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Dimension n Representations of the Braid Group on n Strings

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In 1996, E. Formanek classified all the irreducible complex representations of B_n of dimension at most $n-1$, where B_n is the Artin braid group on n strings. In this paper we extend this classification to the representations of dimension n, for $n \geq 9$. We prove that all such representations are equivalent to the tensor product of a one-dimensional representation and a specialization of a certain one-parameter family of n-dimensional representations which was first discovered in 1996 by Tong, Yang, and Ma. In order to do this, we classify all the irreducible complex representations ρ of B_n for which $rank(\rho(\sigma_i) - 1) = 2$, where the σ_i are the standard generators. © 2001 Academic Press

1. INTRODUCTION

Let B_n be the braid group on n strings. In terms of generators and relations it has the following presentation:

DEFINITION 1. $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, 1 \leq i \leq n-2;\ \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2 \rangle$.

In his paper [3] Formanek classified all of the irreducible representations of B_n of dimension at most $(n-1)$. Since then there have been some attempts to classify irreducible representations of B_n of dimension n. In particular the classification is known for very small n. The case $n = 3$ was done by Formanek [3, Theorem 24]. Woo Lee has classified the four-dimensional irreducible representations of B_4 [5].

In this paper we solve this problem completely for $n \geq 9$.

To describe our results, we need the following definition.

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DEFINITION 2. The **corank** of the representation $\rho : B_n \to GL_r(\mathbb{C})$ is $rank(\rho(\sigma_i) - 1)$, where the σ_i are the standard generators of the group B_n.

Remark 1.1. Because the σ_i are conjugate to each other [2, p. 655], the number $rank(\rho(\sigma_i) - 1)$ does not depend on i, which justifies the above definition.

The corank of specializations of the reduced Burau representation [1, p. 121; 4, p. 338] and of the standard one-dimensional representation is 1.

If one looks at the proof of the classification theorem of Formanek in [3], it can be separated into two parts. The first is to classify all irreducible representations of braid groups of corank 1. The second is to prove that apart from a few exceptions, the irreducible representations of the braid group B_n of dimension at most $(n-1)$ can be obtained as a tensor product of a one-dimensional representation and an irreducible representation of corank 1.

Our proof follows a similar strategy.

The first part of this paper is the classification of all the irreducible complex representations of corank 2. Apart from a number of exceptions for $n \leq 6$, they all are equivalent to specializations for $u \neq 1$, $u \in \mathbb{C}^*$ of the following representation $\rho : B_n \to GL_n(\mathbb{C}[u^{\pm 1}])$, first discovered by Dian-Ming Tong *et al.* in [7],

$$\rho(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & u & \\ & 1 & 0 & \\ & & & I_{n-1-i} \end{pmatrix},$$

for $i = 1, 2, \ldots, n-1$, where I_k is the $k \times k$ identity matrix.

In the second part of the paper we complete the proof of the Main Theorem (Theorem 6.1). We show that for $n \geq 9$ every irreducible representation of B_n of dimension n is equivalent to the tensor product of a one-dimensional representation and a representation of corank 2.

The main tool we use for the classification of the irreducible representations of corank 2 is the *friendship graph* of a representation. The (full) friendship graph of a representation ρ of a braid group B_n is the graph whose vertices are the set of generators (σ_0), $\sigma_1, \ldots, \sigma_{n-1}$ of B_n. Two vertices σ_i and σ_j are joined by an edge if and only if $\mathrm{Im}(\rho(\sigma_i) - 1) \cap \mathrm{Im}(\rho(\sigma_j) - 1) \neq \{0\}$. (See Lemma 2.1 for the definition of σ_0.)

Using the braid relations, we investigate the structure of the friendship graph. It turns out that the friendship graph is a chain for every irreducible representation of B_n of dimension at least n and corank 2, provided that

$n \geq 6$. This means that σ_i and σ_j are joined by an edge if and only if $|i - j| = 1$.

For a given friendship graph it is relatively easy to classify all irreducible complex representations of B_n for which it is the associated friendship graph. When the graph is a chain, we get specializations of the representation discovered by Tong, Yang, and Ma.

This paper is the first in a series of papers aimed at extending the classification by Formanek to irreducible representations of higher dimensions.

Another result, which will appear elsewhere, is that for n large enough there are no irreducible complex representations of B_n of corank 3 and no irreducible complex representations of B_n of dimension $n + 1$.

Based on the above results we would like to make the following two conjectures.

Conjecture 1. For every $k \geq 3$ and n large enough there are no irreducible complex representations of B_n of corank k.

Conjecture 2. For every $k \geq 1$ and n large enough there are no irreducible complex representations of B_n of dimension $n + k$.

We should also note that for the purpose of brevity we did not include in this paper some of the details of the classification of representations of B_n for small n. The full proof can be found in our thesis [6, Chaps. 6 and 7].

The paper is organized as follows. In Sections 2 through 5 we prove the classification theorem for the irreducible complex representations of B_n of corank 2. In Section 2 we introduce some convenient notation that will be used throughout the rest of the paper. In Section 3 we define the friendship graph of the representation and study its structure. We also study the case when the friendship graph is totally disconnected. In Section 4 we prove that for $n \geq 6$ the associated friendship graph is a chain for any irreducible complex representation of B_n of corank 2 and dimension at least n. In Section 5 we determine all irreducible representations of corank 2 whose friendship graph is a chain. In Section 6 we give a complete classification of the irreducible representations of B_n of dimension n for $n \geq 6$.

2. NOTATION AND PRELIMINARY RESULTS

Let B_n be the braid group on n strings.

LEMMA 2.1. *For the braid group B_n set*

$$\tau = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \qquad \textit{and} \qquad \sigma_0 = \tau \sigma_{n-1} \tau^{-1}.$$

Then:

(1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$;

(2) $\sigma_{i+1} = \tau \sigma_i \tau^{-1}$;

(3) $\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2$ *for all* i, j *where indices are taken modulo* n.

Remark 2.2. For $1 \leq i \leq n-2$ the relationship (2) was established in [2, p. 655].

Remark 2.3. Taking into account the above lemma, we also have the following presentation of B_n,

$$B_n = \langle \sigma_0, \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}; \\ \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2; \sigma_0 = \tau \sigma_{n-1} \tau^{-1} \rangle$$

for all i, j where indices are taken modulo n and τ is defined as above.

Let $\rho : B_n \to GL_r(\mathbb{C})$ be a matrix representation of B_n with

$$\rho(\sigma_i) = 1 + A_i, \qquad \text{and} \qquad \rho(\tau) = T \in GL_r(\mathbb{C}).$$

Then for any i (indices are modulo n), the relation

$$\tau \sigma_i \tau^{-1} = \sigma_{i+1}$$

implies that

$$T A_i T^{-1} = A_{i+1}.$$

Hence all the A_i are conjugate to each other, so they have the same rank, spectrum, and Jordan normal form.

LEMMA 2.4. *For a representation* ρ *of* B_n *with*

$$\rho(\sigma_i) = 1 + A_i,$$

we have:

(1) $A_i A_j = A_j A_i$, *for* $|i-j| \geq 2$;

(2) $A_i + A_i^2 + A_i A_{i+1} A_i = A_{i+1} + A_{i+1}^2 + A_{i+1} A_i A_{i+1}$ *for all* $i = 0, 1, \dots, n-1$, *where indices are taken modulo* n.

Proof. This follows easily from the relations on the generators of B_n.

3. THE FRIENDSHIP GRAPH

In this section we define and prove some properties of the *friendship graph* which is a finite graph associated with a representation of B_n. Our graphs are *simple-edged*, which means that there is at most one unoriented edge joining two vertices, and no edges joining a vertex to itself.

We assume throughout this section that we have a representation

$$\rho : B_n \to GL_r(\mathbb{C}),$$

with

$$\rho(\sigma_i) = 1 + A_i \qquad (i = 0, 1, \ldots, n-1).$$

DEFINITION 3. (1) A_i, A_{i+1} are **neighbors** (indices modulo n).

(2) A_i, A_j are **friends** if

$$\mathrm{Im}(A_i) \cap \mathrm{Im}(A_j) \neq \{0\}.$$

(3) A_i, A_j are **true friends** if either

(a) A_i and A_j are not neighbors, and

$$A_i A_j = A_j A_i \neq 0;$$

or

(b) A_i and A_j are neighbors, and

$$A_i + A_i^2 + A_i A_j A_i = A_j + A_j^2 + A_j A_i A_j \neq 0.$$

LEMMA 3.1. *If A, B are true friends, then they are friends.*

Proof. (1) If A and B are not neighbors, then $AB = BA \neq 0$, so,

$$\mathrm{Im}(A) \cap \mathrm{Im}(B) \supseteq \mathrm{Im}(AB) \cap \mathrm{Im}(BA) = \mathrm{Im}(AB) \neq \{0\}.$$

(2) If A and B are neighbors, then

$$\begin{aligned} A(1 + A + BA) &= A + A^2 + ABA \\ &= B + B^2 + BAB = B(1 + B + AB) \neq 0, \end{aligned}$$

and again

$$\mathrm{Im}(A) \cap \mathrm{Im}(B) \supseteq \mathrm{Im}(A + A^2 + ABA) \neq \{0\}.$$

DEFINITION 4. **The full friendship graph** (associated with the representation $\rho : B_n \to GL_r(\mathbb{C})$) is the simple-edged graph with n vertices $A_0, A_1, \ldots, A_{n-1}$ and an edge joining A_i and A_j ($i \neq j$) if and only if A_i and A_j are friends.

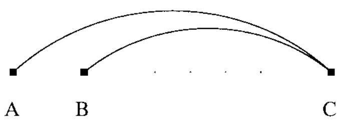
The friendship graph is the subgraph with vertices $A_1, \dots, A_{n-1}$ obtained from the full friendship graph by deleting A_0 and all edges incident to it.

Our main interest is the friendship graph, but it is convenient to introduce the full friendship graph as a tool, because of the following lemma.

LEMMA 3.2. *There is an edge between A_i and A_j in the full friendship graph if and only if there is an edge between A_{i+k} and A_{j+k}, where indices are taken modulo n. In other words, $\mathbb{Z}_n$ acts on the full friendship graph by permuting the vertices cyclically.*

Proof. This follows immediately from the fact that conjugation by $T = \rho(\tau) = \rho(\sigma_1 \cdots \sigma_{n-1})$ permutes $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$ cyclically (Lemma 2.1).

LEMMA 3.3 (Lemma about Friends). *Let A and B be neighbors which are not friends. If C is not a neighbor of A and C is a friend of B then C is a true friend of A.*



Proof. By Lemma 3.1, A and B are not true friends, because they are not friends, that is,

$$A + A^2 + ABA = B + B^2 + BAB = 0.$$

Consider $y \in V$ such that $Cy \in \mathrm{Im}(B)$, $Cy = Bz \neq 0$ (y exists because C and B are friends). Then

$$BACy = BABz = -(B + B^2)z = -(1 + B)Bz \neq 0$$

because $Bz \neq 0$ and $(1 + B)$ is invertible.

So, $AC = CA \neq 0$; that is, A and C are true friends.

THEOREM 3.4. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be a representation. Then one of the following holds.*

(a) *The full friendship graph is totally disconnected (no friends at all).*

(b) *The full friendship graph has an edge between A_i and A_{i+1} for all i.*

(c) *The full friendship graph has an edge between A_i and A_j whenever A_i and A_j are not neighbors.*

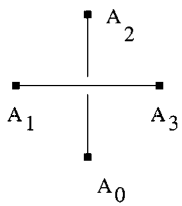
Proof. Suppose neither (a) nor (b) holds. Since the graph is not totally disconnected, there is an edge joining some vertices B and C. Since (b) does not hold, no neighbors are joined by an edge. Lemma 3.3 implies that there is an edge between C and any neighbor of B which is not a neighbor of C. It follows inductively that there is an edge joining C to every vertex which is not a neighbor of C. Then (c) holds, because the full friendship graph is a $\mathbb{Z}_n$-graph.

DEFINITION 5. The friendship graph (the full friendship graph) is **a chain**, if the only edges are between neighbors.

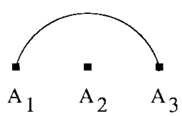
Case (b) of the above theorem can be restated as
(b) *The full friendship graph contains the chain graph.*

COROLLARY 3.5. *For $n \neq 4$, the friendship graph and the full friendship graph are either totally disconnected (no edges) or connected.*

Remark 3.6. For $n = 4$ there is a full friendship graph which is neither totally disconnected nor connected:



The friendship graph in this case is:



By [6, Lemmas 6.2 and 6.3], every representation of B_4 of corank 2 and dimension at least 4 which has this friendship graph is reducible.

Now consider the case when the friendship graph is totally disconnected (that is, statement (a) of Theorem 3.4 holds).

LEMMA 3.7. *If A and B are neighbors and not friends then*:

(a) $A^2B = AB^2$; $BA^2 = B^2A$.

(b) *If* $x \in \mathrm{Im}(A) \cap \mathrm{Ker}(A - \lambda I)$, *then* $B(Bx) = \lambda(Bx)$ *and* $ABx = -(1 + \lambda)x$.

Proof. (a) By Lemma 3.1, A and B are not true friends, so

$$A + A^2 + ABA = B + B^2 + BAB = 0.$$

Multiplying the left hand side by B on the right and the right hand side by A on the left gives

$$AB + A^2B + ABAB = 0 = AB + AB^2 + ABAB.$$

Thus, $A^2B = AB^2$; by a symmetric argument $BA^2 = B^2A$.

(b) Let $x = Ay \in \mathrm{Im}(A) \cap \mathrm{Ker}(A - \lambda I)$. Then

$$B(Bx) = B^2Ay = BA^2y = BAx = \lambda Bx,$$

and

$$0 = (A + A^2 + ABA)y = (1 + A + AB)x = (1 + \lambda)x + ABx.$$

Thus, $ABx = -(1 + \lambda)x$.

THEOREM 3.8. *Let $\rho : B_n \to GL_r(\mathbb{C})$ $(n \geq 2)$ be an irreducible representation, whose associated friendship graph is totally disconnected. Then* $r = \dim V \leq n - 1$.

Proof. If $A_i = 0$, ρ is the trivial representation and $r = 1$.

If $A_i \neq 0$, choose an eigenvalue λ for A_1 and a non-zero vector

$$x_1 \in \mathrm{Im}(A_1) \cap \mathrm{Ker}(A_1 - \lambda I).$$

Set $x_2 = A_2 x_1$, $x_3 = A_3 x_2, \dots, x_{n-1} = A_{n-1} x_{n-2}$, and $U = \mathrm{span}\{x_1, x_2, \dots, x_{n-1}\}$. By induction and Lemma 3.7(b), $x_i \in \mathrm{Im}(A_i) \cap \mathrm{Ker}(A_i - \lambda I)$.

By Lemma 3.7(b) and the fact that $A_i A_j = A_j A_i = 0$, if i and j are not neighbors,

$$A_{i-1}x_i = A_{i-1}A_i x_{i-1} = -(1 + \lambda)x_{i-1}, \qquad i = 2, \dots, n-1,$$

$$A_i x_i = \lambda x_i, \qquad i = 1, \dots, n-1,$$

$$A_{i+1}x_i = x_{i+1}, \qquad i = 1, \dots, n-2,$$

and

$$A_j x_i = A_j A_i x_{i-1} = 0, \qquad j \neq i-1, i, i+1.$$

Thus U is invariant under B_n. Hence $r = \dim U \leq n - 1$, since ρ is irreducible.

COROLLARY 3.9. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where* $r = \dim V \geq n$, $n \neq 4$.

Then the associated friendship graph is connected.

Proof. By Corollary 3.5 the friendship graph of ρ is either totally disconnected or connected. By Theorem 3.8 it is not disconnected.

COROLLARY 3.10. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where $r = \dim V \geq n$, $n \neq 4$. Suppose $\rho(\sigma_i) = 1 + A_i$, where rank$(A_i) = k$.*
Then $r = \dim V \leq (n-1)(k-1) + 1$.
In particular, for $k = 2$, $r = \dim V = n$, where $V = \mathbb{C}^n$.

Proof. By Corollary 3.9, the friendship graph of the representation is connected. Arrange the vertices of the graph in a sequence $A_{i_1}, A_{i_2}, \dots, A_{i_{n-1}}$ such that each term A_{i_j}, $2 \leq j \leq n-1$, is a friend of one of the terms $A_{i_1}, A_{i_2}, \dots, A_{i_{j-1}}$. Then

$$\dim\left(\mathrm{Im}(A_{i_1})\right) = k$$

$$\dim\left(\mathrm{Im}(A_{i_1}) + \mathrm{Im}(A_{i_2})\right) \leq k + (k-1) = 2k - 1$$

$$\cdots$$

$$\dim\left(\mathrm{Im}(A_{i_1}) + \cdots + \mathrm{Im}(A_{i_{n-1}})\right)$$
$$\leq k + (n-2)(k-1) = (n-1)(k-1) + 1.$$

Combining Theorem 3.4 and Corollaries 3.9 and 3.10, we get the following

THEOREM 3.11. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where $r = \dim V \geq n$, $n \neq 4$. Suppose $\rho(\sigma_i) = 1 + A_i$, where rank$(A_i) = 2$.*
Then $r = n$ and one of the following holds.

(a) *The full friendship graph has an edge between A_i and A_{i+1} for all i.*

(b) *The full friendship graph has an edge between A_i and A_j whenever A_i and A_j are not neighbors.*

4. FOR CORANK 2 THE FRIENDSHIP GRAPH IS A CHAIN

In this section, we assume throughout that we have an irreducible representation

$$\rho : B_n \to GL_r(\mathbb{C}),$$

where $r \geq n$, and

$$\rho(\sigma_i) = 1 + A_i, \qquad \mathit{rank}(A_i) = 2, 1 \leq i \leq n-1.$$

THEOREM 4.1. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $r \geq n$ and $n \geq 6$. Let rank$(A_1) = 2$.*

Then $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1}) \neq \{0\}$ *for* $1 \leq i \leq n-2$; *that is, the friendship graph of ρ contains the chain graph.*

Proof. Suppose not. Then by Theorem 3.11(b), $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_j) \neq 0$ whenever A_i and A_j are not neighbors. Consider

$$U = \mathrm{Im}(A_1) + \mathrm{Im}(A_2) + \mathrm{Im}(A_3).$$

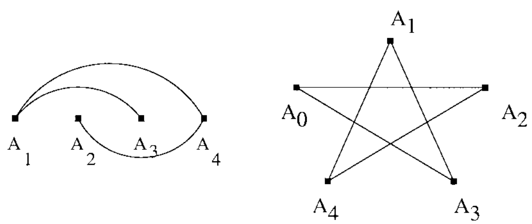
Since $\mathrm{Im}(A_1) \cap \mathrm{Im}(A_3) \neq 0$, $\dim U \leq 5$.

For $i = 4, \dots, n-1$, let a_i, b_i be, respectively, nonzero elements of $\mathrm{Im}(A_1) \cap \mathrm{Im}(A_i)$ and $\mathrm{Im}(A_2) \cap \mathrm{Im}(A_i)$. Since $\mathrm{Im}(A_1) \cap \mathrm{Im}(A_2) = 0$, a_i and b_i are linearly independent, so they form a basis for $\mathrm{Im}(A_i)$, and $\mathrm{Im}(A_i) \subseteq \mathrm{Im}(A_1) + \mathrm{Im}(A_2)$. Thus

$$U = \mathrm{Im}(A_1) + \mathrm{Im}(A_2) + \cdots + \mathrm{Im}(A_{n-1}),$$

which is invariant under $\rho(B_n)$. Thus $r \leq 5$, by the irreducibility of ρ, a contradiction since $r \geq n \geq 6$.

Remark 4.2. For $n = 5$ and ρ satisfying the hypothesis of Theorem 4.1 there are two possible friendship graphs: (1) all neighbors are friends and (2) an exceptional case:



Friendship Graph *Full Friendship Graph*

By [6, Theorem 7.1, part 2], every irreducible representation with the above friendship graph is equivalent to the restriction to B_5 of the Jones representation (see [3, p. 296]).

LEMMA 4.3. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $r \geq n$, $n \geq 5$, and rank$(A_1) = 2$. Suppose that the associated friendship graph contains the chain.*

Then $r = n$ and the associated friendship graph is the chain (*that is, the only edges are between neighbors*).

Proof. By Corollary 3.10, $r = n$. Consider the full friendship graph of ρ. Then

$$\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1}) \neq \{0\}$$

for any i where indices are taken modulo n. If $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1})$ is two-dimensional, then $\mathrm{Im}(A_1) = \mathrm{Im}(A_2) = \cdots$, and $\mathrm{Im}(A_1)$ is a two-dimensional invariant subspace, contradicting the irreducibility of ρ. Hence $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1})$ are one-dimensional.

For any $x \in \mathrm{Im}(A_i)$, $x = A_i y$, $x \neq 0$, we have that

$$Tx = TA_i y = TA_i T^{-1}(Ty) = A_{i+1}(Ty) \in \mathrm{Im}(A_{i+1})$$

for $T = \rho(\tau)$. Moreover, $Tx \neq 0$ because T is invertible.

Choose $x_1 \neq 0$ to be a basis vector for $\mathrm{Im}(A_1) \cap \mathrm{Im}(A_2)$. Define $x_{i+1} = T^i x_1$ for $1 \leq i \leq n-1$. Then x_i is a basis vector for $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1})$.

If for some i, x_i is proportional to x_{i+1} then, because a full friendship graph is a $\mathbb{Z}_n$-graph, all the x_j are proportional to x_1. Then, because we have 5 or more vertices in the full friendship graph, for any A_i there exists j such that both A_j and A_{j+1} are not neighbors of A_i. Then

$$A_i A_j = A_j A_i$$

and

$$A_i A_{j+1} = A_{j+1} A_i.$$

So, if $x \in \mathrm{Im}(A_j) \cap \mathrm{Im}(A_{j+1})$ then $A_i x \in \mathrm{Im}(A_j) \cap \mathrm{Im}(A_{j+1})$. But this means that $\mathrm{span}\{x_1\}$ is an invariant subspace and the representation is not irreducible.

So, if the representation is irreducible, then for any i,

$$x_i \notin \mathrm{span}\{x_{i+1}\}.$$

From this follows that for any i

$$\mathrm{Im}(A_i) = \mathrm{span}\{x_{i-1}, x_i\}$$

and the n vectors $x_0, x_1, \ldots, x_{n-1}$ form a basis of V. Then for any two non-neighbors A_i and A_j

$$\mathrm{Im}(A_i) \cap \mathrm{Im}(A_j) = \{0\}.$$

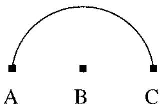
Now, we have the following

THEOREM 4.4. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be irreducible, where $r \geq n$. Suppose that for any generator σ_i, $\rho(\sigma_i) = 1 + A_i$, where rank$(A_i) = 2$.*

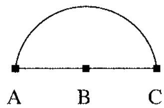
(1) *If $n \geq 6$, then $r = n$ and ρ has a friendship graph which is a chain.*

(2) *If $n = 5$, then $r = 5$ and either ρ has a friendship graph which is a chain or ρ has the exceptional friendship graph (see Remark 4.2).*

(3) *If $n = 4$, then either $r = 4$ and ρ has a friendship graph which is a chain; or ρ has one of the following exceptional friendship graphs*:



or



Proof. (1) If $n \geq 6$, then by Theorem 4.1 the associated friendship graph contains a chain, and, by Lemma 4.3 has no other edges and $r = n$.

(2) If $n = 5$, then by Corollaries 3.9 and 3.10 the friendship graph of ρ is connected and $r = n$. If it contains a chain graph, then, by Lemma 4.3, it has no other edges. If it does not contain a chain graph, we obtain the exceptional case.

(3) If $n = 4$, then by Theorem 3.8 the friendship graph is not totally disconnected. Hence, we have only three possible $\mathbb{Z}_4$-graphs on 4 vertices.

Remark 4.5. It is proven in [6, Chap. 6] that any representation of B_4 with either of the exceptional friendship graphs in (3) of the above theorem is reducible.

5. THE CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF B_n OF CORANK 2

In this section we will describe the representations whose friendship graph is a chain. We start with the following definition:

DEFINITION 6. **The standard representation** is the representation

$$\tau_n : B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$$

defined by

$$\tau_n(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & t & \\ & 1 & 0 & \\ & & & I_{n-1-i} \end{pmatrix},$$

for $i = 1, 2, \ldots, n-1$, where I_k is the $k \times k$ identity matrix.

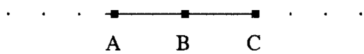
We call the above representation *standard* because of its simplicity. Surprisingly, it does not seem to be well known. In fact it seems that it was first discovered only in 1996 by Dian-Min Tong *et al.* [7, Eq. (19)].

THEOREM 5.1. *Let $\rho : B_n \to GL_n(\mathbb{C})$ be an irreducible representation, where $n \geq 4$. Suppose that $\rho(\sigma_1) = 1 + A_1$, where $rank(A_1) = 2$, and the associated friendship graph of ρ is a chain.*

Then ρ is equivalent to a specialization $\tau_n(u)$ of the standard representation for some $u \in \mathbb{C}^*$.

Before proving the theorem, we will need the following technical lemma.

LEMMA 5.2. *Let A be a friend and a neighbor of B, B be a friend and a neighbor of C, and suppose that A is not a friend of C:*



Let $a \neq 0$ be such that $\operatorname{span}\{a\} = \operatorname{Im}(A) \cap \operatorname{Im}(B)$, *and let* $b = (1 + B)a$. *Then*:

(1) $\operatorname{span}\{b\} = \operatorname{Im}(C) \cap \operatorname{Im}(B)$.

(2) $(1 + B)b \in \operatorname{span}\{a\}$ *and* $(1 + B)b \neq 0$.

(3) *The vectors a and b are linearly independent.*

Proof. First of all, notice that the vector b is non-zero, because $1 + B$ is invertible and $a \neq 0$.

(1) $b = (1 + B)a \in \operatorname{Im}(B)$, because $a \in \operatorname{Im}(B)$.

A and C are not friends, that is, $CA = 0$, so $Ca = 0$. Let $a = Ba_1$. Then

$$(1 + B)a = (1 + B + BC)a = (1 + B + BC)Ba_1 = (B + B^2 + BCB)a_1$$
$$= (C + C^2 + CBC)a_1 \in \operatorname{Im}(C);$$

that is, $b \in \operatorname{Im}(C) \cap \operatorname{Im}(B)$, and because $\operatorname{Im}(C) \cap \operatorname{Im}(B)$ is one-dimensional and $b \neq 0$,

$$\operatorname{span}\{b\} = \operatorname{Im}(C) \cap \operatorname{Im}(B).$$

(2) Clearly, $(1 + B)b \in \operatorname{Im}(B)$.

Note that $Ab = 0$, as $b \in \operatorname{Im}(C)$ by the above, and $AC = 0$. Let $b = Ba'$. Then

$$(1 + B)b = (1 + B + BA)b = (1 + B + BA)Ba'$$
$$= (A + A^2 + ABA)a' \in \operatorname{Im}(A).$$

(3) $a \in \mathrm{Im}(A)$, $b \in \mathrm{Im}(C)$ by part (1), and $\mathrm{Im}(A) \cap \mathrm{Im}(C) = \{0\}$ by the hypothesis of the lemma.

Proof of Theorem 5.1. We include the redundant generator σ_0, and indices are modulo n. Consider $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1})$, which is 0, 1, or 2-dimensional. It is non-zero, because of the hypothesis that the friendship graph is a chain. It is not 2-dimensional, for then

$$\mathrm{Im}(A_0) = \mathrm{Im}(A_1) = \cdots = \mathrm{Im}(A_{n-1})$$

would be a 2-dimensional invariant subspace, contradicting the irreducibility of ρ. Hence, $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1})$ is one-dimensional.

Let a_0 be a basis vector for $\mathrm{Im}(A_0) \cap \mathrm{Im}(A_1)$. Let

$$a_1 = (1 + A_1)a_0, \qquad a_2 = (1 + A_2)a_1, \dots, a_{n-1} = (1 + A_{n-1})a_{n-2}.$$

By induction and Lemma 5.2, part (1), a_i is a basis vector for $\mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1})$, for $0 \le i \le n - 1$. By Lemma 5.2, part (3), a_i and a_{i+1} are linearly independent. Thus $\{a_i, a_{i+1}\}$ is a basis for $\mathrm{Im}(A_i)$.

Since

$$\mathrm{span}\{a_0, \dots a_{n-1}\} = \mathrm{Im}(A_1) + \cdots + \mathrm{Im}(A_{n-1})$$

is invariant under B_n and ρ is an n-dimensional irreducible representation, $\{a_0, \dots a_{n-1}\}$ is a basis for $\mathbb{C}^n$.

We now wish to determine the action of $\rho(\sigma_1), \rho(\sigma_2), \dots, \rho(\sigma_{n-1})$ on this basis.

Consider $a_i \in \mathrm{Im}(A_i) \cap \mathrm{Im}(A_{i+1})$. If $j \neq i$, $i + 1$, then A_j is not a neighbor of one of A_i, A_{i+1} (since $n \ge 4$), say A_k, and then $A_k A_j = A_j A_k = 0$, so $A_j a_i = 0$, and

$$\rho(\sigma_j)a_i = (1 + A_j)a_i = a_i.$$

By our construction

$$\rho(\sigma_{i+1})a_i = (1 + A_{i+1})a_i = a_{i+1}$$

for $0 \le i \le n - 2$.

By Lemma 5.2, part (2),

$$\rho(\sigma_i)a_i = (1 + A_i)a_i = u_i a_{i-1},$$

for $1 \le i \le n - 1$, where $u_i \in \mathbb{C}^*$.

By the above calculations the matrices of $\rho(\sigma_1), \dots, \rho(\sigma_{n-1})$ with respect to the basis $a_0, a_1, \dots, a_{n-1}$ are

$$\rho(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & u_i & \\ & 1 & 0 & \\ & & & I_{n-1-i} \end{pmatrix},$$

for $i = 1, 2, \ldots, n-1$, where I_k is the $k \times k$ identity matrix, and $u_1, \ldots, u_{n-1} \in \mathbb{C}^*$. Since $\sigma_1, \ldots, \sigma_{n-1}$ are conjugate in B_n, the u_i are all equal, and we have the standard representation.

Now let us consider when the standard representation is irreducible.

LEMMA 5.3. *If $u = 1$ then $\tau_n(u)$ is reducible.*

Proof. If $u = 1$ then the vector $v = (1, 1, 1, \ldots, 1)^T$ is a fixed vector.

LEMMA 5.4. *If $u \neq 1$ then $\tau_n(u)$ is irreducible.*

Proof. We need to prove that starting from any non-zero vector $x = \Sigma a_i e_i$, we can generate the whole space. Obviously, it is enough to show that we can generate one of the standard basis vectors e_i. To do this, take i such that $a_i \neq 0$. Consider the operator

$$H = A + A^2 + ABA = B + B^2 + BAB,$$

where $A = \rho(\sigma_{i-1}) - 1$ and $B = \rho(\sigma_i) - 1$. By a direct calculation $Hx = (u-1)a_i e_i$. Because $u \neq 1$ the vector Hx is a non-zero multiple of e_i.

Now, we have the classification theorem for the representations of B_n of corank 2.

THEOREM 5.5. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation of B_n for $n \geq 6$. Let $r \geq n$, and let $\rho(\sigma_1) = 1 + A_1$ with rank$(A_1) = 2$.*
Then $r = n$ and ρ is equivalent to the representation

$$\tau : B_n \to GL_n(\mathbb{C}),$$

$$\tau(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & u & \\ & 1 & 0 & \\ & & & I_{n-1-i} \end{pmatrix},$$

for $i = 1, 2, \ldots, n-1$, where I_k is the $k \times k$ identity matrix, and $u \in \mathbb{C}^*$, $u \neq 1$. These representations are non-equivalent for different values of u.

Proof. By Theorem 4.4 the friendship graph of ρ is a chain. Then, by Theorem 5.1, ρ is equivalent to a standard representation $\tau(u)$ for some $u \in \mathbb{C}^*$. By Lemmas 5.3 and 5.4, $u \neq 1$.

Combining Theorem 5.5 and the classification theorem of Formanek (see [3, Theorem 23]), we get the following

COROLLARY 5.6. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation of B_n for $n \geq 7$. Let* corank$(\rho) = 2$.
Then ρ is equivalent to a specialization of the standard representation $\tau_n(u)$, for some $u \neq 1$, $u \in \mathbb{C}^*$.

6. THE CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF B_n OF DIMENSION n, $n \geq 9$

In this section we will prove the main result of this paper.

THEOREM 6.1 (The Main Theorem). *Suppose that $\rho : B_n \to GL_n(\mathbb{C})$ is an irreducible representation of B_n of dimension $n \geq 9$. Then it is equivalent to the tensor product of a one-dimensional representation and a specialization for $u \neq 0, 1$ of the standard representation.*

We proved in Section 5, Theorem 5.5, and Corollary 5.6 that for $n \geq 7$ every irreducible complex representation of B_n of corank 2 is a specialization of the standard representation (see Definition 6.) To complete the proof of Theorem 6.1 it is enough to show that for $n \geq 9$ every irreducible representation of B_n of dimension n is the tensor product of a one-dimensional representation and a representation of corank 2. This will be done in Theorem 6.5. Before that we need some preparatory results. The key of the proof is the following theorem, which is similar to Theorem 16 of [3].

THEOREM 6.2. *Suppose that $\rho : B_{n+1} \to GL_{n+1}(\mathbb{C})$ is an irreducible representation of B_{n+1} of dimension $n+1$ ($n \geq 4$). Suppose that the restriction of ρ, $\rho \mid B_{n-1} \times \langle \sigma_n \rangle$, stabilizes the one-dimensional subspace $\mathbb{C}v$ of $\mathbb{C}^{n+1}$. Then rank$(\rho(\sigma_1) - yI) = 2$ for some $y \in \mathbb{C}^*$.*

Proof. For notational simplicity we will write σ instead of $\rho(\sigma)$ for $\sigma \in B_n$.

By hypothesis,

$$\rho \mid B_{n-1} \times \langle \sigma_n \rangle : \mathbb{C}v \to \mathbb{C}v$$

is a one-dimensional representation of $B_{n-1} \times B_2$, so there exist $x, y \in \mathbb{C}^*$ such that

$$\sigma_1 v = \sigma_2 v = \cdots = \sigma_{n-2} v = yv, \qquad \sigma_n v = xv.$$

Consider $\theta = \theta_{n+1} = \sigma_1 \sigma_2 \cdots \sigma_n$, $\sigma_0 = \theta \sigma_n \theta^{-1}$,

$$v_n = v, \qquad v_{n+1} = \theta v, \qquad v_1 = \theta^2 v, \ldots, v_{n-1} = \theta^n v.$$

Conjugation by θ permutes $\sigma_1, \ldots, \sigma_n, \sigma_0$ cyclically.

Because ρ is an irreducible representation and θ^{n+1} is central in B_{n+1}, $\rho(\theta^{n+1}) = dI$ for some $d \in \mathbb{C}^*$. Thus, the left action of θ permutes $\mathbb{C}v_1, \mathbb{C}v_2, \ldots, \mathbb{C}v_{n+1}$ cyclically.

We have

$$\sigma_i v_i = x v_i,$$
$$\sigma_i v_{i+j} = y v_{i+j}$$

for

$$i = 1, \dots, n+1, \qquad j = 2, \dots, n-1,$$

where indices are taken modulo $n+1$.

The following table summarizes the above calculations:

	v_1	v_2	v_3	$\cdots$	v_{n-1}	v_n	v_{n+1}
σ_1	xv_1	$*$	yv_3	$\cdots$	yv_{n-1}	yv_n	$*$
σ_2	$*$	xv_2	$*$	$\cdots$	yv_{n-1}	yv_n	yv_{n+1}
σ_3	yv_1	$*$	xv_3	$\cdots$	yv_{n-1}	yv_n	yv_{n+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
σ_{n-1}	yv_1	yv_2	yv_3	$\cdots$	xv_{n-1}	$*$	yv_{n+1}
σ_n	yv_1	yv_2	yv_3	$\cdots$	$*$	xv_n	$*$
σ_0	yv_1	yv_2	yv_3	$\cdots$	yv_{n-1}	$*$	xv_{n+1}

Suppose that $v_1, \dots, v_{n+1}$ are linearly dependent. Consider

$$a_1 v_1 + a_2 v_2 + \cdots + a_t v_t = a_1 v_1 + a_2 \theta v_1 + \cdots + a_t \theta^{t-1} v_1 = 0,$$

a linear dependence relationship with minimal t.

In the equation above, $a_1 \neq 0$, since θ is invertible, and $a_t \neq 0$ by the minimality of t.

We claim that $t \geq n$. Indeed, suppose that $t \leq n-1$. Then v_{n-1} is a linear combination of $v_1, \dots, v_{n-2}$, which are eigenvectors for σ_n with $\sigma_n v_i = y v_i$, $i = 1, \dots, n-2$. So, $\sigma_n v_{n-1} = y v_{n-1}$. Applying θ^3 implies that $\sigma_2 v_1 = y v_1$, which means that $\mathbb{C} v_1$ is B_{n+1}-invariant, which contradicts the irreducibility of ρ. So, $t \geq n$.

Thus, $v_1, \dots, v_{n-1}$ are linearly independent.

Assume that $rank(\sigma_1 - yI) > 2$. Then, as

$$\dim(\operatorname{Ker}(\sigma_1 - yI)) + rank(\sigma_1 - yI) = n + 1,$$

$\dim(\operatorname{Ker}(\sigma_1 - yI)) \leq n - 2$.

Note that $v_3, \dots, v_n$ are $n-2$ linearly independent elements of $L = \operatorname{Ker}(\sigma_1 - yI)$. So, $\dim(\operatorname{Ker}(\sigma_1 - yI)) = n - 2$, and $L = \operatorname{span}\{v_3, \dots, v_n\}$.

Since the vectors $\{v_1, \dots, v_{n-1}\}$ are linearly independent, $\{v_2, \dots, v_n\}$ and $\{v_3, \dots, v_{n+1}\}$ are also linearly independent. Therefore $v_2 \notin L$, and $v_{n+1} \notin L$.

The action of θ implies that for $i = 1, \dots, n+1$

$$\operatorname{Ker}(\sigma_i - yI) = \operatorname{span}\{v_{i+2}, \dots, v_{i-2}\},$$

$v_{i-1} \notin L$, and $v_{i+1} \notin L$, where indices are taken modulo $n+1$.

Since σ_1 commutes with σ_n, and $n \geq 4$,

$$(\sigma_n - yI)\sigma_1 v_2 = \sigma_1(\sigma_n - yI)v_2 = 0.$$

Thus, $\sigma_1 v_2 \in \operatorname{Ker}(\sigma_n - yI)$, so

$$\sigma_1 v_2 = b_1 v_1 + b_2 v_2 + \cdots + b_s v_s,$$

where $1 \leq s \leq n-2$ and $b_s \neq 0$.

We claim that $s \leq 2$. Indeed, if $s \geq 3$, then

$$\begin{aligned} 0 &= \sigma_1(\sigma_{s+1} - yI)v_2 = (\sigma_{s+1} - yI)\sigma_1 v_2 \\ &= (\sigma_{s+1} - yI)(b_1 v_1 + b_2 v_2 + \cdots + b_s v_s) = (\sigma_{s+1} - yI) b_s v_s. \end{aligned}$$

This contradicts the fact that $v_s \notin \operatorname{Ker}(\sigma_{s+1} - yI)$.

Thus,

$$\sigma_1 v_2 = b_1 v_1 + b_2 v_2, \qquad b_1, b_2 \in \mathbb{C}.$$

By a symmetric argument which reverses the roles of σ_1 and σ_n, and starts with the equation

$$(\sigma_1 - yI)\sigma_n v_{n-1} = \sigma_n(\sigma_1 - yI)v_{n-1} = 0,$$

we obtain

$$\sigma_n v_{n-1} = c_1 v_{n-1} + c_2 v_n, \qquad c_1, c_2 \in \mathbb{C}.$$

Using the action of θ, we get the following table:

	v_1	v_2	$\cdots$	v_n	v_{n+1}
σ_1	xv_1	$b_1v_1 + b_2v_2$	$\cdots$	yv_n	$c_1v_{n+1} + c_2v_1$
σ_2	$c_1v_1 + c_2v_2$	xv_2	$\cdots$	yv_n	yv_{n+1}
σ_3	yv_1	$c_1v_2 + c_2v_3$	$\cdots$	yv_n	yv_{n+1}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
σ_{n-1}	yv_1	yv_2	$\cdots$	$b_1v_{n-1} + b_2v_n$	yv_{n+1}
σ_n	yv_1	yv_2	$\cdots$	xv_n	$b_1v_n + b_2v_{n+1}$

Span$\{v_1, v_2, \ldots, v_{n+1}\}$ is B_{n+1}-invariant. Thus, if $\{v_1, v_2, \ldots, v_{n+1}\}$ are linearly dependent, then ρ is reducible. So, $\{v_1, v_2, \ldots, v_{n+1}\}$ are linearly independent, and they form a basis for $\mathbb{C}^{n+1}$.

In this basis,

$$\sigma_1 = \begin{pmatrix} x & b_1 & & & & c_2 \\ 0 & b_2 & & & & \\ & & y & & & \\ & & & \ddots & & \\ & & & & y & \\ & & & & & c_1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} y & & & & & & \\ & c_1 & & & & & \\ & c_2 & x & b_1 & & & \\ & & & b_2 & & & \\ & & & & y & & \\ & & & & & \ddots & \\ & & & & & & y \end{pmatrix}.$$

Using the $(3,2)$-entry of the matrix $\sigma_1\sigma_3 = \sigma_3\sigma_1$, we have

$$b_2 c_2 = y c_2.$$

If $c_2 = 0$, then $\mathbb{C}v_1$ is invariant under B_{n+1}, which contradicts the irreducibility of ρ. So, $c_2 \neq 0$. Thus, $b_2 = y$. Then $rank(\sigma_1 - yI) \leq 2$, a contradiction.

So, $rank(\sigma_1 - yI) \leq 2$. But by [3, Theorem 10], the case $rank(\sigma_1 - yI) = 1$ is impossible. Thus, $rank(\sigma_1 - yI) = 2$.

The following argument is due to Formanek. He also used it in [3, Lemma 17 and Corollary 18]. My original argument was much longer.

Lemma 6.3 below is a corollary of Theorem 23 of [3], which classifies the irreducible representations of B_n of dimension at most $n - 1$.

LEMMA 6.3. *If $\rho : B_n \to GL_r(\mathbb{C})$ is irreducible and $r \leq n - 3$, then ρ is one-dimensional.*

LEMMA 6.4. *Let $\rho : B_n \to GL_r(\mathbb{C})$ be a representation, where $n \geq 6$. Suppose that λ is an eigenvalue of $\rho(\sigma_{n-1})$. Suppose that the largest Jordan block corresponding to λ has size s and multiplicity d.*

If $d \leq n - 5$, then $\rho \mid B_{n-2} \times \langle \sigma_{n-1} \rangle$ has a one-dimensional invariant subspace.

Proof. Let $f(t)$ be the minimal polynomial of $\rho(\sigma_{n-1})$. Set $m(t) = f(t)/(t - \lambda)$. Let V be the image of $\mathbb{C}^r$ under $m(\rho(\sigma_{n-1}))$. Then V is invariant under $\rho \mid B_{n-2} \times \langle \sigma_{n-1} \rangle$, and $\dim V = d$. If $d \leq n - 5$, then by Lemma 6.3, all composition factors of

$$\rho \mid B_{n-2} \times \langle \sigma_{n-1} \rangle : V \to V$$

are one-dimensional.

THEOREM 6.5. *For $n \geq 9$, every n-dimensional complex irreducible representation ρ of the braid group B_n is equivalent to a tensor product of a one-dimensional representation $\chi(y)$, $y \in \mathbb{C}^*$, and an n-dimensional representation of corank* 2.

Proof. Assume not. Then by Theorem 6.2 and Lemma 6.4, the largest Jordan block corresponding to each eigenvalue of $\rho(\sigma_{n-1})$ has multiplicity $\geq n - 4$.

If $\rho(\sigma_{n-1})$ has two or more eigenvalues, we get

$$(n-4) + (n-4) \leq n,$$

a contradiction, since $n \geq 9$. Similarly, if some eigenvalue has the corresponding largest Jordan block of size $s \geq 2$, we get a contradiction

$$2(n-4) \leq n.$$

Thus, $\rho(\sigma_{n-1})$ has only one eigenvalue λ and the Jordan canonical form of $\rho(\sigma_{n-1})$ consists of 1×1 elementary Jordan blocks. But then $\rho(\sigma_{n-1}) = \lambda I$, which contradicts the irreducibility of ρ.

This completes the proof of the theorem, and thus the proof of Theorem 6.1.

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