Strong approximations for stochastic differential equations with boundary conditions

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Abstract

We study the Euler approximation scheme for solutions of stochastic differential equations with boundary conditions in two different examples: (a) the one-dimensional case with linear boundary condition, and (b) the multidimensional case with constant diffusion coefficient and general boundary condition. In both cases the error is measured in the $L^p$-norm.

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1 Introduction

This article deals with Stratonovich stochastic differential equations driven by a Wiener process

$$dX_t = \sigma(X_t) \circ dW_t + b(X_t) \, dt, \quad 0 \leq t \leq 1, \quad (1.1)$$

with a boundary condition of the form

$$h(X_0, X_1) = \bar{h}. \quad (1.2)$$

The solution \{X_t, t \in [0, 1]\}, whenever it exists, is a non-adapted stochastic process, because of (1.2).
Our goal is to prove an Euler scheme for these kinds of equations in two different cases considered in Donati–Martin (1991) and Nualart and Pardoux (1991), respectively:

(a) The one-dimensional case with linear boundary condition;
(b) the multidimensional case with constant coefficient $\sigma$ and a general boundary condition.

We prove $L^p$-convergence of the approximation scheme and obtain the rate of convergence (see Theorems 2.1 and 3.2).

Section 2 is devoted to the analysis of case (a). Here we assume $h(X_0, X_1) = F_0X_0 + F_1X_1, F_0F_1 \geq 0$ and, then, $X_t = \Phi_t(X_0), \text{ where } \{\Phi_t(x), t \in [0, 1]\}$ is the stochastic flow associated with (1.1) and $X_0$ is the unique solution to $F_0X_0 + F_1\Phi_1(X_0) = h_0$. Therefore, the approximation problem is related to the strong convergence of an Euler scheme for (1.1) with some given initial condition $X_0$ studied in Ahn and Kohatsu–Higa (1995). Actually, we have to prove that $X_0 \in \cap_{p \geq 1} L^p$ (this condition ensures the assumption of Ahn and Kohatsu–Higa (1995, Theorem 3.1)) and then provide an approximate scheme for $X_0$ as well.

In Section 3 we study case (b). Following Nualart and Pardoux (1991), the solution $X_t$ is obtained by composition of a linearized equation related to (1.1) and a bijective non-linear transformation $T$ of the paths of the Wiener process. The main difficulty here is to find approximations for $T$ and it is solved reducing the problem to one of approximation of stochastic differential equations where the initial condition is the fixed point of some mapping related with $T$.

Along the paper all constants are denoted by $C$ although they may be different from one expression to another.

2. The Stratonovich stochastic differential equation with linear boundary condition

Consider the stochastic Stratonovich differential equation with linear boundary condition

\[
\begin{aligned}
\{dX_t &= \sigma(X_t) \cdot dW_t + b(X_t) \, dt, \quad 0 \leq t \leq 1, \\
F_0X_0 + F_1X_1 &= h_0.
\end{aligned}
\]

Here $\sigma$ and $b$ are real functions, $F_0, F_1, h_0 \in \mathbb{R}$. Set $b_1 = b + \frac{1}{2} \sigma\sigma'$ and suppose that $b_1, \sigma$ are $C^4$ functions with bounded derivatives and $F_0F_1 > 0$. Then, there exists a unique continuous solution to (2.1) and, moreover, this solution belongs to the space $L_c^{1,\text{loc}}$ (see Theorem 4.1 in Donati–Martin, 1991). More precisely, consider the stochastic flow associated with (2.1),

\[
\Phi_t(x) = x + \int_0^t \sigma(\Phi_s(x)) \cdot dW_s + \int_0^t b(\Phi_s(x)) \, ds,
\]

and the function $G(x) = F_0x + F_1\Phi_1(x)$. There exists a unique measurable $X_0: \Omega \to \mathbb{R}$ such that $G(\omega, X_0(\omega)) = h_0$. Furthermore, $X_t := \Phi_t(X_0)$ is the unique continuous solution of Eq. (2.1) in the space $L_c^{1,\text{loc}}$. 
Our aim is to present a strong Euler approximation scheme for (2.1). Taking into account the results of Ahn and Kohatsu–Higa (1995), the program consists in giving numerical approximations for \( X_0 \) and then checking the assumptions of Theorem 3.1 in Ahn and Kohatsu–Higa (1995).

We introduce some notation. Let \( \pi = \{0 = t_0 < t_1 < \cdots < t_{m+1} = 1\} \) be a partition of \([0, 1]\); as usual, we denote by \( \| \pi \| \) its mesh, that is, \( \| \pi \| = \sup_{0 \leq k \leq m} |t_{k+1} - t_k| \).

We define \( \eta : [0, 1] \to \mathbb{R} \) as follows:

\[
\eta(t) = \sum_{k=0}^{m} t_k I_{[t_k, t_{k+1})}(t).
\]

The approximation of (2.2) is given by the solution to the equation

\[
\Phi_t(x) = x + \int_0^t \sigma(\Phi_{\eta(s)}(x)) \, dW_s + \int_0^t b_1(\Phi_{\eta(s)}(x)) \, ds.
\]  

We will define a numerical approximation for \( X_0 \) associated with \( \pi \), which we will call \( \tilde{X}_0 \). Then we will prove

\[
\lim_{\| \pi \| \to 0} E \left[ \sup_{0 \leq t \leq 1} |\Phi_t(X_0) - \Phi_t(\tilde{X}_0)|^p \right] = 0, \quad p > 1
\]

and we will obtain the rate of convergence.

Fix \( \pi \) with \( \| \pi \| = \delta \). Without loss of generality, we can assume that \( \pi \) is a uniform partition. Let \( M > 0 \); Chebychev’s inequality yields

\[
P \left\{ \sup_{0 \leq k \leq m} |W(\Delta_k)| > M \right\} \leq \frac{1}{M^q} \delta^{q/2 - 1}, \quad q \geq 2,
\]  

where \( W(\Delta_k) \) denotes the increment \( W_{t_{k+1}} - W_{t_k}, k = 0, \ldots, m \). From (2.3) we obtain

\[
\Phi_{t_k}(x) = x + \int_0^{t_k} \sigma(\Phi_{\eta(s)}(x)) \, dW_s + \int_0^{t_k} b(\Phi_{\eta(s)}(x)) \, ds.
\]

and, using a recursive argument, \( \Phi_0(x) = 1 \),

\[
\Phi_{t_{k+1}}(x) = \prod_{i=0}^{k} \left( 1 + \sigma'(\Phi_{t_i}(x)) W(\Delta_i) + b'(\Phi_{t_i}(x)) \delta \right),
\]

\( k = 0, \ldots, m \).

Let \( C := \| \sigma' \|_\infty + \| b' \|_\infty \). Choose \( \delta, M > 0 \) satisfying \( \delta \vee M < 1/4C \). Then, on the set \( L_M = \{ \sup_{0 \leq k \leq m} |W(\Delta_k)| < M \} \), \( \Phi_1(x) \) is strictly positive and, consequently, if \( \omega \in L_M \), the function \( \bar{G}(x, \omega) \) given by

\[
\bar{G}(x, \omega) = F_0 x + F_1 \Phi_1(x, \omega),
\]

is monotone. We denote by \( \tilde{X}_0(\omega) \) the unique solution to \( \bar{G}(x, \omega) = h_0, \omega \in L_M \).

**Remark 2.1.** The following procedure provides approximations for \( \tilde{X}_0(\omega) \). Assume \( F_0 F_1 > 0 \). To simplify the notation we skip the dependence on \( \omega \). Fix \( \tilde{X}_0^{2.0} \leq \tilde{X}_0^{1.0} \).
such that \( G(\tilde{X}^{2,0}_0) \leq h_0, G(\tilde{X}^{1,0}_0) \geq h_0 \). We proceed inductively as follows. Let \( \tilde{X}^{2,i}_0 \leq \tilde{X}^{1,i}_0 \) be such that \( G(\tilde{X}^{2,i}_0) \leq h_0, G(\tilde{X}^{1,i}_0) \geq h_0, i \geq 0 \). Consider \((\tilde{X}^{2,i}_0 + \tilde{X}^{1,i}_0)/2\).

Then,

\[
\begin{align*}
\text{if } G\left(\frac{\tilde{X}^{2,i}_0 + \tilde{X}^{1,i}_0}{2}\right) < h_0, & \quad \text{set } \tilde{X}^{2,i+1}_0 = \frac{\tilde{X}^{2,i}_0 + \tilde{X}^{1,i}_0}{2}, \tilde{X}^{1,i+1}_0 = \tilde{X}^{1,i}_0, \\
\text{if } G\left(\frac{\tilde{X}^{2,i}_0 + \tilde{X}^{1,i}_0}{2}\right) > h_0, & \quad \text{set } \tilde{X}^{2,i+1}_0 = \tilde{X}^{2,i}_0, \tilde{X}^{1,i+1}_0 = \frac{\tilde{X}^{2,i}_0 + \tilde{X}^{1,i}_0}{2}, \\
\text{if } G\left(\frac{\tilde{X}^{2,i}_0 + \tilde{X}^{1,i}_0}{2}\right) = h_0, & \quad \text{set } \tilde{X}_0 = \frac{\tilde{X}^{2,i}_0 + \tilde{X}^{1,i}_0}{2}.
\end{align*}
\]

Notice that \(|\tilde{X}^{2,i}_0 - \tilde{X}^{1,i}_0| \leq |\tilde{X}^{2,0}_0 - \tilde{X}^{1,0}_0|/(2^{i-1}), i \geq 1.\)

Let \( \tilde{X}_0 = \tilde{X}_0 1_{L_M} + 01_{L'_M} \). The integer \( M \) plays the role of a stability index. When the increments of the Brownian motion are too big, then a solution to \( G(x, \omega) = h_0 \) may not exist.

We can now state a fundamental ingredient in the proof of the main result of this section.

**Proposition 2.1.** Assume that \( b_1, \sigma \) are \( C^4 \) functions with bounded derivatives, \( F_0 F_1 > 0 \). Then \( X_0 \) belongs to \( L^p(\Omega) \) for any \( p \geq 1 \).

**Proof.** It suffices to check the following facts:

(a) \( P - \lim_{\delta \to 0} \tilde{X}_0 = X_0 \),

(b) \( \sup_{0 < \delta < 1} E |\tilde{X}_0|^p < +\infty \).

Let us first show (a). We have

\[
(X_0 - \tilde{X}_0) 1_{L_M} = -\frac{F_1(\Phi_1(X_0) - \tilde{\Phi}_1(X_0))}{F_0 + F_1 \tilde{\Phi}_1(Z_0)} 1_{L_M},
\]

where \( Z_0 \) a random point between \( X_0(\omega) \) and \( \tilde{X}_0(\omega), \omega \in L_M \). Indeed, for \( \omega \in L_M \),

\[
F_0 \tilde{X}_0(\omega) + F_1 \tilde{\Phi}_1(\omega, \tilde{X}_0(\omega)) = h_0.
\]

Consequently,

\[
P\{|X_0 - \tilde{X}_0| > \eta\} \leq P\{|X_0 - \tilde{X}_0| > \eta\} \cap L_M\} + P\{L'_M\}, \quad \eta > 0.
\]

Let \( k > 0 \) be such that \( P\{|X_0| > k\} \leq \delta \). Then, (2.8), (2.7) and (2.4) with \( q = 4 \) yield

\[
P\{|X_0 - \tilde{X}_0| > \eta\} \leq P\{\Phi_1(X_0) - \tilde{\Phi}_1(X_0) > \eta', |X_0| \leq k\} + 2\delta
\]

\[
\leq P\left\{ \sup_{|x| \leq k} |\Phi_1(x) - \tilde{\Phi}_1(x)| > \eta' \right\} + 2\delta,
\]

with \( \eta' = \eta F_0/F_1 \).

By Lemmas 3.2 and 3.3 in Ahn and Kohatsu–Higa (1995), the right-hand side of (2.10) is bounded by \((1/\eta')^p C \delta^{p/2} + 2\delta\) and, consequently,

\[
\lim_{\delta \to 0} P\{|X_0 - \tilde{X}_0| > \eta\} = 0, \quad \eta > 0.
\]
We now show (b). The identity (2.8) implies
\[ X_0 1_{L_M} = \frac{h_0 - F_1 \Phi_1(0)}{F_0 + F_1 \Phi_1(Z_0)} 1_{L_M}, \]  
(2.11)
with \( Z_0 \) a random point lying between 0 and \( X_0(\omega) \), \( \omega \in L_M \). Next we check
\[ \sup_{0 \leq \delta \leq 1} \sup_{0 \leq k \leq m+1} E|\Phi_1(x)|^p \leq C(1 + |x|^p), \quad p \geq 1. \]  
(2.12)
Indeed, the linear growth of the coefficients implies
\[ E|\Phi_1(x)|^p \leq C \left( |x|^p + 1 + \int_0^T E|\Phi_{\eta(t)}(x)|^p \, ds \right). \]
Consequently,
\[ \sup_{0 \leq t \leq T} E|\Phi_1(x)|^p \leq C \left( |x|^p + 1 + \sup_{0 \leq r \leq s} E|\Phi_r(x)|^p \, ds \right), \quad T \in [0, 1] \]
and Gronwall's lemma yields (2.12).
Then, (2.11) and (2.12) imply
\[ \sup_{0 < \delta \leq 1} E|X_0|^p = \sup_{0 < \delta \leq 1} \left\{ \frac{h_0 - F_1 \Phi_1(0)}{F_0 + F_1 \Phi_1(Z_0)} 1_{L_M} \right\}^p \leq |F_0|^{-p} \left( |h_0|^p + |F_1|^p \sup_{0 < \delta \leq 1} E|\Phi_1(0)|^p \right) \leq C. \]
The proof of the proposition is complete. \( \square \)

We can now state the result which motivated this section.

**Theorem 2.1.** Suppose that the assumptions of Proposition 2.1 are satisfied. Let \( \pi = \{0 = t_0 < t_1 < \cdots < t_{m+1} = 1\} \) be a partition with \( \|\pi\| \leq \delta \). Then
\[ E \left\{ \sup_{0 \leq t \leq 1} |\Phi_t(X_0) - \Phi_t(\bar{X}_0)|^p \right\} \leq C \delta^{p/2}, \]  
(2.13)
for some constant \( C \) independent of \( \delta \).

**Proof.** Clearly,
\[ E \left\{ \sup_{0 \leq t \leq 1} |\Phi_t(X_0) - \Phi_t(\bar{X}_0)|^p \right\} \leq C(a(\delta) + b(\delta)), \]  
(2.14)
with
\[ a(\delta) = E \left\{ \sup_{0 \leq t \leq 1} |\Phi_t(X_0) - \Phi_t(\bar{X}_0)|^p \right\}, \]
\[ b(\delta) = E \left\{ \sup_{0 \leq t \leq 1} |\Phi_t(X_0) - \Phi_t(\bar{X}_0)|^p \right\}. \]
Proposition 2.1 and the analogue of Theorem 3.1 in Ahn and Kohatsu-Higa (1995) for $L^p$ norms yield
\[ a(\delta) \leq C\delta^{p/2}. \] (2.15)

For $b(\delta)$ we have
\[ b(\delta) \leq Cc(\delta)d(\delta), \]
where
\[ c(\delta)^2 = E\left\{ \sup_{0 \leq t \leq 1} |\Phi_t'(\eta_0)|^{2p} \right\}, \]
\[ d(\delta)^2 = E|X_0 - X_0|^2, \]
with $\eta_0$ a random point between $X_0$ and $\bar{X}_0$.

In Lemma 2.1 we prove that $c(\delta)$ is finite. By the definition of $X_0$,
\[ E|X_0 - \bar{X}_0|^p \leq C(e(\delta) + f(\delta)), \]
with
\[ e(\delta) = E\{|X_0|^p 1_{L_M^c}\}, \]
\[ f(\delta) = E\{|X_0 - \bar{X}_0|^p 1_{L_M}\}. \]

Schwarz's inequality, Proposition 2.1 and the estimate (2.4) with $q = 2p + 2$ yield
\[ e(\delta) \leq (E|X_0|^{2p})^{1/2} (p(L_M))^{1/2} \leq C\delta^{p/2}. \] (2.16)

Finally, since on $L_M$, $\Phi_1(x)$ is positive, using (2.7) and Theorem 3.1 in Ahn and Kohatsu-Higa (1995) we obtain
\[ f(\delta) \leq \frac{F_1}{F_0} E|\Phi_1(X_0) - \bar{\Phi}_1(X_0)|^p \leq C\delta^{p/2}. \] (2.17)

The estimates (2.16) and (2.17) yield $b(\delta) \leq C\delta^{p/2}$. This result, together with (2.15) and (2.14) conclude the proof. \qed

The following lemma has been used in the proof of the preceding theorem.

**Lemma 2.1.** For any random variable $X$,
\[ \sup_{0 \leq \delta \leq 1} E\left\{ \sup_{0 \leq t \leq 1} |\Phi_t'(X)|^p \right\} < \infty, \quad p \geq 2. \]

**Proof.** Taking derivatives in (2.3) we obtain
\[ E\left\{ \sup_{0 \leq t \leq 1} |\Phi_t'(X)|^p \right\} = E\left\{ \sup_{0 \leq k \leq m} \sup_{t \in [t_k, t_{k+1})} |\Phi_t'(X)|^p \right\} \]
\[ \leq C \left( E\left\{ \sup_{0 \leq k \leq m} |\Phi_{t_k}'(X)|^p \right\} \right) \]
\[ + E\left\{ \sup_{0 \leq k \leq m} \sup_{t \in [t_k, t_{k+1})} |\Phi_t'(X)|^p (|W_t - W_{t_k}|^p + \delta^p) \right\}. \]
The identity (2.6) yields
\[
E \left\{ \sup_{0 \leq k \leq m} |\Phi_k(X)|^p \right\} \leq CE \left\{ \prod_{i=0}^{m} (1 + |W(\Delta_i)|^p + \delta^p) \right\}
\leq C(1 + \delta^{p/2})^{1/\delta}.
\]
Notice that the right-hand side of the preceding inequality is bounded uniformly in \( \delta \in (0, 1] \).
In addition, Doob’s maximal inequality implies
\[
E \left\{ \sup_{0 \leq k \leq m} \sup_{t \in [t_k, t_{k+1}]} |W_t - W_{t_k}|^p \right\} \leq C\delta^{p/2-1}.
\]
This ends the proof of the lemma. \( \square \)

3. Multidimensional stochastic differential equations with general boundary conditions

In this section we consider the stochastic differential equation
\[
dX_t + f(X_t) \, dt = B \, dW_t, \quad t \in [0, 1],
\]
with boundary condition
\[
h(X_0, X_1) = \bar{h}.
\]
We assume that \( \{X_t, t \in [0, 1]\} \) is \( \mathbb{R}^d \)-valued, \( \{W_t, t \in [0, 1]\} \) is a \( \mathbb{R}^k \)-valued Brownian motion, \( f: \mathbb{R}^d \rightarrow \mathbb{R}^d \) takes the form
\[
f(x) = Ax + B\bar{f}(x),
\]
where \( A \) is a \( d \times d \) matrix, \( \bar{f}: \mathbb{R}^d \rightarrow \mathbb{R}^k \) is measurable and locally bounded and \( B \) is a \( d \times k \) matrix. Finally \( h: \mathbb{R}^{2d} \rightarrow \mathbb{R}^d \) and \( \bar{h} \in \mathbb{R}^d \). Without loss of generality, we may assume \( k \leq d \). In Nualart and Pardoux (1991) a theorem on existence and uniqueness of solution for this kind of equations has been established. More explicitly, let \( C_0([0, 1]; \mathbb{R}^k) \) be the set of continuous, \( \mathbb{R}^d \)-valued functions vanishing at 0; set \( F = \{ \int_0^t e^{A(t-s)} B \, d\varphi(s); \varphi \in C([0, 1]; \mathbb{R}^k) \} \), where the integrals are defined using integration by parts. Assume
\[
(H1) \text{ For any } z \in F \text{ the equation } h(y, e^{-A}(y + z)) = \bar{h} \text{ has a unique solution } y = g(z).
\]
In order to find the solution of (3.1), (3.2) we consider the linear equation
\[
dY_t + AY_t \, dt = B \, dW_t, \quad t \in [0, 1],
\]
with boundary condition (3.2). This equation has a unique solution given by
\[
Y_t = e^{-At} \left[ g \left( \int_0^1 e^{As} B \, dW_s \right) + \int_0^t e^{As} B \, dW_s \right].
\]
Let $\xi \in \mathcal{C}([0, 1]; \mathbb{R}^k)$, $\xi_t = \xi_0 + \int_0^t A\xi_s \, ds \in \text{Im } B$, $0 \leq t \leq 1$, $h(\xi_0, \xi_1) = \mathbf{1}$. There exists a bijection $\psi: \mathcal{C}_0([0, 1]; \mathbb{R}^k) \rightarrow \Sigma$ such that $Y_t = (\psi(W))_t$. Finally, we define the mapping $T: \mathcal{C}_0([0, 1]; \mathbb{R}^k) \rightarrow \mathcal{C}_0([0, 1]; \mathbb{R}^k)$ by

$$T(\theta) = \theta + \int_0^\cdot \mathcal{F}(\psi(\theta)_s) \, ds. \tag{3.4}$$

**Theorem 3.1.** (Nualart and Pardoux, 1991). Assume $T$ is a bijection and (H1). Then Eq. (3.1) with boundary condition (3.2) possess a unique solution given by

$$X = \psi(T^{-1}(W)). \tag{3.5}$$

As in the preceding section our purpose is to provide numerical approximations for the process $X$ defined by (3.5). The difficulty here comes from the fact that we do not have an explicit expression for the paths $T^{-1}(W)$. Indeed, set $\theta = T^{-1}(W)$; then (3.4) and (3.3) yield

$$\theta_t = W_t - \int_0^t \mathcal{F}(e^{-As}(g(\int_0^1 e^{Au}B \, d\theta_u) + \int_0^s e^{Au}B \, d\theta_u)) \, ds. \tag{3.6}$$

This equation shows that $\theta$ depends on its past as well as its future. Therefore, the approximation of $\theta$ does not seem easy to obtain from (3.6). Instead, we propose the following approach: Set $\varphi_t = \int_0^t e^{Au}B \, d\theta_u$. Then,

$$X_t = e^{-At}g(\int_0^1 e^{Au}B \, d\theta_u) + \int_0^t e^{Au}B \, d\theta_u$$

$$= e^{-At}[g(\varphi_1) + \varphi_t]. \tag{3.7}$$

From (3.6) we easily obtain $\varphi_t = \xi_t + u_t$ with

$$\xi_t = \int_0^t e^{As}B \, dW_s, \tag{3.8}$$

$$u_t = -\int_0^t e^{As}B\mathcal{F}(e^{-As}[g(\xi_t + u_t) + (\xi_s + u_s)]) \, ds. \tag{3.9}$$

Fix $y \in \mathbb{R}$ and assume

$$u_t(y) = -\int_0^t e^{As}B\mathcal{F}(e^{-As}[g(\xi_t + y) + \xi_s + u_s(y)]) \, ds \tag{3.10}$$

has a unique solution. Moreover, suppose that the mapping $y \mapsto u_t(y)$ has a unique fixed point $y^*$. Then, clearly

$$\varphi_t = \xi_t + u_t(y^*). \tag{3.11}$$

Hence, our approximations for $X_t$ will be constructed from approximations of $\xi_t$ and approximations of the flow $u_t(y^*)$. \[^1\]
We first establish a result ensuring that $T$ is bijective. This question is also related to the existence of a unique fixed point for $y \mapsto u_1(y)$. Given a $d \times k$ matrix $M$ we denote by $|M|$ its norm, that is, $|M| = \sup_{x:|x|=1} |Mx|$.

**Proposition 3.1.** Assume $f$ and $g$ are Lipschitz functions with Lipschitz constant $L_f$ and $L_g$, respectively. If

$$K(L_f, L_g) = L_g \left[ \exp \left( \frac{e^{2|A|} - 1}{2|A|} \frac{|B|}{L_f} \right) - 1 \right] < 1,$$

(3.12)

the mapping $T$ defined by (3.4) is bijective.

**Proof.** The assumptions on $f, g$ imply the existence of a unique continuous solution $\{u_t(y), t \in [0, 1]\}$ to (3.10). Then, following Nualart and Pardoux (1991, Proposition 2.5) the proof reduces to show that $y \mapsto u_1(y)$ has a unique fixed point. Let $y_1, y_2 \in \mathbb{R}^d$, $t \in [0, 1]$. The Lipschitz property of $f$ and $g$ yield

$$|u_t(y_1) - u_t(y_2)| \leq |B| L_f L_g \left[ \frac{e^{2|A|t} - 1}{2|A|} \right] |y_1 - y_2|$$

$$+ |B| L_f \int_0^t e^{2|A|s} |u_s(y_1) - u_s(y_2)| \, ds.$$  

(3.13)

**Lemma 3.1 (Gronwall's lemma),** with $y_t = |u_t(y_1) - u_t(y_2)|$, $f(t) = |B| L_f L_g \times ((e^{2|A|} - 1) / 2|A|) |y_1 - y_2|$ and $g(t) = |B| L_f e^{2|A|t}$, implies

$$|u_t(y_1) - u_t(y_2)| \leq L_g \left[ \exp \left( \frac{e^{2|A|t} - 1}{2|A|} \frac{|B|}{L_f} \right) - 1 \right] |y_1 - y_2|.$$  

(3.14)

The assumption (3.12) together with (3.14) for $t = 1$ imply that the mapping $y \mapsto u_1(y)$ is a strict contraction. This completes the proof.  

Notice that condition (3.12) depends on the time interval where Eqs. (3.1) with (3.2) are considered.

**Remark 3.1.** If $d = 1$, condition (3.12) can be replaced by

$$L_g \left[ \exp(|B| L_f) - 1 \right] < 1,$$

(3.15)

which is weaker. Indeed, in this case

$$|u_t(y_1) - u_t(y_2)| \leq L_g \exp(|B| L_f) |y_1 - y_2| + |B| L_f \int_0^t |u_s(y_1) - u_s(y_2)| \, ds.$$  

Hence, by Gronwall's lemma,

$$|u_t(y_1) - u_t(y_2)| \leq L_g \left[ \exp(|B| L_f) - 1 \right] |y_1 - y_2|.$$
Remark 3.2. If \( f \) is linear then \( \bar{f} = 0 \) and therefore (3.12) is automatically satisfied.

Fix a partition \( \pi = \{ 0 = t_0 < t_1 < \cdots < t_{m+1} = 1 \} \). In the sequel we will use the same notation as in the preceding section and assume that the hypotheses of Proposition 3.1 are satisfied. Associated with \( \pi \) we define \( \tilde{\xi}_t, \tilde{u}_t(y) \) by

\[
\tilde{\xi}_t = \int_0^t e^{A\eta(s)} B \, dW_s,
\]

\[
\tilde{u}_t(y) = -\int_0^t e^{A\eta(s)} B \tilde{f}(e^{-A\eta(s)}[g(\tilde{\xi}_s + y) + \tilde{\xi}_s + \tilde{u}_s(y)]) \, ds.
\]

Using similar arguments as in the proof of Proposition 3.1 we prove that \( y \mapsto \tilde{u}_t(y) \) has a unique fixed point, say \( \tilde{y}^* \). Indeed, the analogue of (3.13) is

\[
|\tilde{u}_t(y_1) - \tilde{u}_t(y_2)| \leq \left| B \right| L_f L_g |y_1 - y_2| \int_0^t e^{2|A|\eta(s)} \, ds
\]

\[+ \left| B \right| L_f \int_0^t e^{2|A|\eta(s)} |\tilde{u}_s(y_1) - \tilde{u}_s(y_2)| \, ds,
\]

and, consequently, since \( e^{2|A|\eta(s)} \leq e^{2|A|s} \),

\[
\sup_{0 \leq s \leq t} |\tilde{u}_s(y_1) - \tilde{u}_s(y_2)| \leq \left| B \right| L_f L_g |y_1 - y_2| \frac{e^{2|A|t} - 1}{2|A|}
\]

\[+ \left| B \right| L_f \int_0^t e^{2|A|\eta(s)} \sup_{0 \leq r \leq s} |\tilde{u}_r(y_1) - \tilde{u}_r(y_2)| \, ds.
\]

Then, Gronwall's lemma yields

\[
\sup_{0 \leq s \leq t} |\tilde{u}_s(y_1) - \tilde{u}_s(y_2)| \leq L_g \left[ \exp \left( \frac{(e^{2|A|t} - 1)}{2|A|} \left| B \right| L_f \right) - 1 \right] |y_1 - y_2|.
\]

(3.16)

Set \( \bar{\phi}_t = \bar{\xi}_t + \bar{u}_t(\tilde{y}^*) \). We can now state the main result of this section.

Theorem 3.2. Assume that the assumptions of Proposition 3.1 are satisfied. Fix a partition \( \pi \) of \([0,1]\) and let

\[
\bar{X}_t = e^{-A_t}[g(\bar{\phi}_1) + \bar{\phi}_t].
\]

Then

\[
E \left\{ \sup_{0 \leq t \leq 1} |\bar{X}_t - X_t|^p \right\} \leq C \delta^p, \quad p \geq 2.
\]

(3.17)

where \( \delta := \| \pi \| \) and \( C \) is a constant independent of \( \delta \).

Proof. Taking into account (3.7), the proof of (3.17) reduces to check

\[
E \left\{ \sup_{0 \leq t \leq 1} |\xi_t - \bar{\xi}_t|^p \right\} \leq C \delta^p,
\]

(3.18)

\[
E \left\{ \sup_{0 \leq t \leq 1} |u_t(y^*) - \bar{u}_t(\tilde{y}^*)|^p \right\} \leq C \delta^p, \quad p \geq 2.
\]

(3.19)
Inequality (3.18) follows easily from Doob’s maximal inequality and Burkholder’s inequality.

In order to check (3.19) we decompose \( |u_t(y^*) - \tilde{u}_t(\tilde{y}^*)| \) as follows:

\[
|u_t(y^*) - \tilde{u}_t(\tilde{y}^*)| \leq \sum_{i=1}^{3} |u_i^t|,
\]

with

\[
u_1^t = \int_{0}^{t} e^{At} B(\tilde{f}(e^{-At}[g(\xi_1 + y^*) + \xi_s + u_s(y^*)]) - \tilde{f}(e^{-At}[g(\xi_1 + y^*) + \xi_s + u_{\tau(t)}(y^*)])) ds,
\]

\[
u_2^t = \int_{0}^{t} e^{At} B(\tilde{f}(e^{-At}[g(\xi_1 + y^*) + \xi_s + u_{\tau(t)}(y^*)]) - \tilde{f}(e^{-At}[g(\xi_1 + \tilde{y}^*) + \xi_{\tau(t)} + \tilde{u}_{\tau(t)}(\tilde{y}^*)])) ds,
\]

\[
u_3^t = \int_{0}^{t} \{e^{At} B(\tilde{f}(e^{-At}[g(\xi_1 + \tilde{y}^*) + \xi_{\tau(t)} + \tilde{u}_{\tau(t)}(\tilde{y}^*)]) - e^{At} B(\tilde{f}(e^{-At}[g(\xi_1 + \tilde{y}^*) + \xi_{\tau(t)} + \tilde{u}_{\tau(t)}(\tilde{y}^*)])) ds.
\]

First we prove

\[
E\left\{\sup_{0 \leq t \leq 1} |u_1^t|^p\right\} \leq C \delta^p, \quad p \geq 2.
\] (3.20)

Indeed,

\[
\sup_{0 \leq t \leq 1} |u_1^t| \leq C \int_{0}^{1} |u_s(y^*) - u_{\tau(t)}(y^*)| ds
\]

for some positive constant \( C \) independent of \( \tau \). Using (3.26) and (3.28) we can show

\[
\sup_{0 \leq t \leq 1} E|u_t(y^*) - u_{\tau(t)}(y^*)|^p \leq C \delta^p.
\]

Consequently, (3.20) holds.

For \( u_2^t \) we have

\[
|u_2^t| \leq \int_{0}^{t} e^{2|A|s} |B| L_\tilde{f}(L_{g_\xi}(|\xi_1 - \xi_1^s| + |y^* - \tilde{y}^*|)
\]

\[
+ |\xi_s - \tilde{\xi}_{\tau(s)}| + |u_{\tau(s)}(y^*) - \tilde{u}_{\tau(s)}(\tilde{y}^*)|) ds.
\]

Therefore,

\[
E\left\{\sup_{0 \leq s \leq t} |u_2^s|^p\right\} \leq |B| L_\tilde{f} L_g e^{2|A|t} \frac{-1}{2|A|} (E(|\xi_1 - \xi_1^s|^p + |y^* - \tilde{y}^*|^p))
\]

\[
+ \int_{0}^{t} e^{2|A|s} |B| L_\tilde{f} E[|\xi_s - \tilde{\xi}_{\tau(s)}|^p] ds
\]

\[
+ \int_{0}^{t} e^{2|A|s} |B| L_\tilde{f} E\left\{\sup_{0 \leq r \leq s} |u_r(y^*) - \tilde{u}_r(\tilde{y}^*)|\right\} ds.
\]
Clearly,
\[ E|\tilde{\xi}_1 - \bar{\xi}_1|^p + \sup_{0 \leq s \leq 1} E|\tilde{\xi}_s - \bar{\xi}_{\eta(s)}|^p \leq C\delta^p, \]
with \( C \) independent of \( \delta \). Consequently,
\[
E\left\{ \sup_{0 \leq s \leq t} |u_s|^p \right\} \leq C_1\delta^p + |B|L_fL_g \frac{e^{2|A|t} - 1}{2|A|} E(|y^* - \bar{y}^*)|^p)
+ \int_0^t e^{2|A|s}|B|L_f E\left\{ \sup_{0 \leq r \leq s} |u_r(y^*) - \bar{u}_r(\bar{y}^*)|^p \right\} ds. \tag{3.21}
\]
Finally, we deal with \( u_t^2 \) and we obtain
\[
|u_t^2| \leq \int_0^t |e^{As} - e^{A\eta(s)}||B||\tilde{\xi}(e^{-As}[g(\tilde{\xi}_1 + \bar{y}^*) + \tilde{\xi}_{\eta(s)} + \bar{\xi}_{\eta(s)}(\bar{y}^*)])| ds
+ \int_0^t e^{2A|s|}|B|L_f |e^{-As} - e^{-A\eta(s)}||g(\tilde{\xi}_1 + \bar{y}^*) + \tilde{\xi}_{\eta(s)} + \bar{\xi}_{\eta(s)}(\bar{y}^*)| ds.
\]
Clearly, \( E|\tilde{\xi}_1|^p \leq C_1, \sup_{0 \leq s \leq 1} E|\tilde{\xi}_{\eta(s)}|^p \leq C_2 \) for some positive constants \( C_1, C_2 \) not depending on \( \delta \). Hence, the Lipschitz property of the mapping \( t \mapsto e^{At}, \ t \mapsto e^{-At} \) and the results stated in Remark 3.3 yield
\[
E\left\{ \sup_{0 \leq r \leq t} |u_r|^p \right\} \leq C\delta^p, \ p \geq 2 \tag{3.22}
\]
for some constant \( C \) independent of \( \delta \).

Thus, the estimates (3.20) to (3.22) imply
\[
E\left\{ \sup_{0 \leq r \leq s} |u_r(y^*) - \bar{u}_r(\bar{y}^*)|^p \right\} \leq C\delta^p + |B|L_fL_g \frac{e^{2|A|t} - 1}{2|A|} E(|y^* - \bar{y}^*)|^p)
+ \int_0^t e^{2|A|s}|B|L_f E\left\{ \sup_{0 \leq r \leq s} |u_r(y^*) - \bar{u}_r(\bar{y}^*)|^p \right\} ds.
\]

Lemma 3.1 with
\[
y(t) = E\left\{ \sup_{0 \leq r \leq s} |u_r(y^*) - \bar{u}_r(\bar{y}^*)|^p \right\},
\]
\[
f(t) = C\delta^p + |B|L_fL_g \frac{e^{2|A|t} - 1}{2|A|} E(|y^* - \bar{y}^*)|^p),
\]
\[
g(t) = e^{2|A|t}|B|L_f,
\]
yields
\[
E\left\{ \sup_{0 \leq r \leq s} |u_r(y^*) - \bar{u}_r(\bar{y}^*)|^p \right\} \leq C\delta^p + L_g \left[ \exp\left( \frac{e^{2|A|t} - 1}{2|A|} \right) - 1 \right] E(|y^* - \bar{y}^*)|^p). \tag{3.23}
\]
In particular, for $t = 1$, 
\[
E(|y^* - \bar{y}^*|^p) \leq C(1 - K(L_f, L_g))^{-1} \delta^p
\]  
(3.24)
because $u_1(y^*) = y^*$ and $\bar{u}_1(\bar{y}^*) = \bar{y}^*$. The estimates (3.23) and (3.24) yield (3.19) and finish the proof of the Theorem. □

Let $\bar{y}^{*,m}$ be an approximation of the fixed point $\bar{y}^*$ obtained at step $m \geq 1$ through the bisection method. From (3.30) we get 
\[
|\bar{y}^* - \bar{y}^{*,m}| \leq 2^{-(dm-1)} (1 - K(L_f, L_g))^{-1} \bar{\tau}.
\]  
(3.25)

Set 
\[
\bar{X}^m_t = e^{-\lambda t} [g(\bar{\varphi}^m_t) + \bar{\varphi}^m_t],
\]
with $\bar{\varphi}^m_t = \bar{\tau} + \bar{u}_t(\bar{y}^{*,m})$. Then
\[
E \left\{ \sup_{0 \leq t \leq 1} |X_t - \bar{X}^m_t|^p \right\} \leq C(\delta^p + 2^{-p(dm-1)}).
\]

Indeed, (3.16) and (3.22) ensure 
\[
E \left\{ \sup_{0 \leq t \leq 1} |\bar{u}_t(\bar{y}^*) - \bar{u}_t(\bar{y}^{*,m})|^p \right\} \leq [K(L_f, L_g)]^p (1 - K(L_f, L_g))^{-1} E[\bar{\tau}]^p 2^{-p(dm-1)}.
\]

We end this section with some technical tools that have been used along the proofs.

**Lemma 3.1.** Let $f: [0, 1] \to \mathbb{R}, g: [0, 1] \to \mathbb{R}_+$ be continuous mappings. Consider a continuous function $y: [0, 1] \to \mathbb{R}$ satisfying 
\[
y(t) \leq f(t) + \int_0^t g(s)y(s) \, ds, \quad t \in [0, 1].
\]
Then, 
\[
y(t) \leq f(t) + \int_0^t f(s)g(s)\exp\left(\int_s^t g(u) \, du\right) \, ds, \quad t \in [0, 1].
\]

**Lemma 3.2.** Under the assumptions of Proposition 3.1, the unique fixed point $y^*$ of the mapping $y \mapsto u_1(y)$, with $\{u_t(y), t \in [0, 1]\}$ defined by (3.10), satisfies the following property:

- There exists a random variable $\tau$ possessing moments of any order such that 
\[
|y^*| \leq (1 - K(L_f, L_g))^{-1} \tau,
\]  
(3.26)

where $K(L_f, L_g)$ is the constant given in (3.12).
Proof. The Lipschitz property of the coefficients $f, g$ and Eq. (3.10) yield

$$|u_t(y)| \leq h_1(t) + h_2(t) + h_3(t) + \int_0^t L_f |B| e^{2|A|^s} |u_t(y)| \, ds,$$

with

$$h_1(t) = |B| |f(0)| \frac{e^{2|A|^t} - 1}{|A|},$$

$$h_2(t) = \int_0^t L_f |B| e^{2|A|^s} [|g(0)| + L_g |\xi_1| + |\xi_2|] \, ds,$$

$$h_3(t) = L_f |B| L_g \frac{e^{2|A|^t} - 1}{2|A|} |y|.$$ 

Thus, Lemma 3.1 with $f(t) = \sum_{i=1}^3 h_i(t), g(t) = L_f |B| e^{2|A|^t}, y(t) = |u_t(y)|$ ensures

$$|u_t(y)| \leq \sum_{i=1}^3 h_i(t) + \int_0^t (h_1(s) + h_2(s)) L_f |B| e^{2|A|^s} \exp\left(L_f |B| \frac{e^{2|A|^t} - e^{2|A|^s}}{2|A|}\right) ds$$

$$+ |y| \int_0^t L_f^2 L_g^2 |B| \frac{e^{2|A|^s} - 1}{2|A|} e^{2|A|^s} \exp\left(L_f |B| \frac{e^{2|A|^t} - e^{2|A|^s}}{2|A|}\right) ds.$$

Let

$$\tau(t) = h_1(t) + h_2(t)$$

$$+ \int_0^t (h_1(s) + h_2(s)) L_f |B| e^{2|A|^s} \exp\left(L_f |B| \frac{e^{2|A|^t} - e^{2|A|^s}}{2|A|}\right) ds.$$ (3.27)

Notice that $\tau(t)$ is a random variable in $\bigcap_{p \geq 1} L^p(\Omega), t \in [0, 1]$. Then,

$$|u_t(y)| \leq \tau(t) + K(L_f, L_g) |y|.$$ (3.28)

The estimate (3.26) follows from (3.28) taking $t = 1$ and setting $\tau = \tau(1)$. \qed

Remark 3.3. With parallel arguments the following results can also be proved.

- There exists a random variable $\bar{\tau}(t)$ in $\bigcap_{p \geq 1} L^p(\Omega), t \in [0, 1]$, such that

$$|\bar{u}_t(y)| \leq \bar{\tau}(t) + K(L_f, L_g) |y|.$$ (3.29)

- The unique fixed point $\bar{y}^*$ for the mapping $y \mapsto \bar{u}_1(y)$ satisfies

$$|\bar{y}^*| \leq (1 - K(L_f, L_g))^{-1} \bar{\tau}.$$ (3.30)

where $\bar{\tau} = \bar{\tau}(1)$.

Remark 3.4. For $d = 1$, Eq. (3.1) is a particular case of the Stratonovich equation in (2.1). The boundary condition (3.2) satisfying (H1) includes the boundary condition $F_0 X_0 + F_1 X_1 = h_0, F_0 F_1 > 0$. However, the restriction (3.12) between the Lipschitz
constant of \( \tilde{f} \) and the boundary condition is not necessary to the approach of Section 2.

References

