ON CRACK PROPAGATION IN A NONLINEAR COUPLED THERMO-MECHANICAL SYSTEM

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Abstract. The crack propagation problem in a coupled thermo-mechanical system of nonlinear media have been considered and the related path-independent integrals are given. It is shown the dynamical crack extension force in a coupled thermo-mechanical system equals to this integral, thus it could consider such integrals as a nonlinear fracture criterion for coupled thermo-mechanical fracture dynamics.

Keywords. Crack propagation; coupled thermo-mechanical system; nonlinear media; path-independent integral; fracture criterion.

INTRODUCTION

It is well-known in fracture statics that the J-integral (Rice, 1968)

$$J = \int_\Gamma W\,dy - T_i \frac{\partial u_i}{\partial x} ds \quad . \qquad (1)$$

Here W is strain energy density, $\int \sigma_{ij} de_{ij}$; T_i is the surface traction; u_i is the displacement; and Γ is integration path around the crack tip. For any path Γ around the crack tip, J is path-independent.

Sometimes, in engineering problems thermo-mechanical coupling is important, and cannot be neglected. Here thermo-mechanical coupling effects, dynamical effect and crack propagation phenomemon should be included in the analysis (Ouyang, 1981).

This paper deals with the crack propagation problem for nonlinear coupled thermo-mechanical system. Both the nonlinear elastic and elastic-plastic media are considered, and some related path-independent integrals are worked out. For explaining the physical meaning of such integrals, a notched specimen is used, and we have shown that this integral equals to the dynamical crack extension force. Thus, it is possible to form nonlinear dynamical fracture criterion by using these integrals.

BASIC EQUATIONS FOR COUPLED THERMO-MECHANICAL SYSTEM OF NONLINEAR CONTINUA

We consider a solid body subjected to external forces and heating. Assume that the material is nonlinear elastic or elastic-plastic and that it is stress-free at a uniform reference temperature T_0 when all external forces are removed. Choose the rectangular Cartesian coordinates x_i.

Let u_i, e_{ij}, σ_{ij}, ρ, T, v_i, h_i be the displacements, strain tensor, stress tensor, density, temperature, velocity and heat flux. Then, for a coupled thermo-mechanical system of nonlinear continua, we have the following governing equations:

Strain-displacement equation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \; , \quad i,j = 1,2,3 \qquad (2)$$

Constitutive equation:

$$\sigma_{ij} = f'_{ij}(e_{kl}; T) = f_{ij}(e_{kl}) - \beta_{ij}\theta \qquad (3)$$

$$\theta = T - T_0 \qquad (4)$$

β_{ij} : Thermal moduli

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0 \qquad (5)$$

$$v_i = \frac{\partial u_i}{\partial t} \qquad (6)$$

Equation of motion:

$$\rho \dot{v}_i = \sigma_{ij,j} + X_i \qquad (7)$$

Fourier's law:

$$h_i = -k_{ij} T_{,j} \qquad (8)$$

k_{ij}: Heat conduction coefficients.

Equation of energy conservation:

$$-h_{i,i} = \rho C_v \dot{\theta} + T \beta_{ij} \dot{e}_{ij} \qquad (9)$$

Here C_v is heat capacity per unit mass at constant strain, symbol "$\cdot$" means $\partial/\partial t$, ",j" means partial differentiation about x_j.

Therefore, we get 23 equations (2) - (9) for 23 unknowns u_i, v_i, σ_{ij}, e_{ij}, , T, h_i.

If we introduce vector H_i, proportional to the entropy displacement, such that

$$h_i = \frac{\partial H_i}{\partial t} \qquad (10)$$

and

$$H_i = 0 \ , \quad \text{when} \quad \theta = e_{ij} = 0 \qquad (11)$$

Then (9) may be integrated to give

$$-H_{i,i} = \rho C_v \theta + T_o \beta_{ij} e_{ij} \ . \qquad (12)$$

Here we have assumed that $\theta = T - T_o \ll T_o$, thus the approximation $T \approx T_o$ is valid.

In subsequent discussion about crack propagation, we need the following thermal and mechanical, initial and boundary conditions:

Boundary conditions:

$$u_i = \bar{u}_i \ , \quad \text{on boundary } S_u \qquad (13)$$

$$\overset{\nu}{T}_i = \bar{T}_i \ , \quad \text{on boundary } S \qquad (14)$$

$$\theta = \bar{\theta} \ , \quad \text{on boundary } S_1 \qquad (15)$$

$$h_n = \bar{h}_n \ , \quad \text{on boundary } S-S_1 \qquad (16)$$

$$\overset{\nu}{T}_i = 0, \ \theta = 0, \quad \text{on crack surface} \qquad (17)$$

Initial conditions:

$$u_i = g_i(x_j) \ , \quad t=0, \text{ in } V \qquad (18)$$

$$\frac{\partial u_i}{\partial t} = k_i(x_j) \ , \quad t=0, \text{ in } V \qquad (19)$$

$$\theta = \theta_o(x_j) \ , \quad t=0, \text{ in } V \qquad (20)$$

Here T_i is the surface traction, $h_n = h_i \nu_i$ is the component of the heat flux vector h_i in the direction of the outer normal to the boundary. $\bar{u}_i$, $\bar{T}_i$, $\bar{\theta}$, $\bar{h}_n$ are assigned values on the boundary. g_i, k_i, θ_o are given in V.

CRACK PROPAGATION IN NONLINEAR ELASTIC MEDIA

For the crack propagation in nonlinear elastic media, we may propose the following:

Theorem 1. The integral

$$Y_1 = \int_{t_o}^{t_1} \Big(\int_\Gamma (W + Q - X_i u_i - K) dy - \big(T_i \frac{\partial u_i}{\partial x} + \frac{\theta \nu_i}{T_o} \frac{\partial H_i}{\partial x} \big) dS \Big) dt + \int_V \rho v_i \frac{\partial u_i}{\partial x} dV \Big|_{t_o}^{t_1} + \int_{t_o}^{t_1} \int_V \frac{1}{T_o} \lambda_{ij} \dot{H}_j \frac{\partial}{\partial x} H_i \, dV dt \qquad (21)$$

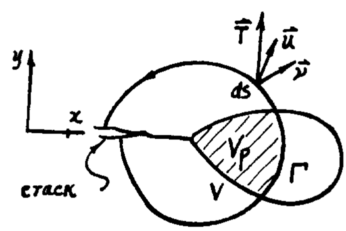

Fig. 1. Crack in Nonlinear Elastic Media.

is path-independent for any path Γ around the crack tip (Fig. 1) and any $t_1 > t_o \geq 0$. Here

$$W = \int \sigma_{ij} de_{ij} \qquad (22)$$

is the strain energy density under uniform temperature,

$$Q = \int \frac{\rho C_v \theta}{T_o} d\theta \qquad (23)$$

is the heat that may be transformed into useful work.

$$K = \frac{1}{2} \rho v_i v_i \qquad (24)$$

is the kinetic energy,

$$\lambda_{ij} = (k_{ij})^{-1} \qquad (25)$$

is the inverse of matrix (k_{ij}). The domain V is bounded by Γ and crack surfaces. Here we assume that X_i is independent of x_i

If we consider moving paths $\Gamma(t)$, then we could obtain the following:

Theorem 2. The integral

$$Y_2=\int_{t_0}^{t_1}\Big(\int_{\Gamma(t)}(W+Q+(\rho a_i-X_i)u_i)dy$$
$$-(T_i\frac{\partial u_i}{\partial x}-\frac{\theta}{T_0}\nu_i\frac{\partial H_i}{\partial x})ds\Big)dt$$
$$-\int_{t_0}^{t_1}\int_{V(t)}\rho u_i\frac{\partial a_i}{\partial x}dvdt$$
$$+\int_{t_0}^{t_1}\int_{V(t)}\frac{1}{T_0}\lambda_{ij}\dot H_j\frac{\partial H_i}{\partial x}dvdt \tag{26}$$

or, simply

$$Y_3=\int_{\Gamma(t)}(W+Q+(\rho a_i-X_i)u_i)dy$$
$$-(T_i\frac{\partial u_i}{\partial x}-\frac{\theta}{T_0}\nu_i\frac{\partial H_i}{\partial x})ds-\int_{V(t)}\rho u_i\frac{\partial a_i}{\partial x}dv$$
$$+\int_{V(t)}\frac{1}{T_0}\lambda_{ij}\dot H_j\frac{\partial H_i}{\partial x}dv \tag{27}$$

CRACK PROPAGATION IN ELASTIC-PLASTIC MEDIA

We introduce the integral

$$Y_4=\int_{t_0}^{t_1}\Big(\int_{\Gamma}(W_e+Q-K-X_iu_i)dy$$
$$-(T_i\frac{\partial u_i}{\partial x}-\frac{\theta}{T_0}\nu_i\frac{\partial H_i}{\partial x})ds\Big)dt$$
$$+\int_{t_0}^{t_1}\int_{V_p}(\sigma_{ij}+\beta_{ij}\theta)\frac{\partial}{\partial x}e_{ij}^{p}dvdt$$
$$+\int_{t_0}^{t_1}\int_{V}\frac{1}{T_0}\lambda_{ij}\dot H_j\frac{\partial H_i}{\partial x}dvdt+\int_V\rho v_i\frac{\partial u_i}{\partial x}dv\Big|_{t_0}^{t_1} \tag{28}$$

Here W_e is the elastic strain energy density,

$$W_e=\int f_{ij}de_{ij}^{e} \tag{29}$$

V_p is the plastic region within path Γ, e_{ij}^p is the plastic strain.

Now we have the following:

Theorem 3. The integral Y_4 is path-independent for any path Γ around the crack tip and $t_1>t_0\geqslant 0$ in the case of elastic-plastic crack propagation.

For moving path $\Gamma(t)$, we may show the following:

Theorem 4. The integral

$$Y_5=\int_{t_0}^{t_1}\Big(\int_{\Gamma(t)}(W_e+Q+(\rho a_i-X_i)u_i)dy$$
$$-(T_i\frac{\partial u_i}{\partial x}-\frac{\theta}{T_0}\nu_i\frac{\partial H_i}{\partial x})ds\Big)dt$$
$$+\int_{t_0}^{t_1}\int_{V_p(t)}(\sigma_{ij}+\beta_{ij}\theta)\frac{\partial}{\partial x}e_{ij}^{p}dvdt$$
$$+\int_{t_0}^{t_1}\int_{V(t)}\frac{1}{T_0}\lambda_{ij}\dot H_j\frac{\partial H_i}{\partial x}dvdt$$
$$-\int_{t_0}^{t_1}\int_{V(t)}\rho u_i\frac{\partial a_i}{\partial x}dvdt \tag{30}$$

or simply

$$Y_6=\int_{\Gamma(t)}(W_e+Q+(\rho a_i-X_i)u_i)dy$$
$$-(T_i\frac{\partial u_i}{\partial x}-\frac{\theta}{T_0}\nu_i\frac{\partial H_i}{\partial x})ds$$
$$+\int_{V_p(t)}(\sigma_{ij}+\beta_{ij}\theta)\frac{\partial}{\partial x}e_{ij}^{p}dv$$
$$+\int_{V(t)}\frac{1}{T_0}\lambda_{ij}\dot H_j\frac{\partial H_i}{\partial x}dv$$
$$-\int_{V(t)}\rho u_i\frac{\partial a_i}{\partial x}dv \tag{31}$$

is path-independent for any path $\Gamma(t)$ around the crack tip and any $t_1>t_0\geqslant 0$.

Theorem 5. The integral

$$Y_7=\int_{t_0}^{t_1}\Big(\int_{\Gamma(t)}(W_e+Q_e+(\rho a_i-X_i)u_i^{e})dy$$

$$-(T_i \frac{\partial u_i^e}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x})ds)dt$$

$$-\int_{t_0}^{t_1}\int_V \rho u_i^e \frac{\partial a_i}{\partial x} dv dt$$

$$+\int_{t_0}^{t_1}\int_V \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv dt \qquad (32)$$

or simply

$$Y_8 = \int_{\Gamma(t)} (W_e + Q_e + (\rho a_i - X_i)u_i^e)dy$$

$$-(T_i \frac{\partial u_i^e}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x})ds - \int_V \rho u_i^e \frac{\partial a_i}{\partial x} dv$$

$$+\int_V \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv \qquad (33)$$

is path-independent for any path $\Gamma(t)$ around the crack tip and any $t_1 > t_0 \geqslant 0$, here

$$Q_e = \int \frac{\rho c_v \theta d\theta^e}{T_0}$$

$$\theta_e = -\frac{1}{\rho c_v}(H_{i,i} + T_0 \beta_{ij} e_{ij}^e) \qquad (34)$$

Y-INTEGRAL AND DYNAMICAL CRACK EXTENSION FORCE

Consider a notched specimen, as shown in Fig. 2. When the notch width tends to zero, we get a cracked body. Suppose boundary traction T_i is given on S_T, and θ is given on $S_1 \subset S$. Let the crack extends a distance Δa. Consider the energy relation during this crack extension.

The work done by the applied traction is

$$\Delta A_T = \int_{S_T} T_i \Delta u_i ds$$

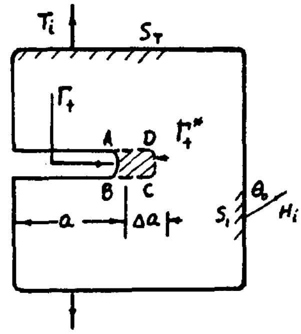

Fig. 2. Crack Extension Force.

Here u_i is the displacement increments after crack extension.

The work done by the applied body force consists of two parts:

a). The one on the shadowed region ΔV, released during crack extension:

$$\Delta A_{B1} = -\int_{\Delta V} (X_i - \rho a_i) u_i^e dv$$

b). The one on the region $V-\Delta V$, it equals to

$$\Delta A_{B2} = \int_{V-\Delta V} (X_i - \rho a_i) \Delta u_i dv$$

The heat flux into the body is

$$\Delta A_H = -\int_{S_1} \frac{\theta}{T_0} \Delta H_i \nu_i ds$$

The work done by the internal force also consists of two parts:

a). The one done in the region ΔV, it equals to the elastic strain energy released during crack extension:

$$\Delta A_I = -\int_{\Delta V} (\int \sigma_{ij} de_{ij}^e) dv$$

b). The one on the region $V-\Delta V$:

$$\Delta A_2 = \int_{V-\Delta V} \sigma_{ij} \Delta e_{ij} \, dv$$

The heat energy absorbed for useful work consists of two parts:

a). $\Delta A_3 = -\int_V \frac{\theta}{T_0} \Delta H_{i,i} \, dv$

b). $\Delta A_4 = \int_{\Delta V} \left(\int \frac{\theta}{T_0} dH_{i,i} \right) dv$

The internal dissipation energy is:
The one in V−ΔV:

$$\Delta A_5 = \int_{V-\Delta V} \frac{1}{T_0} \lambda_{ij} \dot{H}_j \Delta H_i \, dv$$

Thus, the crack extension force is determined by the following equality:

$$\tilde{G} \Delta a = \Delta A_T + \Delta A_{B1} + \Delta A_{B2} + \Delta A_H$$

$$-(\Delta A_1 + \Delta A_2 + \Delta A_3 + \Delta A_4 + \Delta A_5)$$

So

$$\tilde{G} = \int_s T_i \frac{\partial u_i}{\partial a} ds + \int_V (X_i - \rho a_i) \frac{\partial u_i}{\partial a} dv$$

$$-\int_V \sigma_{ij} \frac{\partial e_{ij}}{\partial a} dv - \int_s \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial a} ds$$

$$-\int_V \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial a} dv + \int_V \frac{\theta}{T_0} \frac{\partial H_{i,i}}{\partial a} dv$$

$$+\lim_{\Delta A \to 0} \frac{1}{\Delta a} \int_{\Delta V} \left(\int \sigma_{ij} de_{ij}^e \right) dv - \lim_{\Delta A \to 0} \int_{\Delta V} (X_i - \rho a_i) u_i^e dv$$

$$-\lim_{\Delta A \to 0} \frac{1}{\Delta a} \int_{\Delta V} \left(\int \frac{\theta}{T_0} dH_{i,i} \right) dv$$

From (33), we also have

$$\tilde{Y}_8 = \int_\Gamma (W_e + Q_e + (\rho a_i - X_i) u_i^e) dy \qquad (35)$$

for notched specimen.

Therefore, we have

$$\tilde{G} = \tilde{Y}_8$$

Let the notch width tend to zero, we get

$$G = Y_8 \qquad (36)$$

Thus, we have proved that the integral Y_8 is the dynamical crack extension force. And it is possible to form some nonlinear fracture criterion with this integral in fracture dynamics.

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