotheses, as follows:

Definition. Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $\ell \neq p$ be a prime number. Let Y be a normal variety over $\mathbb{F}_q$, and let $\mathscr{F}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on Y, which is irreducible, and whose determinant is of finite order. We shall say that *hypothesis* (1) *holds for* $(Y, \mathscr{F})$ if:

(1) there exists a number field $E \subset \bar{\mathbb{Q}}_\ell$ such that for every closed point y of Y, the polynomial

$$\det(1 - T\,\mathrm{Frob}_y, \mathscr{F})$$

has coefficients in E.

When hypothesis (1) holds for $(Y, \mathscr{F})$, we shall say that *hypothesis* (3) *holds for* $(Y, \mathscr{F})$ if:

(3) for any place λ' of E lying over a prime number $\ell' \neq p$, and for any algebraic closure $\bar{\mathbb{Q}}_{\ell'}$ of the completion $E_{\lambda'}$ of E at λ', there exists a lisse $\bar{\mathbb{Q}}_{\ell'}$-sheaf $\mathscr{F}'$ on Y, which is irreducible, such that for every closed point y of Y, one has

$$\det(1 - T\,\mathrm{Frob}_y, \mathscr{F}') = \det(1 - T\,\mathrm{Frob}_y, \mathscr{F}) \quad \text{(equality in } E[T]).$$

With this definition, our goal is to prove:

Main Theorem. *Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $\ell \neq p$ be a prime number. Let X be a normal variety over $\mathbb{F}_q$. Assume that*:

for any normal variety Y over $\mathbb{F}_q$ which is finite etale over X, and for any lisse $\bar{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}$ on Y, which is irreducible, and whose determinant is of finite order, hypotheses (1) *and* (3) *hold for the pair* $(Y, \mathscr{F})$.

Let $\mathscr{L}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X, which is irreducible, of rank r, and whose determinant is of finite order. Let $E \subset \bar{\mathbb{Q}}_\ell$ denote the number field given by hypothesis (1) *applied to* $(X, \mathscr{L})$. *Then*:

(3′) *There exists a finite extension F of E in $\bar{\mathbb{Q}}_\ell$ such that for any place λ' of the number field F lying over a prime number $\ell' \neq p$, there exists a lisse $F_{\lambda'}$-sheaf $\mathscr{L}_{\lambda'}$ on X, which is absolutely irreducible, of rank r, such that for every closed point x of X, one has*

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{L}_{\lambda'}) = \det(1 - T\,\mathrm{Frob}_x, \mathscr{L}) \quad \text{(equality in } E[T]).$$

We shall prove this theorem by exploiting properties of the *monodromy groups* associated to these irreducible lisse sheaves. The proof begins in Section 4, after a discussion of the preliminary results we need: Propositions 1 and 2 of Section 1, Corollary 6 of Section 2, and Propositions 7 and 9 of Section 3.

1. Monodromy groups

In this section, we recall some basic properties of monodromy groups of lisse ℓ-adic sheaves on varieties over a finite field; see [D, Sections 1.1 and 1.3] for details.

Let X be a normal, geometrically connected variety over a finite field $\mathbb{F}_q$ of characteristic p. Let $\bar{\eta} \to X$ be a geometric point of X, and let $\bar{\mathbb{F}}_q$ be the algebraic closure $\mathbb{F}_q$ in $\kappa(\bar{\eta})$; we regard $\bar{\eta}$ also as a geometric point of $X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. The profinite groups $\pi_1(X, \bar{\eta})$ and $\pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta})$ are respectively called the *arithmetic fundamental group* of X and the *geometric fundamental group* of X. They sit in a short exact sequence

$$1 \to \pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta}) \to \pi_1(X, \bar{\eta}) \overset{\deg}{\to} \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1.$$

The group $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ has a canonical topological generator $\mathrm{Frob}_{\mathbb{F}_q}$ called the *geometric Frobenius*, which is defined as the inverse of the *arithmetic Frobenius* automorphism $a \mapsto a^q$ of the field $\bar{\mathbb{F}}_q$. We have the canonical isomorphism

$$\hat{\mathbb{Z}} \overset{\cong}{\to} \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q), \quad \text{sending 1 to } \mathrm{Frob}_{\mathbb{F}_q}.$$

For a prime number $\ell \neq p$, the functor

$$\begin{array}{ccl} \{\text{lisse } \bar{\mathbb{Q}}_\ell\text{-sheaves on } X\} & \to & \{\text{finite-dimensional continuous} \\ & & \bar{\mathbb{Q}}_\ell\text{-representations of } \pi_1(X, \bar{\eta})\} \\ \mathscr{L} & \mapsto & \mathscr{L}_{\bar{\eta}} \end{array}$$

is an equivalence of categories; a similar statement holds with $X\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q$ in place of X. Via this equivalence, standard notions associated to representations (e.g. irreducibility, semisimplicity, constituent, etc.) are also applicable to lisse sheaves.

Let $\mathscr{L}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X, corresponding to the continuous monodromy representation

$$\pi_1(X,\bar{\eta})\to \mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

of the arithmetic fundamental group of X. The *arithmetic monodromy group* $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ of $\mathscr{L}$ is the Zariski closure of the image of $\pi_1(X,\bar{\eta})$ in $\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$. The inverse image $\mathscr{L}\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q$ of $\mathscr{L}$ on $X\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q$ is a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on $X\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q$, corresponding to the continuous monodromy representation

$$\pi_1(X\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q,\bar{\eta})\hookrightarrow \pi_1(X,\bar{\eta})\to \mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

of the geometric fundamental group of X, obtained by restriction. The *geometric monodromy group* $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ of $\mathscr{L}$ is the Zariski closure of the image of $\pi_1(X\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q,\bar{\eta})$ in $\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$.

Both $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ and $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ are linear algebraic groups, and it is clear that $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ is a closed normal subgroup of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$. Both $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ and $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ are given with a faithful representation on $\mathscr{L}_{\bar{\eta}}$ corresponding to their realizations as subgroups of $\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$. Thus, if $\mathscr{L}$ is semisimple (as a representation of $\pi_1(X,\bar{\eta})$, and therefore as a representation of $\pi_1(X\otimes_{\mathbb{F}_q}\bar{\mathbb{F}}_q,\bar{\eta})$), then both $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ and $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ are (possibly non-connected) reductive algebraic groups.

Proposition 1. *Let $\mathscr{L}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X.*

(i) *If $\mathscr{L}$ is semisimple, then $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ is a (possibly non-connected) semisimple algebraic group.*
(ii) *If $\mathscr{L}$ is irreducible, and its determinant is of finite order, then $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ is a (possibly non-connected) semisimple algebraic group, containing $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ as a normal subgroup of finite index.*

Assertion (i) is [D, Corollaire (1.3.9)]. For the proof of assertion (ii), we shall make use of the construction in [D, (1.3.7)], which we summarize below.

Recall that the *Weil group* $\mathrm{W}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ of $\mathbb{F}_q$ is the subgroup of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ consisting of integer-powers of $\mathrm{Frob}_{\mathbb{F}_q}$; it is considered as a topological group given with the discrete topology, and we have the canonical isomorphism

$$\mathbb{Z}\xrightarrow{\cong}\mathrm{W}(\bar{\mathbb{F}}_q/\mathbb{F}_q),\quad \text{sending 1 to }\mathrm{Frob}_{\mathbb{F}_q}.$$

The *Weil group* $\mathrm{W}(X,\bar{\eta})$ of X is the preimage of $\mathrm{W}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ in $\pi_1(X,\bar{\eta})$ by the degree homomorphism $\pi_1(X,\bar{\eta})\xrightarrow{\deg}\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$; it is considered as a topological group given with the product topology via the isomorphism

$$\mathrm{W}(X,\bar{\eta}) \cong \pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta}) \rtimes_{\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)} \mathrm{W}(\bar{\mathbb{F}}_q/\mathbb{F}_q),$$

where $\pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta})$ retains its profinite topology, and is an open and closed subgroup of $\mathrm{W}(X,\bar{\eta})$. These groups sit in the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \bar{\eta}) & \longrightarrow & \mathrm{W}(X,\bar{\eta}) & \xrightarrow{\deg} & \mathbb{Z} \cong \mathrm{W}(\overline{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1 \\
 & & \| & & \cap\downarrow & & \cap\downarrow & & \\
1 & \longrightarrow & \pi_1(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \bar{\eta}) & \longrightarrow & \pi_1(X,\bar{\eta}) & \xrightarrow{\deg} & \widehat{\mathbb{Z}} \cong \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1
\end{array}$$

where the right two vertical arrows are inclusion homomorphisms with dense images. (Note that the topologies of $\mathrm{W}(X,\bar{\eta})$ and $\mathrm{W}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ are not the ones induced by the right two vertical arrows!)

Given a lisse $\bar{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}$ on X, the *push-out construction* of [D, (1.3.7)] produces an algebraic group $\mathbf{G}(\mathscr{L},\bar{\eta})$, which is locally of finite type, but not quasi-compact; it is characterized by the fact that it sits in a diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \bar{\eta}) & \longrightarrow & \mathrm{W}(X,\bar{\eta}) & \xrightarrow{\deg} & \mathbb{Z} \cong \mathrm{W}(\overline{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta}) & \longrightarrow & \mathbf{G}(\mathscr{L},\bar{\eta}) & \xrightarrow{\deg} & \mathbb{Z} \cong \mathrm{W}(\overline{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1 \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{GL}(\mathscr{L}_{\bar{\eta}}) & & & &
\end{array}$$

such that the composite of the two continuous homomorphisms

$$\mathrm{W}(X,\bar{\eta}) \to \mathbf{G}(\mathscr{L},\bar{\eta}) \to \mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

is equal to the continuous representation of $\mathrm{W}(X,\bar{\eta})$ on $\mathscr{L}_{\bar{\eta}}$ obtained via restriction:

$$\mathrm{W}(X,\bar{\eta}) \hookrightarrow \pi_1(X,\bar{\eta}) \to \mathrm{GL}(\mathscr{L}_{\bar{\eta}}).$$

Proof of Proposition 1 (ii). From assertion (i), we already know that the group $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ is a semisimple closed normal subgroup of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$. Hence, to prove assertion (ii), it suffices for us to show that $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ contains $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$ as a subgroup of finite index, for then both groups will have the same identity component, which is a connected semisimple algebraic group.

Since $\mathrm{W}(X,\bar{\eta}) \hookrightarrow \pi_1(X,\bar{\eta})$ is an inclusion with dense image, $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ can also be described as the Zariski closure of the image of $\mathrm{W}(X,\bar{\eta})$ in $\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$; likewise, since $\mathrm{W}(X,\bar{\eta}) \hookrightarrow \mathbf{G}(\mathscr{L},\bar{\eta})$ is an inclusion with dense image, $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ is also equal to the

Zariski closure of the image of $\mathbf{G}(\mathscr{L},\bar{\eta})$ in $\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$. Let

$$\rho:\ \mathbf{G}(\mathscr{L},\bar{\eta})\to\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

denote the canonical homomorphism from $\mathbf{G}(\mathscr{L},\bar{\eta})$ into $\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$; then the composite map

$$\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})\hookrightarrow\mathbf{G}(\mathscr{L},\bar{\eta})\overset{\rho}{\to}\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

is just the identity map on $\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta})$. We are thus reduced to showing that $\rho^{-1}(\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta}))$ is a subgroup of $\mathbf{G}(\mathscr{L},\bar{\eta})$ of finite index.

The fundamental fact we need about $\mathbf{G}(\mathscr{L},\bar{\eta})$ is [D, Corollaire (1.3.11)], which asserts that because $\mathscr{L}$ is irreducible (hence semisimple) by hypothesis, there exists some element g in the center of $\mathbf{G}(\mathscr{L},\bar{\eta})$ whose degree is >0 (i.e. g maps to a positive integer under $\mathbf{G}(\mathscr{L},\bar{\eta})\overset{\deg}{\to}\mathbb{Z}\cong\mathrm{W}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$). Therefore, $\rho(g)$ is an element of $\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$ which centralizes $\rho(\mathbf{G}(\mathscr{L},\bar{\eta}))$, and so it centralizes $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$. Since $\mathscr{L}$ is irreducible as a representation of $\pi_1(X,\bar{\eta})$ and hence as a representation of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$, it follows that $\rho(g)$ must be a scalar.

By hypothesis, the determinant of $\mathscr{L}$ is of finite order, which means that the one-dimensional representation of $\pi_1(X,\bar{\eta})$ on the determinant $\det(\mathscr{L}_{\bar{\eta}})$ of $\mathscr{L}_{\bar{\eta}}$ is given by a character of finite order, say d. The same is therefore true for $\det(\mathscr{L}_{\bar{\eta}})$ as a representation of $\mathrm{W}(X,\bar{\eta})$ and of $\mathbf{G}(\mathscr{L},\bar{\eta})$. From this it follows that, if $\mathscr{L}$ has rank r, then $\rho(g)$ is a scalar which is a root of unity of order dividing dr, and so $g^{dr}\in\mathbf{G}(\mathscr{L},\bar{\eta})$ lies in the kernel of ρ. Hence $\rho^{-1}(\mathrm{G}_{\mathrm{geom}}(\mathscr{L},\bar{\eta}))$ contains $\deg^{-1}(\deg(g^{dr}))$ in $\mathbf{G}(\mathscr{L},\bar{\eta})$, which is of finite index in $\mathbf{G}(\mathscr{L},\bar{\eta})$. □

Let $\mathscr{L}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}$ on X. Its arithmetic monodromy group $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$ contains the identity component $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})^0$ as an open normal subgroup; $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})^0$ is a connected algebraic group. The faithful representation

$$\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})\hookrightarrow\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$, when restricted to the subgroup $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})^0$ of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})$, gives a faithful representation

$$\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})^0\hookrightarrow\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})\hookrightarrow\mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})^0$ on $\mathscr{L}_{\bar{\eta}}$. We say that the lisse sheaf $\mathscr{L}$ is *Lie-irreducible* if $\mathscr{L}_{\bar{\eta}}$ is irreducible as a representation of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L},\bar{\eta})^0$. It is clear that Lie-irreducibility implies irreducibility.

Proposition 2. *Let $\mathscr{L}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X, which is Lie-irreducible, and whose determinant is of finite order. Then there exist $\alpha\in\bar{\mathbb{Q}}_\ell$ and a closed point x_0 of X, such* *that α is an eigenvalue of multiplicity one of* Frob_{x_0} *acting on* $\mathscr{L}$; *i.e.* $1/\alpha$ *is a root of multiplicity one of the polynomial*

$$\det(1 - T\,\mathrm{Frob}_{x_0}, \mathscr{L}).$$

Proof. First, we claim that it is a Zariski-open condition for an element of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$ to have an eigenvalue of multiplicity one on $\mathscr{L}_{\bar{\eta}}$; in other words, we claim that the set

$$U := \{g \in \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta}) \colon g \text{ acting on } \mathscr{L}_{\bar{\eta}} \text{ has an eigenvalue of multiplicity one in } \bar{\mathbb{Q}}_\ell\}$$

is a *Zariski-open* subset of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$. We show this as follows. For an element $g \in \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$, let $\mathrm{ch}(g) \in \bar{\mathbb{Q}}_\ell[T]$ denote the characteristic polynomial of g; then the set U can also be described as

$$U = \{g \in \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta}) \colon\ \mathrm{ch}(g) \in \bar{\mathbb{Q}}_\ell[T] \text{ has a root of multiplicity one in } \bar{\mathbb{Q}}_\ell\}.$$

Let r be the rank of $\mathscr{L}_{\bar{\eta}}$; then ch gives rise to a morphism of $\bar{\mathbb{Q}}_\ell$-varieties

$$\mathrm{ch} :\ \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta}) \to \bar{\mathbb{Q}}_\ell[T]^{\mathrm{monic}}_{\deg r}, \quad g \mapsto \mathrm{ch}(g),$$

where $\bar{\mathbb{Q}}_\ell[T]^{\mathrm{monic}}_{\deg r}$ denotes the affine space of monic polynomials in T of degree r. For $g \in \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$, the polynomial $\mathrm{ch}(g)$ has a root of multiplicity one in $\bar{\mathbb{Q}}_\ell$ if and only if it does not divide the square $\mathrm{ch}(g)'^2$ of its derivative $\mathrm{ch}(g)'$ in $\bar{\mathbb{Q}}_\ell[T]$. Thus it suffices for us to show that the set

$$Z := \{f \in \bar{\mathbb{Q}}_\ell[T]^{\mathrm{monic}}_{\deg r} \colon f \text{ divides } f'^2 \text{ in } \bar{\mathbb{Q}}_\ell[T]\}$$

is Zariski-closed in $\bar{\mathbb{Q}}_\ell[T]^{\mathrm{monic}}_{\deg r}$. But for $f \in \bar{\mathbb{Q}}_\ell[T]^{\mathrm{monic}}_{\deg r}$, the Euclidean division algorithm shows that the remainder of dividing f'^2 by f is a polynomial of degree $<r$ whose coefficients are given by certain (universal) $\mathbb{Z}$-polynomial expressions in terms of the coefficients of f; as the set Z above is precisely the zero-set of these polynomial expressions, it is Zariski-closed.

Next, we claim that the set U above is in fact Zariski-open and *non-empty* in $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$. Indeed, by part (ii) of Proposition 1, $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})^0$ is a connected semisimple algebraic group; the representation $\mathscr{L}_{\bar{\eta}}$ of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})^0$ is irreducible by hypothesis, and so by the representation theory of connected semisimple algebraic groups, it is classified by its highest weight, which occurs with multiplicity one. Thus, a generic element of any maximal torus of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})^0$ lies in U.

Finally, by Čebotarev's density theorem, there exist infinitely many closed points x of X whose Frobenius conjugacy classes $\mathrm{Frob}_x \subset \pi_1(X, \bar{\eta})$ are mapped into U under the monodromy representation of $\pi_1(X, \bar{\eta})$ on $\mathscr{L}_{\bar{\eta}}$. Thus we can pick x_0 to be any one of these closed points of X, and pick $\alpha \in \bar{\mathbb{Q}}_\ell$ to be an eigenvalue of multiplicity one of Frob_{x_0} acting on $\mathscr{L}$. $\square$

Remark. In Proposition 2, it is not enough to just assume that the lisse $\bar{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}$ is irreducible; the assumption that it is *Lie-irreducible* is necessary. If $\mathscr{L}$ is irreducible but not Lie-irreducible, it may happen that every element of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$ acting on $\mathscr{L}_{\bar{\eta}}$ has repeated eigenvalues, which is to say that the set $U \subset \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$ in the proof of the proposition is empty. For a specific example, we may take $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$ to be the finite symmetric group on 6 letters, and take $\mathscr{L}_{\bar{\eta}}$ to be the 16-dimensional irreducible representation of this finite group; such a situation can arise geometrically.

2. Dévissage of representations

Let k be an algebraically closed field of characteristic 0—such as $\bar{\mathbb{Q}}_\ell$. In this section, we consider (possibly non-connected) reductive groups over k and their finite-dimensional k-rational representations. If G is such a reductive group, any k-rational representation of G is semisimple (a direct sum of irreducible representations), since k is of characteristic 0. By the quasi-compactness of G, a subgroup H of G is (Zariski-) open if and only if it is (Zariski-) closed of finite index, in which case H necessarily contains the identity component G^0 of G.

The following two results are proved in [I] for representations of finite groups. The same proofs, with minor modifications, work for representations of reductive groups. We reproduce the (modified) arguments below for the sake of completeness.

Lemma 3 (I.M. Isaacs [I, Theorem 6.18]). *Let G be a reductive group, and let K and L be open normal subgroups of G, with $L \subseteq K$. Suppose that K/L is abelian, and that there does not exist a normal subgroup M of G with $L \subsetneq M \subsetneq K$. Let π be an irreducible representation of K whose isomorphism class is invariant under G-conjugation. Then one of the following holds*:

(i) $\mathrm{Res}_L^K(\pi)$ *is isomorphic to a direct sum* $\sigma_1 \oplus \cdots \oplus \sigma_t$ *of* $t := [K : L]$ *many irreducible representations* $\sigma_1, \ldots, \sigma_t$ *of L which are pairwise non-isomorphic*;
(ii) $\mathrm{Res}_L^K(\pi)$ *is an irreducible representation of L*;
(iii) $\mathrm{Res}_L^K(\pi)$ *is isomorphic to* $\sigma^{\oplus e}$, *where σ is an irreducible representation of L, and* $e^2 = [K : L]$.

Proof. Since L is normal in K, the irreducible constituents of $\mathrm{Res}_L^K(\pi)$ are K-conjugate to one another, and each of these constituents occurs in $\mathrm{Res}_L^K(\pi)$ with the same multiplicity. Choose any irreducible constituent σ of $\mathrm{Res}_L^K(\pi)$, and let

$$I := \{g \in G\colon {}^g\sigma \cong \sigma \text{ as representations of } L\}$$

be the open subgroup of G (containing L) which stabilizes the isomorphism type of σ under G-conjugation. Since π is invariant under G-conjugation, every G-conjugate of σ is a constituent of $\mathrm{Res}_L^K(\pi)$, and so every G-conjugate of σ is K-conjugate to σ. It follows that $[G:I] = [K:K\cap I]$, and hence $KI = G$. Since K/L is abelian, $K \cap I$ is normal in K; since K is normal in G, $K\cap I$ is normal in I. As $KI = G$, we see that $K \cap I$ is normal in G. From the hypothesis of the proposition, it follows that $K\cap I$ is either L or K.

Suppose $K \cap I = L$. Then there are $t = [K:L]$ many pairwise non-isomorphic irreducible constituents $\sigma = \sigma_1, \dots, \sigma_t$ of $\mathrm{Res}_L^K(\pi)$, and so we have

$$\mathrm{Res}_L^K(\pi) \cong (\sigma_1 \oplus \cdots \oplus \sigma_t)^{\oplus e}$$

for some multiplicity $e \geqslant 1$. The constituents σ_j of $\mathrm{Res}_L^K(\pi)$ are K-conjugate to one another, and so they have the same rank as σ. Hence

$$\mathrm{rk}(\pi) = \mathrm{rk}(\mathrm{Res}_L^K(\pi)) = et\,\mathrm{rk}(\sigma).$$

But π is a constituent of $\mathrm{Ind}_L^K(\sigma)$, so

$$\mathrm{rk}(\pi) \leqslant \mathrm{rk}(\mathrm{Ind}_L^K(\sigma)) = t\,\mathrm{rk}(\sigma).$$

Thus $e = 1$, and this is case (i).

Henceforth suppose $K \cap I = K$. Then σ is invariant under K-conjugation, so we have

$$\mathrm{Res}_L^K(\pi) \cong \sigma^{\oplus e}$$

for some multiplicity $e \geqslant 1$. Let $\chi_1, \dots, \chi_t$ be the distinct linear characters of the abelian group K/L. Then $\chi_1 \otimes \pi, \dots, \chi_t \otimes \pi$ are irreducible representations of K, each having the same rank as π, and we have

$$\mathrm{Res}_L^K(\chi_j \otimes \pi) \cong \sigma^{\oplus e} \quad \text{for each } j = 1, \dots, t.$$

Suppose $\chi_1 \otimes \pi, \dots, \chi_t \otimes \pi$ are pairwise non-isomorphic representations of K. Then we obtain an inclusion

$$\bigoplus_{j=1}^{t} (\chi_j \otimes \pi)^{\oplus e} \subseteq \mathrm{Ind}_L^K(\sigma).$$

Comparing ranks, we get

$$et\,\mathrm{rk}(\pi) \leqslant \mathrm{rk}(\mathrm{Ind}_L^K(\sigma)) = t\,\mathrm{rk}(\sigma),$$

and so

$$e\,\mathrm{rk}(\pi) \leqslant \mathrm{rk}(\sigma).$$

But

$$e\,\mathrm{rk}(\sigma) = \mathrm{rk}(\mathrm{Res}^K_L(\pi)) = \mathrm{rk}(\pi).$$

Thus $e = 1$, and this is case (ii).

In the remaining situation, at least two of the representations $\chi_1 \otimes \pi, \ldots, \chi_t \otimes \pi$ are isomorphic; this implies that $\pi \cong \chi \otimes \pi$ for some non-trivial linear character χ of K/L. Let $M = \mathrm{Ker}(\chi)$; we have $L \subseteq M \subsetneq K$. First, consider the representation π, with trace-function

$$\mathrm{Tr} \circ \pi : K \to k, \quad x \mapsto \mathrm{Tr}(\pi(x)).$$

On $K - M$, the linear character χ takes values different from 1; since $\mathrm{Tr} \circ \pi = \mathrm{Tr} \circ (\chi \otimes \pi) = \chi \cdot (\mathrm{Tr} \circ \pi)$, it follows that $\mathrm{Tr} \circ \pi$ vanishes on $K - M$. Since the representation π is invariant under G-conjugation, it follows that $\mathrm{Tr} \circ \pi$ vanishes on $K - gMg^{-1}$ for all $g \in G$. The normal subgroup $\bigcap_{g \in G} gMg^{-1}$ of G contains L and is properly contained in K, so it must be equal to L by hypothesis. Thus $\mathrm{Tr} \circ \pi$ vanishes on $K - L$. Next, consider the representation $\mathrm{Ind}^K_L(\mathrm{Res}^K_L(\pi)) \cong \mathrm{Ind}^K_L(1) \otimes \pi$, with its trace-function

$$\mathrm{Tr} \circ \mathrm{Ind}^K_L(\mathrm{Res}^K_L(\pi)) : K \to k, \quad x \mapsto \mathrm{Tr}(\mathrm{Ind}^K_L(1)(x))\,\mathrm{Tr}(\pi(x)).$$

Since the trace-function of $\mathrm{Ind}^K_L(1)$ is 0 on $K - L$ and is t on L, it follows that the trace-function of $\mathrm{Ind}^K_L(\mathrm{Res}^K_L(\pi))$ vanishes on $K - L$, and its values on L are t times those of $\mathrm{Tr} \circ \pi$. Comparing the trace-functions of π and $\mathrm{Ind}^K_L(\mathrm{Res}^K_L(\pi))$, we see that

$$\mathrm{Tr} \circ (\pi^{\oplus t}) = \mathrm{Tr} \circ \mathrm{Ind}^K_L(\mathrm{Res}^K_L(\pi)).$$

By the trace comparison theorem of Bourbaki (cf. [B, Section 12, no. 1, Propositon 3]), this implies

$$\pi^{\oplus t} \cong \mathrm{Ind}^K_L(\mathrm{Res}^K_L(\pi))$$

as representations of K. Hence

$$e^2 = \dim \mathrm{Hom}_{\mathrm{L}}(\mathrm{Res}^K_L(\pi), \mathrm{Res}^K_L(\pi)) = \dim \mathrm{Hom}_K(\pi, \mathrm{Ind}^K_L(\mathrm{Res}^K_L(\pi))) = t = [K : L]$$

and this is case (iii). □

Proposition 4 (I.M. Isaacs [I, Theorem 6.22]). *Let G be a reductive group, and let N be an open normal subgroup of G such that G/N is a nilpotent finite group. Let ρ be an irreducible representation of G. Then there exists an open subgroup H of G with $N \subseteq H \subseteq G$, and an irreducible representation σ of H, such that $\rho \cong \mathrm{Ind}^G_H(\sigma)$, and such that $\mathrm{Res}^H_N(\sigma)$ is an irreducible representation of N.*

Remark. The proposition holds in slightly greater generality: we need only to assume that G/N is a solvable finite group whose *chief factors* are of square-free orders; see [I]. This technical condition is automatically verified when G/N is nilpotent or supersolvable.

Proof of Proposition 4. The theorem is clear when $G = N$. We proceed by induction on $\#(G/N)$; hence assume that the theorem holds for any proper subgroup of G containing N. If $\mathrm{Res}_N^G(\rho)$ is irreducible, then the theorem holds with $H = G$ and $\sigma = \rho$. Hence suppose $\mathrm{Res}_N^G(\rho)$ is reducible.

Since G/N is finite, we can find an open normal subgroup K of G which is minimal for the conditions that $N \subseteq K$ and $\mathrm{Res}_K^G(\rho)$ is irreducible. Then $N \subsetneq K$ necessarily, and so we can find an open normal subgroup L of G which is maximal for the conditions that $N \subseteq L \subsetneq K$. Since G/N is nilpotent, it follows that K/L is cyclic of prime order, say t.

The isomorphism class of the irreducible representation $\pi = \mathrm{Res}_K^G(\rho)$ of K is invariant under G-conjugation, since π is the restriction of an irreducible representation ρ of G. Thus we may apply Lemma 3 to the representation π of K. By the choice of L and K, $\mathrm{Res}_L^K(\pi)$ is not irreducible, so case (ii) cannot occur; since $t = [K : L]$ is a prime number, case (iii) cannot occur. Hence we are in case (i), and it follows that $\mathrm{Res}_L^G(\rho)$ is isomorphic to a direct sum $\sigma_1 \oplus \cdots \oplus \sigma_t$ of t many irreducible representations $\sigma_1, \ldots, \sigma_t$ of L which are pairwise non-isomorphic.

Let

$$I := \{g \in G : \ {}^g\sigma_1 \cong \sigma_1 \text{ as representations of } L\}$$

be the open subgroup of G (containing L) which stabilizes the isomorphism type of σ_1 under G-conjugation. Thus $[G : I] = t$ is >1, and $\rho \cong \mathrm{Ind}_I^G(\rho')$ for some irreducible representation ρ' of I. Applying the induction hypothesis to I, we obtain an open subgroup H of I with $N \subseteq H \subseteq I$, and an irreducible representation σ of H, such that $\rho' \cong \mathrm{Ind}_H^I(\sigma)$ and $\mathrm{Res}_N^H(\sigma)$ is an irreducible representation of N. Then $\rho \cong \mathrm{Ind}_H^G(\sigma)$, which completes the proof of the proposition. □

If G is a reductive group over k, we let $\mathrm{K}(G)$ denote the Grothendieck group of the abelian category of finite-dimensional k-rational representations of G. It is clear that $\mathrm{K}(G)$ as a $\mathbb{Z}$-module is freely generated by the irreducible representations of G. The tensor product of representations gives rise to a commutative ring structure on $\mathrm{K}(G)$, whose unit element is the class 1 of the trivial representation of G. If $H \subseteq G$ is an open subgroup, then induction of representations from H to G gives rise to a homomorphism of $\mathbb{Z}$-modules

$$\mathrm{Ind} : \ \mathrm{K}(H) \to \mathrm{K}(G).$$

The projection formula shows that the Ind-image of $\mathrm{K}(H)$ in $\mathrm{K}(G)$ is an ideal.

Recall that, for p a prime number, a finite group G is called *p-elementary* if it is isomorphic to a direct product $A \times B$, where A is a cyclic group of order prime to p, and B is a p-group. A finite group G is called *elementary* if it is p-elementary for some prime number p. It is clear that an elementary finite group is nilpotent.

Let G be a reductive group, and N be an open normal subgroup of G. We say that, for a prime number p, an open subgroup H of G is *p-elementary modulo N* if one has the inclusions $N \subseteq H \subseteq G$ and furthermore the finite quotient H/N is p-elementary; we say that H is *elementary modulo N* if it is p-elementary modulo N for some prime number p.

Proposition 5 (R. Brauer). *Let G be a reductive group, and let N be an open normal subgroup of G. Then the* $\mathbb{Z}$*-homomorphism*

$$\mathrm{Ind} : \bigoplus_{\substack{H \subseteq G \\ \text{elem.mod } N}} \mathrm{K}(H) \to \mathrm{K}(G)$$

is surjective (the direct sum is over all subgroups H of G which are elementary modulo N).

Proof. Recall that Brauer's theorem on induced characters for finite groups (see [I, Theorem 8.4] or [H, Theorem 34.2] for instance) states that if G is a finite group, then the $\mathbb{Z}$-homomorphism

$$\mathrm{Ind} : \bigoplus_{\substack{H \subseteq G \\ \text{elem.}}} \mathrm{K}(H) \to \mathrm{K}(G)$$

is surjective; the key point is that the unit element 1 of $\mathrm{K}(G)$ lies in the ideal generated by the Ind-images of $\mathrm{K}(H)$ where H runs over all elementary subgroups of G. Therefore, the proposition follows from applying Brauer's theorem to the finite group G/N. □

Corollary 6. *Let G be a reductive group, and let N be an open normal subgroup of G. Let ρ be a representation of G. Then there exist a finite list of pairs:*

$$(H_1, \sigma_1), \ldots, (H_s, \sigma_s), \tag{$*$}$$

where, for each $i = 1, \ldots, s$,

(a) H_i *is an open subgroup of G with* $N \subseteq H_i \subseteq G$,
(b) σ_i *is an irreducible representation of* H_i, *and in fact*,
(c) $\mathrm{Res}_N^{H_i}(\sigma_i)$ *is an irreducible representation of N*,

such that one has an isomorphism of representations of G of the form

$$\rho \oplus \left(\bigoplus_{i=1}^{t} \operatorname{Ind}_{H_i}^{G}(\sigma_i) \right) \cong \left(\bigoplus_{j=t+1}^{s} \operatorname{Ind}_{H_j}^{G}(\sigma_j) \right) \qquad (**)$$

for some t with $1 \leqslant t \leqslant s$.

Remark. If one takes N to be the identity component G^0 of G, then property (c) asserts that each σ_i is Lie-irreducible. This is the situation which we shall encounter later in Section 4.

Proof of Corollary 6. Proposition 5 tells us that we can find a finite list of pairs as in $(*)$, such that an isomorphism of form $(**)$ holds, such that properties (a) and (b) are verified, and such that each H_i is elementary modulo N. Since each H_i/N is then a nilpotent finite group, Proposition 4 allows us to replace each H_i by a subgroup containing N and each σ_i by an irreducible representation of the corresponding subgroup, so that, furthermore, property (c) is also verified. This proves the corollary. □

3. Descent of representations

Let Γ be a group, let k_0 be a field of characteristic zero, and let k be a field extension of k_0. In this section, we prove two criteria (Propositions 7 and 9) for descending a k-representation of Γ to a k_0-representation.

Proposition 7. *Let ρ be a finite-dimensional k-representation of Γ, which is absolutely irreducible* (*i.e. irreducible over an algebraic closure of k*). *Assume*:

(i) *ρ is defined over a finite Galois extension K of k_0 in k;*
(ii) *for every $\gamma \in \Gamma$, the trace $\operatorname{Tr}(\rho(\gamma))$ of γ with respect to ρ lies in k_0;*
(iii) *there exists some $\alpha \in k_0$ and some $\gamma_0 \in \Gamma$ such that α is an eigenvalue of multiplicity one of γ_0 with respect to ρ.*

Then ρ is defined over k_0.

Proof. By (i), we may assume that ρ is given as a K-matrix representation of Γ:

$$\rho : \Gamma \to \mathrm{GL}_r(K),$$

and we let $\Sigma = \mathrm{Gal}(K/k_0)$ be the finite Galois group. According to (iii), we may choose an eigenvector $v \in K^{\oplus r}$ of $\rho(\gamma_0)$ with eigenvalue α. By changing basis, we may assume that v is the first basis vectors of $K^{\oplus r}$; thus the matrix $\rho(\gamma_0)$

has the form

$$\begin{pmatrix} \alpha & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}.$$

Each $\sigma \in \Sigma$ defines a K-representation

$$\sigma\rho : \Gamma \xrightarrow{\rho} \mathrm{GL}_r(K) \xrightarrow{\mathrm{GL}_r(\sigma)} \mathrm{GL}_r(K).$$

Since $\alpha \in k_0$ is invariant under Σ, the matrices $\sigma\rho(\gamma_0)$ also have the same form as $\rho(\gamma_0)$ above; thus v is also an eigenvector with eigenvalue α of each $\sigma\rho(\gamma_0)$, $\sigma \in \Sigma$.

Assumption (ii) and the invariance of k_0 under Σ gives the equality in k_0:

$$\mathrm{Tr}(\sigma\rho(\gamma)) = \mathrm{Tr}(\rho(\gamma)) \quad \text{for any } \sigma \in \Sigma, \text{ any } \gamma \in \Gamma.$$

Therefore, by the trace comparison theorem of Bourbaki (cf. [B, Section 12, no. 1, Proposition 3]), the K-representations $\sigma\rho$ of Γ, for various $\sigma \in \Sigma$, are all isomorphic over K to ρ. Choose such isomorphisms over K:

$$a(\sigma) : (\sigma\rho, K^{\oplus r}) \xrightarrow{\cong} (\rho, K^{\oplus r}), \quad \sigma \in \Sigma.$$

Since ρ is absolutely irreducible by hypothesis, any automorphism of it must be a scalar in K. It follows that each $a(\sigma) \in \mathrm{GL}_r(K)$ is determined up to a K-scalar multiple. For any $\sigma, \sigma' \in \Sigma$, the two different ways of expressing $\sigma'\sigma\rho$ in terms of ρ then gives

$$a(\sigma'\sigma) = (\text{scalar in } K) \cdot a(\sigma') \cdot \sigma' a(\sigma).$$

We shall now rigidify the situation. For each $\sigma \in \Sigma$, we have the equality

$$a(\sigma) \cdot \sigma\rho(\gamma_0) = \rho(\gamma_0) \cdot a(\sigma),$$

and the fact that $v \in K^{\oplus r}$ is an eigenvector of $\sigma\rho(\gamma_0)$ with eigenvalue α; it follows that $a(\sigma)v \in K^{\oplus r}$ is an eigenvector of $\rho(\gamma_0)$ with eigenvalue α. Thanks to the multiplicity-one hypothesis (iii) on α, $a(\sigma)v$ is necessarily a K-scalar multiple of v itself. Since we are free to adjust $a(\sigma) \in \mathrm{GL}_r(K)$ by any K-scalar multiple, we may and do assume that each $a(\sigma)$ maps v to itself. Thus the matrices $a(\sigma)$, for $\sigma \in \Sigma$, have the form

$$\begin{pmatrix} 1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix},$$

and it follows that the matrices $\sigma' a(\sigma)$, for $\sigma, \sigma' \in \Sigma$, also have the same form above, which implies that each $\sigma' a(\sigma)$ also maps v to itself. Therefore, we now have

$$a(\sigma'\sigma) = a(\sigma') \cdot \sigma' a(\sigma) \quad \text{for any } \sigma, \sigma' \in \Sigma.$$

By Hilbert Theorem 90 for GL_r, there exists some $b \in \mathrm{GL}_r(K)$ such that

$$a(\sigma) = b \cdot \sigma b^{-1} \quad \text{for each } \sigma \in \Sigma.$$

Using $b^{-1} \in \mathrm{GL}_r(K)$ for a change of basis, we obtain the K-representation $\tilde{\rho} := b^{-1}\rho b$ defined by

$$\tilde{\rho} : \Gamma \to \mathrm{GL}_r(K), \quad \gamma \mapsto b^{-1}\rho(\gamma)b,$$

which is isomorphic over K to ρ. A straightforward computation now shows that the matrices

$$\tilde{\rho}(\gamma) \in \mathrm{GL}_r(K) \quad \text{for } \gamma \in \Gamma,$$

are all fixed under the action of the Galois group Σ; in other words, $\sigma\tilde{\rho} = \tilde{\rho}$ for any $\sigma \in \Sigma$. Thus the representation $\tilde{\rho}$ factorizes as

$$\Gamma \to \mathrm{GL}_r(k_0) \hookrightarrow \mathrm{GL}_r(K).$$

So $\tilde{\rho}$ is defined over k_0, and the same is therefore true for ρ. □

Lemma 8. *Let M, N be k_0-representations of Γ.*

(i) *The canonical homomorphism of k-vector spaces*

$$k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(M, N) \to \mathrm{Hom}_{k\Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N)$$

is injective; it is surjective if M is finitely generated as a left $k_0\Gamma$-module.

(ii) *The canonical homomorphism of k-vector spaces*

$$k \otimes_{k_0} \mathrm{Ext}^1_{k_0\Gamma}(M, N) \to \mathrm{Ext}^1_{k\Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N)$$

is injective if M is finitely generated as a left $k_0\Gamma$-module.

Remark. (a) If M is *finitely presented* as a left $k_0\Gamma$-module, the lemma follows from the well-known "change of rings" isomorphisms applied to $k_0\Gamma \hookrightarrow k\Gamma$ (see [R, Theorem 2.39] for instance). Of course, if M is a finite-dimensional k_0-representation of Γ, then it is automatically a finitely generated left $k_0\Gamma$-module; however, it need not be finitely presented as a left $k_0\Gamma$-module.

(b) When Γ is a finite group, the group ring $k_0\Gamma$ is left-noetherian, so a finite-dimensional k_0-representation M of Γ is finitely presented as a left $k_0\Gamma$-module, and the lemma follows from (a) above. But since we will use the lemma when Γ is a profinite group, and we could not identify a satisfactory reference for the corresponding result, we find it prudent to give a complete proof here.

(c) The proof below actually shows that the lemma holds in slightly greater generality: it suffices to assume that k_0 is any commutative ring, and that k is a k_0-algebra which is *free as a k_0-module*.

Proof of Lemma 8. We first show that the canonical homomorphism

$$\begin{array}{ccc} k\otimes_{k_0}\mathrm{Hom}_{k_0\Gamma}(M,N) & \to & \mathrm{Hom}_{k\Gamma}(k\otimes_{k_0}M, k\otimes_{k_0}N) \\ \alpha\otimes\phi & \mapsto & (\beta\otimes m\mapsto\alpha\beta\otimes\phi(m)) \end{array}$$

is injective. Choose a basis $\{e_i\in k: \ i\in I\}$ of k as a k_0-vector space. Then the $k_0\Gamma$-module $k\otimes_{k_0}N$ is the direct sum of the $k_0\Gamma$-submodules $e_i\otimes N$:

$$k\otimes_{k_0}N\cong\bigoplus_{i\in I} e_i\otimes N;$$

likewise, the k_0-vector space $k\otimes_{k_0}\mathrm{Hom}_{k_0\Gamma}(M,N)$ is the direct sum of the corresponding k_0-subspaces $e_i\otimes\mathrm{Hom}_{k_0\Gamma}(M,N)$:

$$k\otimes_{k_0}\mathrm{Hom}_{k_0\Gamma}(M,N)\cong\bigoplus_{i\in I} e_i\otimes\mathrm{Hom}_{k_0\Gamma}(M,N).$$

Any $\phi\in k\otimes_{k_0}\mathrm{Hom}_{k_0\Gamma}(M,N)$ is therefore equal to a sum

$$\phi=\sum_{i\in I} e_i\otimes\phi_i$$

for some uniquely determined $\phi_i\in\mathrm{Hom}_{k_0\Gamma}(M,N)$, $i\in I$, all but finitely of which are the zero-map. Suppose ϕ lies in the kernel of the canonical homomorphism. Then for any $m\in M$, one has

$$\sum_{i\in I} e_i\otimes\phi_i(m)=0 \qquad \text{in} \qquad k\otimes_{k_0}N\cong\bigoplus_{i\in I} e_i\otimes N,$$

so $\phi_i(m)=0$ in N for each $i\in I$. It follows that $\phi=0$, which is what we want.

If M is finite free as a left $k_0\Gamma$-module, then it follows from the functorial properties of Hom and $\otimes$ that the canonical homomorphism is an isomorphism. In general, if M is finitely generated as a left $k_0\Gamma$-module, let

$$0\to K\to F\to M\to 0$$

be a short exact sequence of left $k_0\Gamma$-modules with F finite free. Then

$$0 \to \mathrm{Hom}_{k_0\Gamma}(M, N) \to \mathrm{Hom}_{k_0\Gamma}(F, N) \to \mathrm{Hom}_{k_0\Gamma}(K, N)$$

is an exact sequence of k_0-vector spaces. From this and the fact that k is flat over k_0, we obtain the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(M, N) & \longrightarrow & \mathrm{Hom}_{k\Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(F, N) & \xrightarrow{\cong} & \mathrm{Hom}_{k\Gamma}(k \otimes_{k_0} F, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(K, N) & \hookrightarrow & \mathrm{Hom}_{k\Gamma}(k \otimes_{k_0} K, k \otimes_{k_0} N) \\
\downarrow & & \downarrow
\end{array}$$

where the middle horizontal arrow is an isomorphism and the bottom horizontal arrow is injective, by what we have already shown. A diagram chase shows that the top horizontal arrow is surjective. This proves part (i).

For part (ii), we write down the next terms in the above commutative diagram:

$$\begin{array}{ccc}
k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(F, N) & \xrightarrow{\cong} & \mathrm{Hom}_{k\Gamma}(k \otimes_{k_0} F, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(K, N) & \hookrightarrow & \mathrm{Hom}_{k\Gamma}(k \otimes_{k_0} K, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
k \otimes_{k_0} \mathrm{Ext}^1_{k_0\Gamma}(M, N) & \longrightarrow & \mathrm{Ext}^1_{k\Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

By part (i), the top horizontal arrow is an isomorphism and the middle horizontal arrow is injective. A diagram chase shows that the bottom horizontal arrow is injective. This proves part (ii). □

Proposition 9 (E. Noether–M. Deuring). *Let ρ, τ and π be semisimple finite-dimensional k-representations of Γ such that*

$$\rho \oplus \tau \cong \pi.$$

Suppose τ and π are defined over k_0. Then ρ is also defined over k_0.

Proof. Our argument here is adapted from that given for representations of finite groups (see [H, Theorem 37.6] for instance). The proposition is clear when $\tau = 0$. We proceed by induction on the rank $\mathrm{rk}(\tau)$ of τ; hence assume that $\mathrm{rk}(\tau) \geqslant 1$. By hypothesis, there exist k_0-representations τ_0, π_0 of Γ such that

$$\tau \cong k \otimes_{k_0} \tau_0, \quad \pi \cong k \otimes_{k_0} \pi_0.$$

For any finite-dimensional k_0-representations M, N of Γ, we have the canonical inclusion:

$$\mathrm{Ext}^1_{k_0\Gamma}(M,N) \hookrightarrow k \otimes_{k_0} \mathrm{Ext}^1_{k_0\Gamma}(M,N) \underset{\text{Lemma } 8}{\hookrightarrow} \mathrm{Ext}^1_{k\Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} M);$$

this fact and the hypothesis that τ, π are semisimple as k-representations of Γ imply that τ_0, π_0 are semisimple as k_0-representations of Γ.

Let $\sigma_0 \subseteq \tau_0$ be an irreducible constituent of the k_0-representation τ_0 of Γ. Then

$$k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(\sigma_0, \pi_0) \underset{\text{Lemma } 8}{\overset{\cong}{\rightarrow}} \mathrm{Hom}_{k\Gamma}(\sigma, \pi) \cong \mathrm{Hom}_{k\Gamma}(\sigma, \rho \oplus \tau)$$

contains

$$\mathrm{Hom}_{k\Gamma}(\sigma, \tau) \underset{\text{Lemma } 8}{\overset{\cong}{\leftarrow}} k \otimes_{k_0} \mathrm{Hom}_{k_0\Gamma}(\sigma_0, \tau_0) \neq 0,$$

whence $\mathrm{Hom}_{k_0\Gamma}(\sigma_0, \pi_0) \neq 0$. Thus σ_0 is also an irreducible constituent of the k_0-representation π_0 of Γ. Therefore,

$$\tau_0 \cong \tau_0' \oplus \sigma_0, \quad \pi_0 \cong \pi_0' \oplus \sigma_0$$

for some semisimple k_0-representations τ_0' and π_0' of Γ. Letting

$$\tau' \coloneqq k \otimes_{k_0} \tau_0', \quad \pi' \coloneqq k \otimes_{k_0} \pi_0', \quad \sigma \coloneqq k \otimes_{k_0} \sigma_0,$$

we obtain an isomorphism

$$\rho \oplus \tau' \oplus \sigma \cong \pi' \oplus \sigma$$

of semisimple k-representations of Γ, and hence an equality of their k-valued trace functions:

$$\mathrm{Tr}(\rho(g)) + \mathrm{Tr}(\tau'(g)) + \mathrm{Tr}(\sigma(g)) = \mathrm{Tr}(\pi'(g)) + \mathrm{Tr}(\sigma(g)) \quad \text{for every } g \in \Gamma.$$

Applying the trace comparison theorem of Bourbaki (cf. [B, Section 12, no. 1, Proposition 3]) to the equality

$$\mathrm{Tr}(\rho(g)) + \mathrm{Tr}(\tau'(g)) = \mathrm{Tr}(\pi'(g)) \quad \text{for every } g \in \Gamma,$$

we obtain an isomorphism

$$\rho \oplus \tau' \cong \pi'$$

of semisimple k-representations of Γ. Since $\mathrm{rk}(\tau') < \mathrm{rk}(\tau)$, our induction hypothesis shows that ρ is defined over k_0. $\square$

4. Proof of main theorem

We shall now prove the main theorem stated in the introduction.

Thus, let $\mathbb{F}_q$ be a finite field of characteristic p, let $\ell \neq p$ be a prime number, let X be a normal variety over $\mathbb{F}_q$, and let $\mathscr{L}$ be a lisse $\bar{\mathbb{Q}}_\ell$-sheaf on X, which is irreducible, and whose determinant is of finite order. Let $E \subset \bar{\mathbb{Q}}_\ell$ denote the number field given by hypothesis (1) applied to $(X, \mathscr{L})$; thus for every closed point x of X, the polynomial

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{L})$$

has coefficients in E. We may replace the finite field $\mathbb{F}_q$ by its algebraic closure in the function field $\kappa(X)$ of X, and hence assume that X is geometrically connected over $\mathbb{F}_q$; this allows us to use the results in Section 1. Let $\bar{\eta} \to X$ be a geometric point of X, and set

$$\Gamma := \pi_1(X, \bar{\eta}), \quad G := \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta}).$$

Let

$$\rho_{\mathscr{L}} : \Gamma \to \mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

denote the monodromy $\bar{\mathbb{Q}}_\ell$-representation of Γ corresponding to $\mathscr{L}$, and let

$$\rho : G \hookrightarrow \mathrm{GL}(\mathscr{L}_{\bar{\eta}})$$

denote the faithful representation of $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})$ on $\mathscr{L}_{\bar{\eta}}$.

By Proposition 1 (ii), G is a (possibly non-connected) semisimple algebraic group. We apply Corollary 6 to the representation ρ of G, with $N := G^0 = \mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})^0$, to obtain a finite list of pairs as in $(*)$, satisfying properties (a)–(c) listed there, such that an isomorphism of representations of G of the form $(**)$ holds.

Consider any pair (H_i, σ_i) in $(*)$. By property (a), the identity component H_i^0 of H_i is a connected semisimple algebraic group (in fact it is $\mathrm{G}_{\mathrm{arith}}(\mathscr{L}, \bar{\eta})^0$), which is therefore equal to its own commutator subgroup; hence the one-dimensional representation $\det(\sigma_i)$ of H_i, given by the determinant of σ_i, factors through H_i/H_i^0, and so is given by a character of H_i of finite order. This and properties (b) and (c) show that each σ_i is a Lie-irreducible representation of H_i, and its determinant is of finite order.

Set

$$\Gamma_i := (\rho_{\mathscr{L}})^{-1}(H_i) \subseteq \Gamma.$$

Then Γ_i is an open subgroup of Γ, corresponding to a finite etale cover $X_i \to X$ of X by a connected variety X_i pointed by the geometric point $\bar{\eta}$; we identify Γ_i with the arithmetic fundamental group $\pi_1(X_i, \bar{\eta})$ of X_i. If V_i is the representation space of σ_i, then the composite homomorphism

$$\sigma_{\mathscr{F}_i} : \Gamma_i \overset{\rho_{\mathscr{L}}}{\to} H_i \overset{\sigma_i}{\to} \mathrm{GL}(V_i)$$

is a $\bar{\mathbb{Q}}_\ell$-representation of Γ_i which corresponds to a lisse $\bar{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}_i$ on the variety X_i. It follows from the corresponding properties of σ_i that $\mathscr{F}_i$ is Lie-irreducible, and its determinant is of finite order. By hypothesis (1) applied to $(X_i, \mathscr{F}_i)$, there is a number field $E_i \subset \bar{\mathbb{Q}}_\ell$ such that for every closed point x of X_i, the polynomial

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{F}_i)$$

has coefficients in E_i; and by Proposition 2, there is some $\alpha_i \in \bar{\mathbb{Q}}_\ell$ and some closed point $x_0^{(i)}$ of X_i such that α_i is an eigenvalue of multiplicity one of $\mathrm{Frob}_{x_0^{(i)}}$ acting on $\mathscr{F}_i$. It follows that α_i is algebraic over the number field E_i.

Let

$$\rho_{\mathscr{L}_i} := \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\sigma_{\mathscr{F}_i})$$

be the $\bar{\mathbb{Q}}_\ell$-representation of Γ induced from $\sigma_{\mathscr{F}_i}$, and let

$$F := \text{composite of } E_1(\alpha_1), \ldots, E_s(\alpha_s) \text{ and } E \text{ in } \bar{\mathbb{Q}}_\ell.$$

It is clear that F is a finite extension of E in $\bar{\mathbb{Q}}_\ell$. The isomorphism $(**)$ implies that for any closed point x of X, one has

$$\mathrm{Tr}(\rho_{\mathscr{L}}(\mathrm{Frob}_x)) + \sum_{i=1}^{t} \mathrm{Tr}(\rho_{\mathscr{L}_i}(\mathrm{Frob}_x)) = \sum_{j=t+1}^{s} \mathrm{Tr}(\rho_{\mathscr{L}_j}(\mathrm{Frob}_x)) \quad (\text{equality in } F \subset \bar{\mathbb{Q}}_\ell). \qquad (***)$$

We shall now show that the number field F satisfies the conclusion of assertion (3′).

To that end, pick a place λ' of F lying over a prime number $\ell' \neq p$, and choose an algebraic closure $\bar{\mathbb{Q}}_{\ell'}$ of $F_{\lambda'}$. By hypothesis (3) applied to $(X, \mathscr{L})$ and each $(X_i, \mathscr{F}_i)$, there exist irreducible lisse $\bar{\mathbb{Q}}_{\ell'}$-sheaves $\mathscr{L}'$ on X and $\mathscr{F}'_i$ on X_i, which are *compatible with* $\mathscr{L}$ *and* $\mathscr{F}_i$, respectively; i.e. for each closed point x of X, one has

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{L}') = \det(1 - T\,\mathrm{Frob}_x, \mathscr{L}) \quad (\text{equality in } F[T]), \qquad (1)$$

and for each $i = 1, \ldots, s$ and each closed point x of X_i, one has

$$\det(1 - T\,\mathrm{Frob}_x, \mathscr{F}'_i) = \det(1 - T\,\mathrm{Frob}_x, \mathscr{F}_i) \quad (\text{equality in } F[T]). \tag{2}$$

It follows that $\mathscr{L}'$ has the same rank as $\mathscr{L}$ (and each $\mathscr{F}'_i$ has the same rank as $\mathscr{F}_i$). It also follows from these compatibility relations that

$$\alpha_i \in F \subset F_{\lambda'} \subset \bar{\mathbb{Q}}_{\ell'} \quad \text{is an eigenvalue of multiplicity one of } \mathrm{Frob}_{x_0^{(i)}} \text{ acting on } \mathscr{F}'_i. \tag{3}$$

Let $\rho_{\mathscr{L}'}$ denote the irreducible monodromy $\bar{\mathbb{Q}}_{\ell'}$-representation of Γ corresponding to $\mathscr{L}'$, and let $\sigma_{\mathscr{F}'_i}$ denote the irreducible monodromy $\bar{\mathbb{Q}}_{\ell'}$-representation of Γ_i corresponding to $\mathscr{F}'_i$. Let

$$\rho_{\mathscr{L}'_i} \coloneqq \mathrm{Ind}_{\Gamma_i}^{\Gamma}(\sigma_{\mathscr{F}'_i})$$

be the $\bar{\mathbb{Q}}_{\ell'}$-representation of Γ induced from $\sigma_{\mathscr{F}'_i}$. From (1) and (2), we deduce that for each closed point x of X, one has

$$\mathrm{Tr}(\rho_{\mathscr{L}'}(\mathrm{Frob}_x)) = \mathrm{Tr}(\rho_{\mathscr{L}}(\mathrm{Frob}_x)) \quad (\text{equality in } F), \tag{4}$$

and for each $i = 1, \ldots, s$ and each closed point x of X_i, one has

$$\mathrm{Tr}(\sigma_{\mathscr{F}'_i}(\mathrm{Frob}_x)) = \mathrm{Tr}(\sigma_{\mathscr{F}_i}(\mathrm{Frob}_x)) \quad (\text{equality in } F), \tag{5}$$

whence for each $i = 1, \ldots, s$ and each closed point x of X, one has

$$\mathrm{Tr}(\rho_{\mathscr{L}'_i}(\mathrm{Frob}_x)) = \mathrm{Tr}(\rho_{\mathscr{L}_i}(\mathrm{Frob}_x)) \quad (\text{equality in } F). \tag{6}$$

Combining equalities (4) and (6) with $(***)$, we see that for any closed point x of X, one has

$$\begin{aligned}\mathrm{Tr}(\rho_{\mathscr{L}'}(\mathrm{Frob}_x)) &+ \sum_{i=1}^{t} \mathrm{Tr}(\rho_{\mathscr{L}'_i}(\mathrm{Frob}_x)) \\ &= \sum_{j=t+1}^{s} \mathrm{Tr}(\rho_{\mathscr{L}'_j}(\mathrm{Frob}_x)) \quad (\text{equality in } F \subset \bar{\mathbb{Q}}_{\ell'}).\end{aligned} \tag{$***'$}$$

By Čebotarev's density theorem, this equality of traces, as an equality in $\bar{\mathbb{Q}}_{\ell'}$, holds for every element of Γ. Therefore, by the trace comparison theorem of Bourbaki (cf. [B, Section 12, no. 1, Proposition 3]), we obtain an isomorphism of semisimple $\bar{\mathbb{Q}}_{\ell'}$-representations of Γ:

$$\rho_{\mathscr{L}'} \oplus \left(\bigoplus_{i=1}^{t} \rho_{\mathscr{L}'_i} \right) \cong \left(\bigoplus_{j=t+1}^{s} \rho_{\mathscr{L}'_j} \right). \tag{7}$$

Consider the (absolutely) irreducible $\bar{\mathbb{Q}}_{\ell'}$-representation $\sigma_{\mathscr{F}'_i}$ of Γ_i. We wish to apply Proposition 7 to this representation; so let us check that the hypotheses there are verified.

(i) By the definition of lisse $\bar{\mathbb{Q}}_{\ell'}$-sheaves (cf. [D, (1.1.1)]—alternatively, apply [KSa, Remark 9.0.7]), the $\bar{\mathbb{Q}}_{\ell'}$-representation $\sigma_{\mathscr{F}'_i}$ is defined over a finite extension of $\mathbb{Q}_{\ell'}$, which we may of course assume to be finite Galois over $F_{\lambda'}$.

(ii) From (5), we see that for every closed point x of X_i, the trace $\mathrm{Tr}(\sigma_{\mathscr{F}'_i}(\mathrm{Frob}_x))$ of $\mathrm{Frob}_x \subset \Gamma_i$ with respect to $\sigma_{\mathscr{F}'_i}$ lies in $F_{\lambda'}$; so it follows from Čebotarev's density theorem that the trace $\mathrm{Tr}(\sigma_{\mathscr{F}'_i}(\gamma))$ of every element $\gamma \in \Gamma_i$ with respect to $\sigma_{\mathscr{F}'_i}$ lies in $F_{\lambda'}$.

(iii) Finally, from (3), we know that $\alpha_i \in F_{\lambda'}$ is an eigenvalue of multiplicity one of $\mathrm{Frob}_{x_0^{(i)}} \subset \Gamma_i$ with respect to $\sigma_{\mathscr{F}'_i}$.

Hence Proposition 7 shows that $\sigma_{\mathscr{F}'_i}$ is defined over $F_{\lambda'}$. Then each $\rho_{\mathscr{L}'_i}$, being induced from $\sigma_{\mathscr{F}'_i}$, is also defined over $F_{\lambda'}$. Therefore, in (7), the two representations in parentheses are defined over $F_{\lambda'}$. Proposition 9 now shows that $\rho_{\mathscr{L}'}$ is also defined over $F_{\lambda'}$, and hence the lisse $\bar{\mathbb{Q}}_{\ell'}$-sheaf $\mathscr{L}'$ is defined over $F_{\lambda'}$; in other words, there exists a lisse $F_{\lambda'}$-sheaf $\mathscr{L}_{\lambda'}$ on X such that $\mathscr{L}' \cong \mathscr{L}_{\lambda'} \otimes_{F_{\lambda'}} \bar{\mathbb{Q}}_{\ell'}$. The asserted properties of $\mathscr{L}_{\lambda'}$ follow from this isomorphism and (1).

This completes the proof of our main theorem.

Acknowledgments

I am grateful to P. Deligne, J. de Jong and especially N. Katz for many fruitful conversations, from which I have learned much of the material presented here.

References

[B] N. Bourbaki, Algèbre. Chapitre 8: Modules et anneaux semi-simples, Hermann, Paris, 1958.

[D] P. Deligne, La conjecture de Weil II, Inst. Hautes Études Sci. Publ. Math. 52 (1980) 137–252.

[H] B. Huppert, Character theory of finite groups, de Gruyter, Berlin, 1998.

[I] I.M. Isaacs, Character theory of finite groups (Corrected reprint of the 1976 original, Academic Press, New York), Dover, New York, 1994.

[K] N.M. Katz, Wild ramification and some problems of "independence of ℓ", Amer. J. Math. 105 (1) (1983) 201–227.

[KSa] N.M. Katz, P. Sarnak, Random matrices, Frobenius eigenvalues, and monodromy, Amer. Math. Soc, Providence, RI, 1999.

[L] L. Lafforgue, Chtoucas de Drinfeld et correspondance de Langlands, Invent. Math. 147 (1) (2002) 1–241.

[R] I. Reiner, Maximal orders, Academic Press, London, 1975.

[Se] J.-P. Serre, Abelian ℓ-adic representations and elliptic curves, Revised reprint of the 1968 original, A K Peters, Wellesley, MA, 1998.